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Quantum Bruhat Graphs and Tilted Richardson Varieties

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Abstract. The quantum Bruhat graph is a weighted directed graph on a finite Weyl group first defined by Brenti–Fomin–Postnikov. It encodes quantum Monk's rule and can be utilized to study the 3-point Gromov–Witten invariants of the flag variety. In this paper, we provide a combinatorial formula for the minimal weights between any pair of permutations on the quantum Bruhat graph, and consequently obtain an Ehresmann-like characterization for the tilted Bruhat order. We define the tilted Richardson variety $\mathcal{T}_{u,v}$, with a stratification that gives a geometric meaning to intervals in the tilted Bruhat order. We prove some fundamental geometric properties of this new family of varieties, including their dimensions, closure relations, irreducibility, and a Deodhar-like decomposition. We demonstrate their equivalence to the two-point curve neighborhoods of Schubert varieties X_u and X^v in the minimal degree, and relate their cohomology classes to quantum products of Schubert classes.

Keywords: quantum Bruhat graphs, quantum Schubert calculus, Richardson varieties.

1 Introduction

Hilbert's fifteenth problem, Schubert calculus, concerns the full flag variety

$$\mathrm{Fl}_n = \{0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \mid \dim V_i = i \text{ for } i = 1, \ldots, n-1\}$$

and its Schubert decomposition $\mathrm{Fl}_n = \bigsqcup_{w \in S_n} X_w^\circ$. The cohomology ring $H^*(\mathrm{Fl}_n)$ has a linear basis given by the Schubert varieties $\{[X_w]\}_{w \in S_n}$. The corresponding structure constants $c_{u,w}^v$'s, also referred to as the *generalized Littlewood–Richardson numbers*, are known to be nonnegative integers from transversal intersection. It has been a long standing open problem to find a combinatorial interpretation of these numbers. The study of flag varieties, Schubert varieties and the structure constants is central in algebraic geometry and algebraic combinatorics.

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The small quantum cohomology ring $QH^*(\mathrm{Fl}_n)$ is a deformation of the cohomology ring. The structure constants of $QH^*(\mathrm{Fl}_n)$ with respect to the Schubert basis are known to be the *3-point genus-0 Gromov–Witten invariants*. They extend the generalized Littlewood–Richardson numbers in the “quantum" direction.

The problem of multiplying Schubert classes in $QH^*(\mathrm{Fl}_n)$ can be naturally encoded via the *quantum Bruhat graph*, first defined by Brenti–Fomin–Postnikov [1] and utilized by Postnikov [13]. The quantum Bruhat graph can be seen as a graphical representation of the *quantum Monk's rule* and enjoys very rich algebraic and combinatorial properties. In particular, the minimal degree q^d that appears in the quantum product $[X_u] \star [X^v]$ is the weight of any shortest directed path from u to v [13]. The quantum Bruhat graph directly gives rise to the *tilted Bruhat order* [1]. These are our main combinatorial objects of interest for this paper.

Our first set of main results are about the weights in the quantum Bruhat graph. Specifically, we provide:

1. An explicit combinatorial formula for the minimal weight between any pair of permutations u to v (Theorem 2.6).

2. An Ehresmann-like characterizaiton for the tilted Bruhat order (Theorem 2.12).

We remark that Theorem 2.6 was also obtained via a combination of Postnikov's toric Schur polynomials [12] on the quantum cohomology ring of the Grassmannian, and a geometric result by Buch–Chung–Li–Mihalcea [4] (see also [7]). Our proof is independent and purely combinatorial.

While weights on the quantum Bruhat graph encode important information in the quantum cohomology of the flag variety, we present a novel geometric interpretation of intervals in the tilted Bruhat order with a more classical flavor. For any pair of permutations $u, v \in S_n$, we define the *tilted Richardson variety* $\mathcal{T}_{u,v}$ and the *open tilted Richardson variety* $\mathcal{T}^\circ_{u,v}$ (Definition 3.2), which reduces to the well-known (open) Richardson variety if $u \leq v$ in the Bruhat order.

The tilted Richardson varieties are our central geometric objects of study. Our second set of results concern geometric properties of tilted Richardson varieties (Theorem 3.7). We prove:

1. a stratification $\mathcal{T}_{u,v} = \bigsqcup_{[x,y]\subseteq[u,v]} \mathcal{T}^\circ_{x,y}$ that relates the tilted Bruhat order;

2. $\dim \mathcal{T}_{u,v} = \dim \mathcal{T}^\circ_{u,v} =$ the length of any shortest paths from u to v in the quantum Bruhat graph;

3. the closure relation $\overline{\mathcal{T}^\circ_{u,v}} = \mathcal{T}_{u,v}$;

4. irreducibility of $\mathcal{T}^\circ_{u,v}$ and $\mathcal{T}_{u,v}$.

Deodhar [6] introduced his decomposition of open Richardson varieties as a tool for understanding *Kazhdan–Lusztig polynomials* [10]. As a key ingredient in proving irreducibility of the tilted Richardson varieties, we introduce a decomposition of $\mathcal{T}^{\circ}_{u,v}$, generalizing Deodhar's decomposition. The decomposition also allows us to compute the $\mathbb{F}_q$-point count of open tilted Richardson varieties, a generalization of R-polynomials. More specifically, we give:

1. A decomposition of the (open) tilted Richardson varieties into simple pieces of the form $(\mathbb{C}^*)^a \times \mathbb{C}^b$, extending Deodhar's decomposition (Theorem 4.5);

2. A formula for the $\mathbb{F}_q$-point counts of open tilted Richardson varieties using Hecke algebras (Theorem 4.7).

In [2], Buch–Chaput–Mihalcea–Perrin introduced the *two-point curve neighborhoods* $\Gamma_d(X_u, X^v)$ in their study of the quantum cohomology ring $QH^*(\mathrm{Fl}_n)$ (see also [3]). They encode information about the Gromov–Witten invariants of degree d, but little in the way of explicit descriptions of the curve neighborhoods was known [11, 5]. We present the first concrete combinatorial description in the minimal degree case by establishing a connection with tilted Richardson varieties. We show that:

1. The tilted Richardson variety $\mathcal{T}_{u,v}$ is equal to the minimal degree two-point curve neighborhood $\Gamma_{d_{\min}}(X_u, X^v)$ (Theorem 5.1);

2. The cohomology classes $[\mathcal{T}_{u,v}] = [\Gamma_{d_{\min}}(X_u, X^v)] \in H^*(\mathrm{Fl}_n)$ are equal to the minimal quantum degree component in the quantum product $[X_u] \star [X^v]$ (Theorem 5.2).

2 Minimal weights in the quantum Bruhat graphs

Let S_n be the symmetric group on n elements, generated by the *simple transpositions* $\{s_1, \ldots, s_{n-1}\}$. We typically write a permutation w using its *one-line* notation $w_1 w_2 \cdots w_n$. For $w \in S_n$, let $\ell(w)$ be the *Coxeter length* of w, which is the smallest ℓ such that $w = s_{\alpha_1} \cdots s_{\alpha_\ell}$ is a product of ℓ simple transpositions. Let $t_{ij} = (i\ j)$ be a transposition in S_n.

Definition 2.1 ([1])**.** The **quantum Bruhat graph** Γ_n is a weighted directed graph on S_n with the following two types of edges:

$$\begin{cases} w \to wt_{ij} \text{ of weight } 1 & \text{if } \ell(wt_{ij}) = \ell(w) + 1, \\ w \to wt_{ij} \text{ of weight } q_i q_{i+1} \cdots q_{j-1} & \text{if } \ell(wt_{ij}) = \ell(w) + 1 - 2(j-i), \end{cases}$$

where $1 \le i < j \le n$. Write $\mathrm{wt}(w \to wt_{ij}) \in \mathbb{Z}[q_1, \ldots, q_{n-1}]$ for the weight.

The quantum Bruhat graph for $n = 3$ is shown in Figure 1.

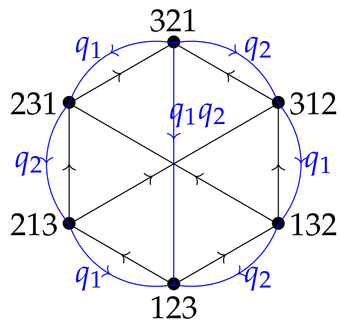

Figure 1: Quantum Bruhat graph Γ_3 (unlabeled edges have weight 1)

Definition 2.2. For a directed path $P : w^{(0)} \to w^{(1)} \to \cdots \to w^{(k)}$ in Γ_n, we say that P has length k, with weight

$$\mathrm{wt}(P) := \prod_{i=1}^{k} \mathrm{wt}(w^{(i-1)} \to w^{(i)}).$$

For $u, v \in S_n$, let $\ell(u, v)$ be the length of a shortest path from u to v.

Postnikov [13] established nice properties regarding weights of shortest paths. Write q^d for $q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$.

Lemma 2.3 ([13])**.** *For any $u, v \in S_n$, all shortest paths from u to v have the same weight $q^{d_{u,v}}$. Moreover, the weight of any path from u to v is divisible by $q^{d_{u,v}}$.*

Remark 2.4. The weight $q^{d_{u,v}}$ is the unique minimum quantum degree that appears in the quantum product of two Schubert classes $\sigma_u \star \sigma_{w_0 v}$ in the quantum cohomology ring $QH^*(\mathrm{Fl}_n)$ [13]. We further discuss this connection to $QH^*(\mathrm{Fl}_n)$ in Section 5.

Our first theorem provides an explicit formula that computes the minimal weight $q^{d_{u,v}}$ for any pair of permutation $u, v \in S_n$.

Definition 2.5. For $A, B \subset [n]$ with $|A| = |B|$, we construct a lattice path $P(A, B)$ starting at $(0, 0)$ and ending at $(n, 0)$ with n steps as follows. For each $i = 1, \ldots, n$, the i^{th} step is

- an upstep $(1, 1)$ if $i \in A$ and $i \notin B$,
- a downstep $(1, -1)$ if $i \notin A$ and $i \in B$,
- a horizontal step $(1, 0)$ if $i \in A \cap B$ or $i \notin A \cup B$.

Define its **depth**, denoted as $\mathrm{depth}(A, B)$, to be the largest number $y \geq 0$ such that $P(A, B)$ passes through $(x, -y)$ for some x.

Theorem 2.6. *Let $u, v \in S_n$. All shortest paths from u to v in the quantum Bruhat graph Γ_n have weight $q^{d_{u,v}} = q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$ where $d_k = \operatorname{depth}(u[k], v[k])$. Here, $w[k] := \{w_1, \ldots, w_k\}$.*

Example 2.7. Consider $u = 7364152$ and $v = 2513746$ in S_7. To figure out $d_4(u, v)$, we need to construct a lattice path $P_4(u[4], v[4])$ with upsteps $u[4] = \{3, 4, 6, 7\}$ and downsteps $v[4] = \{1, 2, 3, 5\}$, as shown in Figure 2 with $\operatorname{depth}(u[4], v[4]) = 2$. In the end, we arrive at $q^{d_{u,v}} = q_1 q_2 q_3^2 q_4^2 q_5 q_6$.

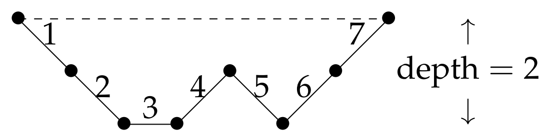

Figure 2: The lattice path $P(u[4], v[4])$

Definition 2.8 ([1]). For $u \in S_n$, define the **tilted Bruhat order** D_u to be the graded partial order on S_n such that $w \leq_u v$ if

$$\ell(u, w) + \ell(w, v) = \ell(u, v). \tag{2.1}$$

Equivalently, $w \leq_u v$ if there is a shortest path in the quantum Bruhat graph from u to v that passes through w. For $w \leq_u v$, define the **tilted Bruhat interval** to be

$$[w, v]_u = \{x \in S_n : w \leq_u x \leq_u v\}.$$

The tilted Bruhat order D_{132} is shown in Figure 3.

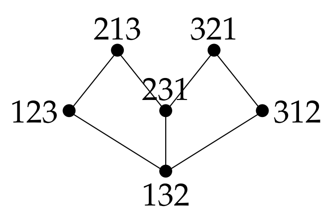

Figure 3: The tilted Bruhat order D_{132}

Remark 2.9. It follows from (2.1) that $[w, v]_u = [w, v]_{u'}$ as long as $w \leq_u v$ and $w \leq_{u'} v$. Since $w \leq_w v$, we omit the subscript and write $[w, v]$ instead of $[w, v]_u$ from now on.

Remark 2.10. When $u = id$, the tilted Bruhat order D_u is the strong Bruhat order.

Our second theorem provides a combinatorial criterion for the tilted Bruhat order.

Definition 2.11. For $r \in [n]$, let $\leq_r$ be the **shifted linear order** on $[n]$ given by $r <_r \cdots <_r n <_r 1 <_r \cdots <_r r-1$. Define the **shifted Gale order** $\leq_r$ on $\binom{[n]}{k}$ as

$$\{a_1 <_r \cdots <_r a_k\} \leq_r \{b_1 <_r \cdots <_r b_k\} \iff a_i \leq_r b_i \text{ for all } i \in [k].$$

For a sequence $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in [n]^n$, define the **a-shifted order** $\leq_{\mathbf{a}}$ on S_n as

$$u \leq_{\mathbf{a}} v \iff u[k] \leq_{a_k} v[k] \text{ for all } k \in [n].$$

Theorem 2.12. *For $u, v, w \in S_n$, the following are equivalent:*

1. *$w \leq_u v$;*
2. *$w \in [u, v]$;*
3. *for any sequence $\mathbf{a} = (a_1, \ldots, a_n)$ such that $u \leq_{\mathbf{a}} v$, we have $u \leq_{\mathbf{a}} w \leq_{\mathbf{a}} v$;*
4. *there exists a sequence $\mathbf{a} = (a_1, \ldots, a_n)$ such that $u \leq_{\mathbf{a}} w \leq_{\mathbf{a}} v$.*

Example 2.13. Let $u = 132$, $w = 123$, $v = 213$. From Figure 3, we can see $w \leq_u v$. Let $\mathbf{a} = (1, 3, 1)$, we have $u \leq_{\mathbf{a}} w \leq_{\mathbf{a}} v$ since

$$\{1\} \leq_1 \{1\} \leq_1 \{2\} \qquad \{3, 1\} \leq_3 \{1, 2\} \leq_3 \{1, 2\} \qquad [3] \leq_1 [3] \leq_1 [3].$$

Alternatively, $u \leq_{\mathbf{a}'} w \leq_{\mathbf{a}'} v$ also for $\mathbf{a}' = (3, 3, 2)$ since

$$\{1\} \leq_3 \{1\} \leq_3 \{2\} \qquad \{3, 1\} \leq_3 \{1, 2\} \leq_3 \{1, 2\} \qquad [3] \leq_2 [3] \leq_2 [3].$$

3 Tilted Richardson varieties

Let $G = GL_n(\mathbb{C})$ and $B, B_- \subset G$ be the Borel and opposite Borel subgroup of G consisting of invertible upper and lower triangular matrices respectively. Let $T = B \cap B_-$ be the maximal torus of diagonal matrices in G.

The *complete flag variety* is defined to be $\mathrm{Fl}_n = G/B$. Fixing a basis of $\mathbb{C}^n$, we can identify a point $gB \in G/B$ with a *flag* $F_\bullet = 0 = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq F_n = \mathbb{C}^n$ where $F_k \in \mathrm{Gr}(k, n)$ is the span of the first k column vectors of any $n \times n$ matrix representative $M_F \in gB$.

Definition 3.1. For $a, b \in [n]$, define the **cyclic interval** $[a, b)_c := \{a, \ldots, b-1\}$ if $a \leq b$ and $[a, b)_c := \{a, \ldots, n\} \cup \{1, \ldots, b-1\}$ if $a > b$.

Motivated by our characterization of tilted Bruhat order (Theorem 2.12), we define the (open) tilted Richardson varieties:

Definition 3.2. For S any subset of $[n]$, let $\mathrm{Proj}_S : \mathbb{C}^n \twoheadrightarrow \mathbb{C}^{|S|}$ be the projection onto the coordinates with indices in S. For $u, v \in S_n$ and a sequence $\mathbf{a}$ where $u \leq_{\mathbf{a}} v$, define the **tilted Richardson variety** by:

$$\mathcal{T}_{u,v,\mathbf{a}} = \left\{ F_\bullet \in \mathrm{Fl}_n : \begin{array}{c} \dim(\mathrm{Proj}_{[a_i,j)_c}(F_i)) \leq \#\{u[i] \cap [a_i,j)_c\}, \\ \dim(\mathrm{Proj}_{[j,a_i)_c}(F_i)) \leq \#\{v[i] \cap [j,a_i)_c\} \end{array} \forall i,j \in [n] \right\}, \tag{3.1}$$

and the **open tilted Richardson variety** $\mathcal{T}^\circ_{u,v,\mathbf{a}}$ by replacing all "$\leq$" with "$=$" in (3.1).

Example 3.3. Let $u = 4231, v = 3142$ and $\mathbf{a} = (4,2,2,3)$. Then $u \leq_{\mathbf{a}} v$. In Figure 4, the $\star$ and $\bullet$ represent u and v respectively, and the red horizontal line segment in column k represents the cutoff of $[n]$ under $\leq_{a_k}$ for each $k \in [4]$.

For $F_\bullet \in \mathcal{T}_{u,v,\mathbf{a}}$, there are 8 rank conditions imposed on F_2 as in (3.1). These conditions can be interpreted as rank conditions on submatrices of M_F in the first two columns with (cyclically) consecutive rows starting or ending at the red line. For example, the condition $\dim(\mathrm{Proj}_{\{1,2,3\}}(F_2)) \leq 2 = \#\{v[2] \cap \{1,2,3\}\}$ is interpreted as the rank of the shaded submatrix of M_F in Figure 4(A) being at most 2, the number of $\bullet$ in said region. The condition $\dim(\mathrm{Proj}_{\{2,3\}}(F_2)) \leq 1 = \#\{u[2] \cap \{2,3\}\}$ is the rank of shaded submatrix in Figure 4(B) being at most 1, the number of $\star$ in said region.

(A) $\dim(\mathrm{Proj}_{\{3,4,1\}}(F_2)) \leq 2$ (B) $\dim(\mathrm{Proj}_{\{2,3\}}(F_2)) \leq 1$

Figure 4: Rank conditions on $\mathcal{T}_{u,v,\mathbf{a}}$ for $u = 4231$, $v = 3142$, and $\mathbf{a} = (4,2,2,3)$

It follows immediately from Theorem 2.12 that the set of T-fixed points on $\mathcal{T}_{u,v,\mathbf{a}}$ is

$$\{e_w := wB/B \in \mathcal{T}_{u,v,\mathbf{a}}\} = [u,v]. \tag{3.2}$$

In particular, $e_w \in \mathcal{T}_{u,v,\mathbf{a}}$ is independent of the choice of $\mathbf{a}$. In fact,

Proposition 3.4. *$\mathcal{T}_{u,v,\mathbf{a}}$ and $\mathcal{T}^\circ_{u,v,\mathbf{a}}$ are both independent of $\mathbf{a}$ as long as $u \leq_{\mathbf{a}} v$.*

As a result, we omit the index $\mathbf{a}$ and write $\mathcal{T}^\circ_{u,v}$ and $\mathcal{T}_{u,v}$ for the (open) tilted Richardson varieties.

Remark 3.5. If $u \leq v$ in strong Bruhat order, namely $u \leq_{\mathbf{a}} v$ where $\mathbf{a} = (1,\ldots,1)$, then $\mathcal{T}^\circ_{u,v}$ and $\mathcal{T}_{u,v}$ are the (open) Richardson variety $\mathcal{R}^\circ_{u,v}$ and $\mathcal{R}_{u,v}$ respectively.

Remark 3.6. In [9], we give three other equivalent definitions of the (open) tilted Richardson varieties using cyclically rotated Grassmannian Richardson varieties, vanishing loci of multi-Plücker coordinates, and intersections of tilted Schubert cells.

We prove the following geometric properties of the tilted Richardson varieties, analogous to those of the classical Richardson varieties.

Theorem 3.7. *For $u, v \in S_n$,*

(1) (stratification) $\mathcal{T}_{u,v} = \bigsqcup_{[x,y]\subseteq[u,v]} \mathcal{T}^{\circ}_{x,y}$;

(2) (dimension) $\dim(\mathcal{T}^{\circ}_{u,v}) = \dim(\mathcal{T}_{u,v}) = \ell(u,v)$;

(3) (closure relation) $\mathcal{T}_{u,v} = \overline{\mathcal{T}^{\circ}_{u,v}}$;

(4) (irreducibility) $\mathcal{T}_{u,v}$ and $\mathcal{T}^{\circ}_{u,v}$ are irreducible.

4 Tilted Deodhar decomposition

In [10], Kazhdan–Lusztig introduced the *Kazhdan–Lusztig R-polynomial* $R_{u,v}(q)$ for $u \leq v$. They are used to give a recursive formula for the *Kazhdan–Lusztig polynomials*. The polynomial $R_{u,v}(q)$ is determined by $\mathbb{F}_q$-point counts on the open Richardson varieties

$$R_{u,v}(q) = \#\mathcal{R}^{\circ}_{u,v}(\mathbb{F}_q).$$

For any pair of permutations $u, v \in S_n$, we define a generalization called **tilted R-polynomial** $R^{\mathrm{tilt}}_{u,v}(q)$, which gives the $\mathbb{F}_q$-point counts on open tilted Richardson varieties

$$R^{\mathrm{tilt}}_{u,v}(q) = \#\mathcal{T}^{\circ}_{u,v}(\mathbb{F}_q).$$

To understand $R_{u,v}(q)$, Deodhar [6] introduced a decomposition of $\mathcal{R}^{\circ}_{u,v}$ into simple pieces D_α that are isomorphic to $\mathbb{C}^a \times (\mathbb{C}^*)^b$. This gives an explicit formula for $R_{u,v}(q)$:

$$\mathcal{R}^{\circ}_{u,v} = \bigsqcup_{\alpha} \mathbb{C}^a \times (\mathbb{C}^*)^b \implies R_{u,v}(q) = \sum_{\alpha} q^a (q-1)^b.$$

In this section, we extend Deodhar's decomposition to tilted Richardson varieties

$$\mathcal{T}^{\circ}_{u,v} = \bigsqcup_{\mathbf{u} \prec \mathbf{v}} D_{\mathbf{u},\mathbf{v}}, \tag{4.1}$$

This decomposition allows us to prove irreducibility of tilted Richardson varieties, and give an explicit formula for the tilted R-polynomials (Corollary 4.6).

We now define the *tilted reduced expression* $\mathbf{v}$ and the *distinguished subexpression* $\mathbf{u}$ indexing the pieces in (4.1). Let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in [n]^n$ be a sequence. Define

$$J_{\mathbf{a}} = \{j_1 > j_2 > \cdots > j_t\} := \{j \in [n] : a_j \neq a_{j+1}\}, \quad (a_{n+1} = 1).$$

Definition 4.1. An **a-tilted reduced expression v** for $v \in S_n$ is a sequence of black or blue (overlined) simple transpositions, whose product is v and of the form

$$\mathbf{v} = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_{N_1}}\overline{s_{\alpha_{N_1+1}}\cdots s_{\alpha_{N_2}}}s_{\alpha_{N_2+1}}\cdots s_{\alpha_{N_3}}\overline{s_{\alpha_{N_3+1}}\cdots s_{\alpha_{N_4}}}\cdots s_{\alpha_{N_{2t}+1}}\cdots s_{\alpha_N}$$

satisfying the following properties:

1. $\mathbf{v}$ is the concatenation of $2t+1$ subexpressions, alternatively colored black and blue, such that each subexpression is a reduced expression for some permutation;
2. for all $k \in [t]$, $\alpha_i < j_k$ if $i > N_{2k-1}$;
3. let $v_{(d)} = \prod_{i=1}^{d} s_{\alpha_i}$. Then the first j_k entries of $v_{(N_{2k-1})}$ are increasing under the shifted order $<_{a_{j_k+1}}$, and the first j_k entries of $v_{(N_{2k})}$ are increasing under the shifted order $<_{a_{j_k}}$, for all $k \in [t]$.

Definition 4.2. Given an **a**-tilted reduced expression $\mathbf{v} = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_N}$, a **distinguished subexpression** for $u \in S_n$ is a sequence $\mathbf{u} = u_1 \ldots u_N$ whose product is u and each factor $u_i \in \{1, s_{\alpha_i}, \overline{s_{\alpha_i}}, \widetilde{s_{\alpha_i}}\}$ satisfies the following rules:

$$u_i = \begin{cases} 1 \text{ or } s_{\alpha_i} & \text{if } s_{\alpha_i} \text{ is black in } \mathbf{v} \text{ and } \ell(u_{(i-1)}s_{\alpha_i}, v_{(i)}) = \ell(u_{(i-1)}, v_{(i-1)}), \\ \widetilde{s_{\alpha_i}} & \text{if } s_{\alpha_i} \text{ is black in } \mathbf{v} \text{ and } \ell(u_{(i-1)}s_{\alpha_i}, v_{(i)}) > \ell(u_{(i-1)}, v_{(i-1)}), \\ \overline{s_{\alpha_i}} & \text{if } \overline{s_{\alpha_i}} \text{ is blue in } \mathbf{v}. \end{cases}$$

Here we set $u_{(i)}$ to be the product of the leftmost i factors of $\mathbf{u}$. We write $\mathbf{u} \prec \mathbf{v}$ if $\mathbf{u}$ is a distinguished subexpression of $\mathbf{v}$.

Example 4.3. If $v = 246513$ and $\mathbf{a} = (5,5,5,1,1,1)$, the set $J_{\mathbf{a}} = \{3\}$, and an example of an **a**-tilted reduced expression for v is $\mathbf{v} = s_3s_4s_5s_1s_2s_3s_4s_3\overline{s_2s_1}s_1s_2$. There are 4 distinguished subexpressions for $u = 512346$:

$$\mathbf{u} = 111111s_4s_3\overline{s_2s_1}11 \qquad \mathbf{u} = 111s_111s_4s_3\overline{s_2s_1}1\widetilde{s_2}$$
$$\mathbf{u} = s_31111\widetilde{s_3}s_4s_3\overline{s_2s_1}11 \qquad \mathbf{u} = s_311s_11\widetilde{s_3}s_4s_3\overline{s_2s_1}1\widetilde{s_2}$$

Unfortunately, we don't know whether $u \leq_{\mathbf{a}} v$ will ensure the existence of a distinguished subexpression for u in $\mathbf{v}$. This motivates the following Lemma/Definition, which gives a sufficient condition.

Lemma/Definition 4.4. For any $u, v \in S_n$, there exists **a** such that $u \leq_{\mathbf{a}} v$, and $u[k] \leq_{a_{k+1}} v[k]$ for all $k \in [n]$. We say such **a** is **flat**. In this case, any **a**-tilted reduced expression **v** has a distinguished subexpression for u.

For each $j \in [n-1]$, define $y_j(p), x_j(m)$ and $\dot{s}_j \in GL_n$ to be

$$y_j(p) = \phi_j\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \qquad \dot{s}_j(p) = \phi_j\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad x_j(m) = \phi_j\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix},$$

where $\phi_j\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ replaces the 2×2 block of the identity matrix in the j and $(j+1)$-th row/column by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The following is our main theorem of this section.

Theorem 4.5. *Let $u \le_{\mathbf{a}} v$ where $\mathbf{a}$ is flat. Let $\mathbf{v}$ be an $\mathbf{a}$-tilted reduced expression for v. Then*

$$\mathcal{T}^{\circ}_{u,v} = \bigsqcup_{\mathbf{u} \prec \mathbf{v}} D_{\mathbf{u},\mathbf{v}}, \text{ where each } D_{\mathbf{u},\mathbf{v}} \cong (\mathbb{C}^*)^{\#1\text{'s in } \mathbf{u}} \times \mathbb{C}^{\#\widetilde{s_{\alpha_i}}\text{'s in } \mathbf{u}}.$$

Each tilted Deodhar cell $D_{\mathbf{u},\mathbf{v}}$ for $\mathbf{u} = u_1 \cdots u_N \prec \mathbf{v}$ is parametrized by

$$D_{\mathbf{u},\mathbf{v}} := \left\{ gB = g_1 g_2 \cdots g_N B \;\middle|\; \begin{array}{ll} g_i = y_{\alpha_i}(p_i), & \text{if } u_i = 1 \\ g_i = \dot{s}_{\alpha_i}, & \text{if } u_i = s_{\alpha_i} \text{ or } \overline{s_{\alpha_i}} \\ g_i = x_{\alpha_i}(m_i)\dot{s}_{\alpha_i}^{-1}, & \text{if } u_i = \widetilde{s_{\alpha_i}} \end{array} \right\} / B.$$

Here $p_i \in \mathbb{C}^*$ and $m_i \in \mathbb{C}$ are parameters.

Corollary 4.6. *There is a unique tilted Deodhar cell $D_{\mathbf{u},\mathbf{v}}$ of maximal dimension $\ell(u,v)$, hence $\mathcal{T}^{\circ}_{u,v}$ is irreducible. The $\mathbb{F}_q$-point counts of $\mathcal{T}^{\circ}_{u,v}$ is*

$$R^{tilt}_{u,v}(q) = \#\mathcal{T}^{\circ}_{u,v}(\mathbb{F}_q) = \sum_{\mathbf{u} \prec \mathbf{v}} (q-1)^{\#1\text{'s in } \mathbf{u}} \times q^{\#\widetilde{s_{\alpha_i}}\text{'s in } \mathbf{u}}.$$

We give an alternate formula for $R^{\mathrm{tilt}}_{u,v}(q)$ using Hecke algebra, generalizing classical results for $R_{u,v}(q)$ (see [8, Section 2.3]). The *Hecke algebra* $\mathcal{H}$ of S_n is a $\mathbb{C}[q^{\pm 1}]$-algebra generated by the set $\{T_i\}_{i \in [n-1]}$ satisfying the braid relations and the *Hecke relation*:

$$(T_i + q)(T_i - 1) = 0 \text{ for } i \in [n-1].$$

For $w \in S_n$, let $T_w := T_{\alpha_1} T_{\alpha_2} \ldots T_{\alpha_{\ell(w)}}$ for any reduced expression $w = s_{\alpha_1} \cdots s_{\alpha_{\ell(w)}}$. The set $\{T_w : w \in S_n\}$ forms a $\mathbb{C}[q^{\pm 1}]$-basis of $\mathcal{H}$. The *trace map* $\epsilon : \mathcal{H} \to \mathbb{C}[q^{\pm 1}]$ is the $\mathbb{C}[q^{\pm 1}]$-linear map defined by

$$\epsilon(T_w) = \begin{cases} 1 & \text{if } w = id \\ 0 & \text{otherwise} \end{cases}.$$

For an expression $\mathbf{w} = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_N}$, denote $T_{\mathbf{w}} := T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_N}$.

Theorem 4.7. *Fix $u \le_{\mathbf{a}} v$ where $\mathbf{a}$ is flat. Let $\mathbf{u}, \mathbf{v}$ be $\mathbf{a}$-tilted reduced expressions for u, v respectively. We have*

$$R^{tilt}_{u,v}(q) = q^{\ell(u,v)} \epsilon(T_{\mathbf{v}}^{-1} T_{\mathbf{u}}).$$

5 Curve neighborhoods and the cohomology classes $[\mathcal{T}_{u,v}]$

For $w \in S_n$, define the *Schubert variety* $X^w = \overline{BwB/B}$ and the *opposite Schubert variety* $X_w = \overline{B_- wB/B}$. Let $\rho_i \in H_2(\mathrm{Fl}_n)$ be the homology class of X^{s_i}. For an effective degree $d = \sum_{i=1}^{n-1} d_i \rho_i \in H_2(\mathrm{Fl}_n)$, the *Kontsevich moduli space* $\mathcal{M}_d = \overline{\mathcal{M}}_{0,3}(\mathrm{Fl}_n, d)$ parametrizes the set of degree d, genus zero stable curves in Fl_n with 3 marked points. We will represent d by the vector $(d_1, \ldots, d_{n-1})$. The space $\mathcal{M}_d$ is naturally equipped with evaluation maps $ev_i : \mathcal{M}_d \to \mathrm{Fl}_n$ sending a curve to its i-th marked point for $i = 1, 2, 3$.

The *quantum cohomology ring* $QH^*(\mathrm{Fl}_n) \cong H^*(\mathrm{Fl}_n) \otimes_{\mathbb{Z}} \mathbb{Z}[q_1, \ldots, q_{n-1}]$ is a free $\mathbb{Z}[q]$-module generated by the Schubert classes $\{\sigma_w := [X_w] : w \in S_n\}$, where the products are given by

$$\sigma_u \star \sigma_v = \sum_{w \in S_n,\ d \in \mathbb{N}^{n-1}} \langle \sigma_u, \sigma_v, \sigma_{w_0 w} \rangle_d \, q^d \sigma_w. \tag{5.1}$$

Here $\langle \sigma_u, \sigma_v, \sigma_{w_0 w} \rangle_d$ are the *Gromov–Witten invariants* of the flag variety defined as

$$\langle \sigma_u, \sigma_v, \sigma_{w_0 w} \rangle_d = \int_{\mathcal{M}_d} ev_1^*(\sigma_u) \cdot ev_2^*(\sigma_v) \cdot ev_3^*(\sigma_{w_0 w}).$$

Informally, these invariants count the equivalence classes of degree d rational curves passing through the given Schubert varieties in general positions.

Fulton–Woodward [7] initiated the study of minimal quantum degrees d appearing in (5.1). Postnikov [13] showed that $d_{u, w_0 v}$ is the unique such minimal degree. Our Theorem 2.6 gives an explicit combinatorial description for this degree.

In their study of the quantum K-theory of homogeneous spaces G/P, Buch–Chaput–Mihalcea–Perrin [3] introduced the *Gromov–Witten variety* $M_d(X_u, X^v) := ev_1^{-1}(X_u) \cap ev_2^{-1}(X^v)$ and the *curve neighborhood* $\Gamma_d(X_u, X^v) := ev_3(M_d(X_u, X^v))$. These varieties encode information about the Gromov–Witten invariants [2].

The main theorems of this section relate tilted Richardson varieties with curve neighborhoods and quantum Schubert calculus.

Theorem 5.1. *For any $u, v \in S_n$, $\Gamma_{d_{u,v}}(X_u, X^v) = \mathcal{T}_{u,v}$.*

Theorem 5.2. *The cohomology class $[\mathcal{T}_{u,v}] = [\Gamma_{d_{u,v}}(X_u, X^v)] \in H^*(\mathrm{Fl}_n)$ is equal to*

$$[q^{d_{u,v}}]\sigma_u \star \sigma_{w_0 v} = \sum_{w \in S_n} \langle \sigma_u, \sigma_{w_0 v}, \sigma_{w_0 w} \rangle_{d_{u,v}} \sigma_w. \tag{5.2}$$

Proof Sketch. For Theorem 5.1, we first show that the two varieties share the same set of T-fixed points $\{e_w : w \in [u, v]\}$, using (3.2) and results in [7]. We then prove $\mathcal{T}_{u,v}$ is the largest T-invariant subvariety of Fl_n with T-fixed points $[u, v]$, hence $\mathcal{T}_{u,v} \supseteq \Gamma_{d_{u,v}}(X_u, X^v)$. Finally, the theorem follows from the fact that both varieties have the same dimension and $\mathcal{T}_{u,v}$ is irreducible (Theorem 3.7 and [7]).

For Theorem 5.2, by the projection formula,

$$[\mathcal{T}_{u,v}] = [\Gamma_{d_{u,v}}(X_u, X^v)] = \frac{1}{c}[q^{d_{u,v}}]\sigma_u \star \sigma_{w_0 v} \text{ for some } c \in \mathbb{Z}_{>0}.$$

Postnikov [13] gave a polynomial representative of (5.2) using *path Schubert polynomials* $\mathfrak{S}_{u,v}(x, q)$. We show that the leading monomial of $\mathfrak{S}_{u,v}(x, q)$ has coefficient 1, implying $c = 1$. □

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