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# Orbits in the affine flag variety of type $A$

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**Abstract.** It is a classical result that the set $K\backslash G/B$ is finite, where $G$ is a reductive algebraic group over an algebraically closed field with characteristic not equal to two, $B$ is a Borel subgroup of $G$, and $K = G^\theta$ is the fixed point subgroup of an involution of $G$. In this work, we investigate the affine counterpart of the aforementioned set, where $G$ is the general linear group over formal Laurent series, $B$ is an Iwahori subgroup of $G$, and $K$ is either the orthogonal group, the symplectic group or product group over formal Laurent series. We construct explicit bijections between the double cosets $K\backslash G/B$ and certain twisted affine involutions or affine $(p,q)$-clans, which are affine involutions with plus or minus signs assigned to the fixed-points. This is the first combinatorial description of $K$-orbits in the affine flag variety of type A.

**Keywords:** affine flag variety, extended affine symmetric group, $K$-orbits

# 1 Introduction

## 1.1 Classical background

Let $G$ be a connected reductive algebraic group over the field of complex numbers $\mathbb{C}$, and let $B \subset G$ be a Borel subgroup of $G$. Then it is a classical result that the set $B\backslash G/B$ is finite, with a distinct set of double coset representatives forming the Weyl group $W$ [1, Chapter 27]. Thus, the *Bruhat decomposition* can be written as

$$G = \bigsqcup_{w \in W} BwB.$$

Now let $\theta$ be a holomorphic involution of $G$. Let $K = G^\theta = \{g \in G : \theta(g) = g\}$ be the corresponding fixed point subgroup. Then it is again a classical result that the set $K\backslash G/B$ is finite [14]. The set $K\backslash G/B$ is treated either as a set of $K$-orbits in the *flag variety* $G/B$, or as $B$-orbits in the *symmetric variety* $K\backslash G$ [22], or simply as $(B \times K)$-double cosets.

The classification of $K\backslash G/B$ when $G$ is any classical linear group of any Lie type is well known [15]. Denote $\mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$ and fix $n \in \mathbb{Z}_{>0}$. We focus here on the case when $G = \mathsf{GL}(n, \mathbb{C})$. Denote $1_n$ to be the $n$-by-$n$ identity matrix. Then there are

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only three types of involutions $\theta$ up to conjugacy [19, Chapter 5, Section 1.5], namely $\theta(g) = (g^T)^{-1}$, $\theta(g) = (-Jg^TJ)^{-1}$ with $n$ is even and $J = \begin{pmatrix} 0 & 1_{n/2} \\ -1_{n/2} & 0 \end{pmatrix}$, and

$$\theta(g) = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix} g \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}$$

with $n = p + q$ for nonnegative integers $p$ and $q$. The corresponding fixed point subgroups $K = G^\theta$ are the orthogonal group $\mathsf{O}(n, \mathbb{C})$, the symplectic group $\mathsf{Sp}(n, \mathbb{C})$, and the product group

$$\mathsf{GL}(p, \mathbb{C}) \times \mathsf{GL}(q, \mathbb{C}) = \left\{ \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} : k_1 \in \mathsf{GL}(p, \mathbb{C}), k_2 \in \mathsf{GL}(q, \mathbb{C}) \right\}$$

respectively. The $K$-orbits of the flag variety $G/B$ in these three cases are in bijection with sets of (signed) involutions in the symmetric group $S_n$ [15]. More precisely, the $K$-orbits are in bijection with involutions in $S_n$ for $K = \mathsf{O}(n, \mathbb{C})$, fixed-point-free involutions in $S_n$ for $K = \mathsf{Sp}(n, \mathbb{C})$ [22], and certain signed involutions called *$(p, q)$-clans* for $K = \mathsf{GL}(p, \mathbb{C}) \times \mathsf{GL}(q, \mathbb{C})$ [29].

The earliest record of studying $K$-orbits in the flag variety is probably due to Gelfand and Graev [8]. They showed that one can construct new irreducible (unitary) representations or discrete series representations of non-compact real forms of $G$ from spaces of functions on these $K$-orbits [9].

In the 1980s, Matsuki and Ōshima [15] gave a concrete combinatorial description of $K\backslash G/B$ for complex classical Lie groups $G$ in terms of *clans*, but without giving a detailed proof. Richardson and Springer [22] and Yamamoto [29] later gave detailed proofs for the three cases of $K$ with $G = \mathsf{GL}(n, \mathbb{C})$. Yamamoto [29] also considered some of the cases in type B, C, D. A useful overview of the $K$-orbit classifications in these cases are provided in the Ph.D. thesis of Wyser [27].

These studies of $K$-orbits lead to applications in the representation theory of real groups and other topics. Richardson and Springer considered the weak order of the closure of such orbits [20, 21, 22]. Fulton related these results to Schubert calculus [5, 6, 7], and Graham [10] and Wyser [27] related Fulton's work to certain torus-equivariant cohomologies of the flag variety.

In recent years, Can, Joyce, and Wyser [3] studied the maximal chains in the weak order poset for the three types of $K$-orbits when $G = \mathsf{GL}(n, \mathbb{C})$ and specifically described a formula for Schubert classes. Woo, Wyser [25] and Burks, Pawlowski [2] studied the pattern avoidance and reduced words of clans and relate them to the closure of the $K$-orbits in the flag variety. More applications can be found in [4, 16, 17, 28, 26], among many others.

## 1.2 Affine analogs

This extended abstract is about affine generalizations of the $K$-orbits above in the following sense.

One can consider affine analogs of the $B$-orbits and $K$-orbits of the flag varieties, and consider their applications in cohomology of the affine flag variety [11]. Let $\mathbb{K}$ be a quadratically closed field, i.e. a field of characteristic not equal to 2 in which every element has a square root. Let $\mathbb{K}((t))$ be the field of formal Laurent series in $t$ consisting of all the formal sums $\sum_{i\geq N}^{\infty} a_i t^i$, in which $N \in \mathbb{Z}$ and $a_i \in \mathbb{K}$ for $i \geq N$. Let $\mathbb{K}[[t]]$ be the ring of formal power series consisting of all the formal sums $\sum_{i\geq 0}^{\infty} a_i t^i$, in which $a_i \in \mathbb{K}$.

We redefine $G = \mathsf{GL}(n, \mathbb{K}((t)))$ to be the group of invertible $n$-by-$n$ matrices over $\mathbb{K}((t))$ and redefine $B$ to be the subgroup consisting of all upper triangular modulo $t$ matrices in $\mathsf{GL}(n, \mathbb{K}[[t]])$, that is, invertible matrices with entries in $\mathbb{K}[[t]]$ that become upper triangular if we set $t = 0$ for these matrices. Then $G$ is the *(algebraic) loop group* of $\mathsf{GL}(n, \mathbb{K})$ and $B$ is an *Iwahori subgroup*. In this setting, the *affine Bruhat decomposition* is written as

$$G = \bigsqcup_{w \in \widetilde{W}} BwB,$$

where $\widetilde{W}$ is the *affine Weyl group* of $G$, which is isomorphic to a semidirect product of the symmetric group $S_n$ of permutations of $n$ elements and $\mathbb{Z}^n$ of $n$-tuples of integers. The set of cosets $G/B$ is often called the *affine flag variety*, and has been studied in [11, 12, 18], for example.

Most constructions related to orbits in flag varieties have affine analogs [11, 12, 18]. However, the subject of $K$-orbits in the affine flag variety is mostly unexplored in the literature, with the important exception of the Ph.D. thesis of Mann [13]. Mann's work gives a type-independent classification of the $K$-orbits in $G/B$ in terms of certain conjugacy classes of triples $(H, B, \mu)$. However, this general classification is non-constructive and its computation is complicated even for matrix groups with small dimensions. By contrast, the results in this article provide explicit combinatorial descriptions of $K$-orbits in the affine flag variety of type A. It is not straightforward to obtain our results as special cases of Mann's theorem.

The results of this extended abstract concern the $K$-orbits in the affine flag variety $G/B$, where $G = \mathsf{GL}(n, \mathbb{K}((t)))$ is the (algebraic) loop group of $\mathsf{GL}(n, \mathbb{K})$ and $B$ is the Iwahori subgroup as in the setting of affine Bruhat decomposition. In type A, the loop group analogs of $K$ are given by $\mathsf{O}(n, \mathbb{K}((t)))$, $\mathsf{Sp}(n, \mathbb{K}((t)))$ and $\mathsf{GL}(p, \mathbb{K}((t))) \times \mathsf{GL}(q, \mathbb{K}((t)))$. We also consider the $\mathsf{SO}(n, \mathbb{K}((t)))$-orbits in the affine flag variety of $\mathsf{SL}(n, \mathbb{K}((t)))$. The following section describe our main theorems in these four cases.

**Remark 1.1.** In this work, by orbits in affine flag variety we mean $K$-orbits in $G/B$. However, it is sometimes more convenient to consider the $B$-orbits in $K\backslash G$ or the $(K, B)$-double cosets in $G$, which are the orbits for the obvious action of $K \times B$ on $G$. It is clear that there are canonical bijections between these three kinds of orbits.

# 2 Main results

We present the following results regarding different orbits in the affine flag variety and omit the proofs. The complete version can be accessed in [23] and [24].

## 2.1 Orbits of the orthogonal group

Continue to let $G = \mathsf{GL}(n, \mathbb{K}((t)))$. In this subsection we state our first main theorem classifying the orbits of

$$K = \mathsf{O}(n, \mathbb{K}((t))) = \{g \in G : g^T g = 1_n\}$$

in the affine flag variety $G/B$. Here the involution $\theta$ on $G$ is defined as $\theta(g) = (g^T)^{-1}$ and $K = G^\theta$.

Recall that a *monomial matrix* is a matrix with only one non-zero entry in each row and column. We define an *affine permutation matrix* to be an $n$-by-$n$ monomial matrix with integral powers of $t$ as non-zero entries. Below we define two subsets of the set of affine permutation matrices.

**Definition 2.1.** Define $\mathsf{SymAPM}_n$ to be the set of all symmetric $n$-by-$n$ affine permutation matrices. Define $\mathsf{eSymAPM}_n$ to be the set of elements in $\mathsf{SymAPM}_n$ for which the sum of the powers of $t$ is even.

The main theorem for the case $K = \mathsf{O}(n, \mathbb{K}((t)))$ is the following:

**Theorem 2.2.** In the case where $K = \mathsf{O}(n, \mathbb{K}((t)))$ and $G = \mathsf{GL}(n, \mathbb{K}((t)))$, for each double coset $\mathcal{O} \in K\backslash G/B$, there exists a unique $w \in \mathsf{eSymAPM}_n$ such that $g^T g = w$ for some $g \in \mathcal{O}$. Moreover, for each $w \in \mathsf{eSymAPM}_n$, the set of matrices $g$ satisfying $g^T g = w$ is non-empty and its elements lie in the same double coset.

This theorem is non-constructive but for each $w \in \mathsf{eSymAPM}_n$, we provide an explicit formula for a matrix $g_w \in G$ such that $g_w^T g_w = w$. See [23, Definition 3.8].

In terms of this notation, the above theorem implies the following corollary.

**Corollary 2.3.** The map $w \mapsto K g_w B$ is a bijection between $\mathsf{eSymAPM}_n$ and $K\backslash G/B$.

**Example 2.4.** Suppose $n = 3$. Then matrices in $\mathsf{eSymAPM}_3$ are in one of the following forms:

$$w_1 = \begin{pmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix}, w_2 = \begin{pmatrix} 0 & t^a & 0 \\ t^a & 0 & 0 \\ 0 & 0 & t^b \end{pmatrix}, \quad \begin{pmatrix} t^b & 0 & 0 \\ 0 & 0 & t^a \\ 0 & t^a & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & t^a \\ 0 & t^b & 0 \\ t^a & 0 & 0 \end{pmatrix}.$$

In all of the above forms, the exponents $a, b, c$ are integers. The sum $a + b + c$ is even for the first form and the integer $b$ is even in the remaining forms. For example if $a, b$ are odd and $c$ is even in $w_1$, then

$$g_{w_1} = \begin{pmatrix} t^{\frac{a-1}{2}} & -(t-1)^{\frac{1}{2}} t^{\frac{b-1}{2}} & 0 \\ t^{\frac{a-1}{2}}(t-1)^{\frac{1}{2}} & t^{\frac{b-1}{2}} & 0 \\ 0 & 0 & t^{\frac{c}{2}} \end{pmatrix} \quad \text{and} \quad g_{w_2} = \begin{pmatrix} i & -it^a/2 & 0 \\ 1 & t^a/2 & 0 \\ 0 & 0 & t^{\frac{b}{2}} \end{pmatrix}.$$

**Remark 2.5.** Define $^*$ to be the automorphism on affine permutation matrices by substituting $t^{-1}$ in the places with $t$. Then the set $\mathsf{SymAPM}_n$ consists of all affine permutations matrices $w$ with the property $w^* = w^{-1}$, and therefore we can call these affine permutation matrices as *extended affine twisted involutions*. Similarly, the set $\mathsf{eSymAPM}_n$ consists of all affine permutation matrices in $\mathsf{SymAPM}_n$ for which the sum of the powers of $t$ is an even integer. Therefore we can also refer to elements in $\mathsf{eSymAPM}_n$ as *even extended affine twisted involutions*.

## 2.2 Orbits of the special orthogonal group

For any commutative ring $R$, denote $\mathsf{SL}(n, R)$ to be the special linear group over $R$. In this subsection we let $G$ be the group $\mathsf{SL}(n, \mathbb{K}((t)))$ and $B$ be the Iwahori subgroup consists of upper triangular matrices modulo $t$ in $\mathsf{SL}(n, \mathbb{K}[[t]])$.

Now we state a theorem classifying the orbits of

$$K = \mathsf{SO}(n, \mathbb{K}((t))) = \{g \in \mathsf{SL}(n, \mathbb{K}((t))) : g^T g = 1_n\}$$

in the affine flag variety $G/B$. Here $K = G^\theta$ for $\theta(g) = (g^T)^{-1}$.

Below, we give the definition of the indexing set for this case.

**Definition 2.6.** Define $i\mathsf{SymAPM}_n \subset \mathsf{SL}(n, \mathbb{K}((t)))$ to be the set of symmetric monomial matrices such that

(i) if there are any non-zero entries on the diagonal, these diagonal entries are of the form $t^a$, and the non-diagonal non-zero entries are of the form $i = \sqrt{-1}$ times $t^a$, where $a \in \mathbb{Z}$.

(ii) if there are no non-zero entries on the diagonal, then the non-zero entries in the first row and first column are of the form $\pm it^a$, while the remaining non-zero entries are of the form $it^a$, where $a \in \mathbb{Z}$.

The following is the main theorem in this subsection:

**Theorem 2.7.** In the case where $K = \mathsf{SO}(n, \mathbb{K}((t)))$ and $G = \mathsf{SL}(n, \mathbb{K}((t)))$, for each double coset $\mathcal{O} \in K\backslash G/B$, there exists a unique $w \in i\mathsf{SymAPM}_n$ such that $g^T g = w$ for some $g \in \mathcal{O}$. Moreover, for each $w \in i\mathsf{SymAPM}_n$, the set of matrices $g$ satisfying $g^T g = w$ is non-empty and its elements lie in the same double coset.

For each $w \in i\mathsf{SymAPM}_n$, we define explicitly $g_w \in \mathsf{SL}(n, \mathbb{K}((t)))$ as a double coset representative satisfying $g_w^T g_w = w$ by [23, Definition 3.12, Lemma 3.13], which follows similar procedures as in Example 2.4. The above theorem implies the following corollary.

**Corollary 2.8.** The map $w \mapsto K g_w B$ is a bijection between $i\mathsf{SymAPM}_n$ and $K \backslash G / B$.

**Example 2.9.** Suppose $n = 4$. Then matrices in $i\mathsf{SymAPM}_4$ are in one of the following forms:

$$w_1 = \begin{pmatrix} t^a & 0 & 0 & 0 \\ 0 & t^b & 0 & 0 \\ 0 & 0 & t^c & 0 \\ 0 & 0 & 0 & t^d \end{pmatrix}, \begin{pmatrix} 0 & it^a & 0 & 0 \\ it^a & 0 & 0 & 0 \\ 0 & 0 & 0 & it^b \\ 0 & 0 & it^b & 0 \end{pmatrix}, \begin{pmatrix} 0 & -it^a & 0 & 0 \\ -it^a & 0 & 0 & 0 \\ 0 & 0 & 0 & it^b \\ 0 & 0 & it^b & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & it^a & 0 \\ 0 & 0 & 0 & it^b \\ it^a & 0 & 0 & 0 \\ 0 & it^b & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & -it^a & 0 \\ 0 & 0 & 0 & it^b \\ -it^a & 0 & 0 & 0 \\ 0 & it^b & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & it^a \\ 0 & 0 & it^b & 0 \\ 0 & it^b & 0 & 0 \\ it^a & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -it^a \\ 0 & 0 & it^b & 0 \\ 0 & it^b & 0 & 0 \\ -it^a & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & it^a & 0 & 0 \\ it^a & 0 & 0 & 0 \\ 0 & 0 & t^b & 0 \\ 0 & 0 & 0 & t^c \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & it^a & 0 \\ 0 & t^b & 0 & 0 \\ it^a & 0 & 0 & 0 \\ 0 & 0 & 0 & t^c \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & it^a \\ 0 & t^b & 0 & 0 \\ 0 & 0 & t^c & 0 \\ it^a & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} t^b & 0 & 0 & 0 \\ 0 & 0 & it^a & 0 \\ 0 & it^a & 0 & 0 \\ 0 & 0 & 0 & t^c \end{pmatrix}, \begin{pmatrix} t^b & 0 & 0 & 0 \\ 0 & 0 & 0 & it^a \\ 0 & 0 & t^c & 0 \\ 0 & it^a & 0 & 0 \end{pmatrix}, \begin{pmatrix} t^b & 0 & 0 & 0 \\ 0 & t^c & 0 & 0 \\ 0 & 0 & 0 & it^a \\ 0 & 0 & it^a & 0 \end{pmatrix}.$$

In all of the above forms, the exponents in $t$'s are integers and add up to zero. Suppose $a, b$ are odd, and $c, d$ are even in $w_1$. Then

$$g_{w_1} = \begin{pmatrix} t^{\frac{a-1}{2}} & -(t-1)^{\frac{1}{2}} t^{\frac{b-1}{2}} & 0 & 0 \\ t^{\frac{a-1}{2}}(t-1)^{\frac{1}{2}} & t^{\frac{b-1}{2}} & 0 & 0 \\ 0 & 0 & t^{\frac{c}{2}} & 0 \\ 0 & 0 & 0 & t^{\frac{d}{2}} \end{pmatrix}.$$

Denote the first two matrices in the second row as $w_2$ and $w_3$ respectively. Then it holds that

$$g_{w_2} = \begin{pmatrix} t^a/2 & i & 0 & 0 \\ it^a/2 & 1 & 0 & 0 \\ 0 & 0 & t^b/2 & i \\ 0 & 0 & it^b/2 & 1 \end{pmatrix} \quad \text{and} \quad g_{w_3} = \begin{pmatrix} i & -t^a/2 & 0 & 0 \\ 1 & -it^a/2 & 0 & 0 \\ 0 & 0 & t^b/2 & i \\ 0 & 0 & it^b/2 & 1 \end{pmatrix}.$$

**Remark 2.10.** The matrices in $i\mathsf{SymAPM}_n$ can be indexed by *affine twisted involutions*, which are symmetric affine permutation matrices with sum of powers of $t$ equal to 0. For more details, refer to [23, Definition 3.10, 3.11].

## 2.3 Orbits of the symplectic group

In this subsection let $G = \mathsf{GL}(2n, \mathbb{K}((t)))$. We state our main theorem classifying the orbits of

$$K = \mathsf{Sp}(2n, \mathbb{K}((t))) = \{g \in \mathsf{GL}(2n, \mathbb{K}((t))) : g^T J g = J\}$$

in the affine flag variety $G/B$, where $J = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)$. Here $K = G^\theta$ for $\theta(g) = (-Jg^T J)^{-1}$. Below we define another subset of $G$ consisting skew-symmetric matrices.

**Definition 2.11.** The set $\mathsf{SkewAPM}_{2n}$ consists of all skew-symmetric $2n$-by-$2n$ monomial matrices whose non-zero entries above the diagonal are integral powers of $t$.

The main theorem for the case $K = \mathsf{Sp}(2n, \mathbb{K}((t)))$ is the following:

**Theorem 2.12.** *In the case where $K = \mathsf{Sp}(2n, \mathbb{K}((t)))$ and $G = \mathsf{GL}(2n, \mathbb{K}((t)))$, for each double coset $\mathcal{O} \in K\backslash G/B$, there exists a unique $w \in \mathsf{SkewAPM}_{2n}$ such that $g^T J g = w$ for some $g \in \mathcal{O}$. Moreover, for each $w \in \mathsf{SkewAPM}_{2n}$, the set of matrices $g$ satisfying $g^T J g = w$ is non-empty and its elements lie in the double coset.*

For each $w \in \mathsf{SkewAPM}_{2n}$, we give an explicit formula for a matrix $g_w \in \mathsf{GL}(2n, \mathbb{K}((t)))$ such that $g_w^T J g_w = w$. See [23, Definition 4.4] for the precise definition of $g_w$. The above theorem implies the following corollary.

**Corollary 2.13.** *The map $w \mapsto K g_w B$ is a bijection between $\mathsf{SkewAPM}_{2n}$ and $K\backslash G/B$.*

**Example 2.14.** Suppose $n = 2$. Then matrices in $\mathsf{SkewAPM}_4$ are in one of the following forms:

$$w_1 = \begin{pmatrix} 0 & t^a & 0 & 0 \\ -t^a & 0 & 0 & 0 \\ 0 & 0 & 0 & t^b \\ 0 & 0 & -t^b & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & t^a & 0 \\ 0 & 0 & 0 & t^b \\ -t^a & 0 & 0 & 0 \\ 0 & -t^b & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & t^a \\ 0 & 0 & t^b & 0 \\ 0 & -t^b & 0 & 0 \\ -t^a & 0 & 0 & 0 \end{pmatrix}.$$

Here $a$ and $b$ are integers. It holds that

$$g_{w_1} = \begin{pmatrix} t^a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t^b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

**Remark 2.15.** The matrices in $\mathsf{SkewAPM}_{2n}$ can obviously be indexed by the set of *fixed-point-free extended affine twisted involutions*, consisting of symmetric affine permutation matrices with no non-zero diagonal entries. Refer to [23, Definition 4.3] for more details.

## 2.4 Orbits of the product group

Matsuki and Oshima [15] introduced the notion of *clans* and classified the set $K\backslash G/B$ as clans under various conditions for $G$ being any classical linear group of any Lie type, albeit without providing proofs. Yamamoto [29] gave details of the proofs of $\mathsf{GL}(p, \mathbb{C}) \times \mathsf{GL}(q, \mathbb{C})$-orbits in the flag variety $Fl_n$ and parametrized the orbits as *$(p, q)$-clans*. Below, we recall the outline of Yamamoto's work.

A *$(p, q)$-clan* is a $n$-tuple $(c_1, c_2, \ldots c_n)$ such that each $c_i$ is either $+$, $-$ or a natural number, such that every natural number, if appears, appears exactly twice, and the number of $+$ signs minus the number of $-$ signs must be $p - q$.

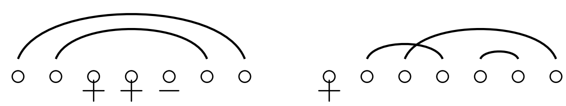
Two such $n$-tuples $c = (c_1, c_2, \dots c_n)$ and $d = (d_1, d_2, \dots d_n)$ are equivalent if the following holds:

(i) For all $i \in [1, n]$, $c_i = +$ if and only if $d_i = +$, and $c_i = -$ if and only if $d_i = -$.

(ii) For all $i, j \in [1, n]$ with $i \neq j$, $c_i = c_j \in \mathbb{N}$ if and only if $d_i = d_j \in \mathbb{N}$.

We do not distinguish between $(p, q)$-clans that are equivalent in the above sense. As a remark, a $(p, q)$-clan contains the same data as an involution in the permutation group $S_n$ with fixed-points labeled by $+$ or $-$.

The $(p, q)$-clans can be represented by arc diagrams, which contains $n$-points lying in a row labelled as $1, 2, \dots, n$, with an arc joining points $i$ and $j$ if $c_i = c_j \in \mathbb{N}$, and plus or minus signs labelled on the points $k$ if $c_k = +$ or $-$ respectively. The following are the arc diagrams for the $(4, 3)$-clans $(1, 2, +, +, -, 2, 1)$ and $(+, 1, 2, 1, 3, 3, 2)$ respectively:



In [29, Proposition 2.2.6], Yamamoto gave an explicit inductive algorithm to produce a $(p, q)$-clan $c(x) = (c_1, c_2, \dots, c_n)$ from a flag $x = (V_0, V_1, \dots, V_n)$, The algorithm involves some dimensional $K$-invariants in $x$. More details are given in [29, Section 2.2]. Nonetheless we follow similar ideas for the affine version in this work. Yamamoto proved that the algorithm [29, Theorem 2.2.8] is a bijection between the $K$-orbits on the set $Fl_n$ and $(p, q)$-clans, which was first discovered by Matsuki and Oshima [15].

In this work we study the affine analog of $\mathsf{GL}(p, \mathbb{C}) \times \mathsf{GL}(q, \mathbb{C})$-orbits in the flag variety in the following sense. Recall $G = \mathsf{GL}(n, \mathbb{K}((t)))$ as the group of invertible $n$-by-$n$ matrices over $\mathbb{K}((t))$ and redefine $B$ to be the subgroup consisting of all upper triangular modulo $t$ matrices in $\mathsf{GL}(n, \mathbb{K}[[t]])$. The set of cosets $G/B$ is the affine flag variety.

We study the $\mathsf{GL}(p, \mathbb{K}((t))) \times \mathsf{GL}(q, \mathbb{K}((t)))$-orbits in the affine flag variety by using an analogue of Yamamoto's work in the classical case. An *affine $(p, q)$-clan* is a $\mathbb{Z}$-indexed sequence $c = (\dots, c_1, c_2, c_3, \dots)$ with $n = p + q$ such that

(i) each $c_i$ is either $+$, $-$ or an integer.

(ii) for $k \in \mathbb{Z}$, $c_{i+kn} = c_i + kn$ if $c_i$ is an integer and $c_{i+n} = c_i$ if $c_i$ is $+$ or $-$,

(iii) $\#\{i \in [n] : c_i = +\} - \#\{i \in [n] : c_i = -\} = p - q$, and every integer, if appears, appears exactly twice in the sequence.

Two such $\mathbb{Z}$-indexed sequences $c = (\dots, c_1, c_2, c_3, \dots)$ and $d = (\dots, d_1, d_2, d_3, \dots)$ are equivalent if the following holds:

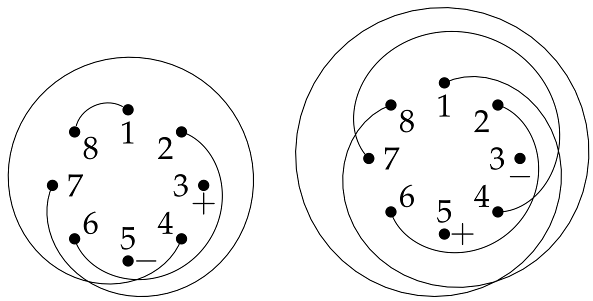
(i) For all $i \in \mathbb{Z}$, $c_i = +$ if and only if $d_i = +$, and $c_i = -$ if and only if $d_i = -$.

(ii) For all $i, j \in \mathbb{Z}$ with $i \neq j$, $c_i = c_j \in \mathbb{Z}$ if and only if $d_i = d_j \in \mathbb{Z}$.

We do not distinguish between affine $(p,q)$-clans that are equivalent in the above sense.

**Example 2.16.** The affine $(1,1)$-clans are $(+,-)$, $(-,+)$ and $(1, 1+2k)$ for $k \in \mathbb{Z}$. The affine $(2,1)$-clans are $(+,+,-)$, $(+,-,+)$, $(-,+,+)$, $(1,1+3k,+)$, $(+,1,1+3k)$ and $(1,+,1+3k)$ for $k \in \mathbb{Z}$.

A useful graphical method of representing the affine $(p,q)$-clans is through the *winding diagrams* of affine $(p,q)$-clans:




Here the numbers $1, 2, \dots n$ are arranged in order around a circle. A curve is travelling $k$ times clockwise connecting $i < j$ if $c_j = c_i - kn$, and travelling $k-1$ times anticlockwise connecting $i < j$ if $c_j = c_i + kn$. The examples above are the affine $(4,4)$-clans with $(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8) = (1, 2, +, 3, -, 2, 2-8, 1+8)$ and $(1, 2, -, 3, +, 2, 2+8, 1-8)$ respectively. Therefore, we can treat affine $(p,q)$-clans as *affine involutions* with $+$ or $-$ signs assigned to the fixed points.

For every affine $(p,q)$-clan $c = (\dots, c_1, c_2, \dots, c_n, \dots)$, the following defines an *affine $(p,q)$-clan matrix* in $\mathsf{GL}(n, \mathbb{K}((t)))$. Define inductively the $(n+1-i)$-th column $v_i$ of a matrix $g_c$ in $\mathsf{GL}(n, \mathbb{K}((t)))$ as follows: Suppose we have already obtained $v_1, v_2, \dots, v_{i-1}$. Denote $\Lambda_{i-1} = \mathrm{span}_{\mathbb{K}[\![t]\!]}\{v_1, v_2, \dots, v_{i-1}\}$.

(a) If $c_i = +$, then set $v_i = e_s \in V_+$, where $s$ is the largest index between 1 and $p$ such that $e_s \notin \pi_+(\Lambda_{i-1})$.

(b) If $c_i = -$, then set $v_i = e_t \in V_+$, where $t$ is the largest index between $p+1$ and $n$ such that $e_t \notin \pi_-(\Lambda_{i-1})$.

(c) If $c_i \equiv c_j \mod n$ for some $i < j \leq n$ and $c_i < c_j$, set $v_i = e_s + t^{-m} e_t$, where $s$ and $t$ are as in $(a)$ and $(b)$, and $m = (c_j - c_i)/n$.

(d) If $c_i \equiv c_j \mod n$ for some $i < j \leq n$ and $c_i \geq c_j$, set $v_i = e_t$, where $t$ is as in $(b)$.

(e) If $c_i \equiv c_j \mod n$ for some $1 \leq j < i$ and $c_i \leq c_j$, set $v_i = e_s + t^{-m} e_t$, where $s$ and $t$ are the same as defined for $v_j$, and $m = (c_j - c_i)/n$.

(f) If $c_i \equiv c_j \mod n$ for some $1 \leq j < i$ and $c_i > c_j$, set $v_i = e_t$, where $t$ is the same as defined for $v_j$.

Finally set $g_c = (v_n, v_{n-1}, \ldots, v_2, v_1) \in \mathsf{GL}(n, \mathbb{K}((t)))$.

The main theorem is the following.

**Theorem 2.17.** Suppose $G = \mathsf{GL}(n, \mathbb{K}((t)))$, $B$ the Iwahori subgroup of $G$, and $K = \mathsf{GL}(p, \mathbb{K}((t))) \times \mathsf{GL}(q, \mathbb{K}((t)))$. The affine $(p,q)$-clan matrices are distinct double coset representatives of the double cosets in $K \backslash G / B$.

**Example 2.18.** Suppose $n = 3$, $p = 2$ and $q = 1$, and $a \in \mathbb{Z}_{\leq 0}$, $b \in \mathbb{Z}_{<0}$. Then the following affine $(2,1)$-clan matrices are distinct double coset representatives in $K \backslash G / B$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t^a & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & t^b \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ t^a & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & t^b & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t^a & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & t^b \end{pmatrix}.$$

The affine $(2,1)$-clan matrices in the first row above correspond to the affine $(2,1)$-clans $(-,+,+)$, $(+,-,+)$ and $(+,+,-)$ while the matrices in the second row correspond to $(1, 1+3a, +)$, $(1, 1-3b, +)$, $(+, 1, 1+3a)$, $(+, 1, 1-3b)$, $(1, +, 1+3a)$ and $(1, +, 1-3b)$ respectively.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