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# On a super version of Thrall's problem

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**Abstract.** Thrall's problem asks for the irreducible decompositions of the Lie modules $\mathcal{L}_\lambda$, which decompose the tensor algebra as a general linear group module. In this extended abstract of [2], we describe a super generalization of Thrall's problem and develop new super tableau combinatorics in order to extend known results to this new setting. As a sample of our results, we obtain a combinatorial interpretation of a $q, t$-hook formula of Macdonald.

**Keywords:** Lie modules, major index, super algebras, free Lie algebras

## 1 Introduction

The free Lie algebra $\mathcal{L}(V)$ was famously studied by Thrall [19], who used it to obtain a certain decomposition of the tensor algebra as a general linear group module. *Thrall's problem* is to determine the irreducible decompositions of the resulting components, called *Lie modules*, and it has remained open in general since Thrall originally posed it in 1942. We describe a generalization of Thrall's problem involving the *free Lie superalgebra*, and extend the only known case of Thrall's problem to the super setting. To do this, we employ a new combinatorial statistic smaj on *super tableaux*, which we use to derive new supersymmetric function identities. Below, we give a short history of Thrall's problem and describe some of the work surrounding it, before outlining our results.

### 1.1 Thrall's problem

Some familiarity with GL-representation theory and symmetric functions will be assumed. See e.g. [8] and [17, Chapter 7], respectively, for missing definitions.

Given a complex vector space $V = \mathbb{C}^N$, its tensor algebra $\mathrm{T}(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ contains the *free Lie algebra* $\mathcal{L}(V)$, which is the Lie subalgebra of $\mathrm{T}(V)$ generated by $V$. The free Lie algebra inherits a grading from $\mathrm{T}(V)$ via $\mathcal{L}_n(V) := \mathcal{L}(V) \cap V^{\otimes n}$ which is compatible with the $\mathrm{GL}(\mathbb{C}^N)$-action on $\mathrm{T}(V)$, endowing $\mathcal{L}(V) = \bigoplus_{n \geq 1} \mathcal{L}_n(V)$ with the structure of a

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graded $\mathrm{GL}(\mathbb{C}^N)$-module. Thrall [19] determined the following $\mathrm{GL}(\mathbb{C}^N)$-decomposition of $\mathrm{T}(V)$ coming from the free Lie algebra:

$$\mathrm{T}(V) \cong \bigoplus_{\lambda} \mathcal{L}_\lambda(V), \tag{1.1}$$

where the sum ranges over all integer partitions $\lambda$. The $\mathrm{GL}(\mathbb{C}^N)$-modules $\mathcal{L}_\lambda(V)$ are called *higher Lie modules*, and are defined by:

$$\mathcal{L}_\lambda(V) := S^{m_1}(\mathcal{L}_1(V)) \otimes S^{m_2}(\mathcal{L}_2(V)) \otimes \cdots,$$

where $\lambda = (1^{m_1}2^{m_2}\cdots)$. Here $S(W)$ denotes the symmetric algebra of a vector space $W$.

*Thrall's problem* is to determine the irreducible $\mathrm{GL}(\mathbb{C}^N)$-decomposition of the higher Lie modules $\mathcal{L}_\lambda(V)$. Thrall's problem remains open in general, but has received significant attention in the literature ([1, 4, 7, 9, 10, 12, 13, 16, 18]) since its inception in 1942.

On the level of characters, this problem can be stated as follows.

**Problem 1.1** (Thrall's problem)**.** Determine the coefficients $a_\mu \in \mathbb{Z}_{\geq 0}$ in the Schur expansion of $\mathrm{Ch}(\mathcal{L}_\lambda(V))$:

$$\mathrm{Ch}(\mathcal{L}_\lambda(V)) = \sum_{\mu} a_\mu s_\mu(x_1, \ldots, x_N).$$

Here $\mathrm{Ch}(\mathcal{L}_\lambda(V))$ denotes the $\mathrm{GL}(\mathbb{C}^N)$-character of $\mathcal{L}_\lambda(V)$ and $s_\mu(x_1, \ldots, x_N)$ denotes the *Schur polynomial* indexed by $\mu$. When passing to characters, we will often implicitly let $N \to \infty$ so that we can instead work with the Schur *functions* $s_\lambda(\mathbf{x}) := s_\lambda(x_1, x_2, \ldots)$. In doing so, we will omit reference to the underlying vector space and write $\mathcal{L}(V) = \mathcal{L}$ (resp. $\mathcal{L}_n, \mathcal{L}_\lambda$) when there is no possibility for confusion.

## 1.2 The one-row case

The one-row case of Thrall's problem, which concerns the $\mathrm{GL}(\mathbb{C}^N)$-module structure of the graded pieces $\mathcal{L}_{(n)} = \mathcal{L}_n$ of the free Lie algebra, is of particular importance. An expression for $\mathrm{Ch}(\mathcal{L}_n)$ in terms of *power-sum symmetric functions* $p_d(\mathbf{x}), d \geq 1$ was first found by Brandt:

**Theorem 1.2** ([7])**.** *For any $n \geq 1$,*

$$\mathrm{Ch}(\mathcal{L}_n) = \frac{1}{n} \sum_{d|n} \mu(d) p_d(\mathbf{x})^{\frac{n}{d}},$$

*where $\mu(-)$ denotes the Möbius function.*

The power-sum expansion of $\mathrm{Ch}(\mathcal{L}_n)$ suggests that the Schur–Weyl dual of $\mathcal{L}_n$ is an $S_n$-representation induced from the cyclic group $C_n$. This was proved by Klyachko:

**Theorem 1.3** ([12]). *For any $n \geq 1$, the Schur–Weyl dual of $\mathcal{L}_n$ is $(\chi^1) \uparrow_{C_n}^{S_n}$, where $\chi^1$ is the representation of $C_n = \langle \sigma \rangle$ given by $\chi^1(\sigma) = \exp(2\pi i/n)$. That is,*

$$\mathrm{Ch}(\mathcal{L}_n) = \mathrm{FrobCh}((\chi^1) \uparrow_{C_n}^{S_n}),$$

*where* FrobCh$(-)$ *denotes the* Frobenius characteristic.

Building on Klyachko's work, Kráskiewicz–Weyman found the Schur expansion of $\mathrm{Ch}(\mathcal{L}_n)$, proving the one-row case of Thrall's problem.

**Theorem 1.4** ([13]). *For any $n \geq 1$, we have*

$$\mathrm{Ch}(\mathcal{L}_n) = \sum_{\mu \vdash n} a_{\mu,1} s_\mu(\mathbf{x}),$$

*where $a_{\mu,1} = |\{T \in \mathrm{SYT}(\mu) : \mathrm{maj}(T) \equiv_n 1\}|$.*

Here $\mathrm{SYT}(\mu)$ denotes the set of *standard tableaux* of shape $\mu$, maj denotes the *major index*, and $\equiv_n$ means congruence modulo $n$.

## 1.3 A super generalization

In Theorem 2.2, we extend Thrall's problem to the supersymmetric setting by finding a $\mathrm{GL}(\mathbb{C}^N) \oplus \mathrm{GL}(\mathbb{C}^M)$-decomposition of the tensor superalgebra of a super vector space $V = \mathbb{C}^N \oplus \mathbb{C}^M$ coming from the free Lie superalgebra $\widetilde{\mathcal{L}}(V) = \bigoplus_{n,m} \widetilde{\mathcal{L}}_{n,m}(V)$. In turn, we generalize the three above results in the one-row case:

1. in Theorem 2.4 we find the power-sum expansion of $\mathrm{Ch}(\widetilde{\mathcal{L}}_{n,m})$,
2. in Theorem 2.5 we determine the Schur–Weyl dual of $\widetilde{\mathcal{L}}_{n,m}$ as an induced representation, and
3. in Theorem 3.1 we determine the irreducible decomposition of $\widetilde{\mathcal{L}}_{n,m}$.

These results extend Theorem 1.2, Theorem 1.3, and Theorem 1.4, respectively.

Our proof of Theorem 3.1 involves a new major index statistic smaj on objects called *super tableaux*. In Proposition 3.9 and Theorem 3.12, we find interesting super analogs of known symmetric function identities using smaj. In particular, we relate this new statistic to a $q,t$-hook formula of Macdonald, as was announced in [6]:

**Theorem 1.5.** *For any $\lambda \vdash n$,*

$$\sum_{\mathcal{T} \in \mathrm{SYT}_\pm(\lambda)} q^{\mathrm{smaj}(\mathcal{T})} t^{\mathrm{neg}(\mathcal{T})} = [n]_q! \prod_{(r,c) \in \lambda} \frac{q^{r-1} + t q^{c-1}}{[h(r,c)]_q} \tag{1.2}$$

*where $[n]_q := 1 + q + \cdots + q^{n-1}$ and $[n]_q! := [1]_q [2]_q \cdots [n]_q$.*

Here $h(r,c)$ denotes the *hook length* of the cell at position $(r,c)$, and $\mathrm{SYT}_\pm(\lambda)$ denotes the set of *standard super tableaux* of shape $\lambda$ (see Section 3.1 for the definitions of smaj and $\mathrm{SYT}_\pm(\lambda)$).

# 2 Super Thrall's problem

We begin by generalizing Thrall's problem to the setting of free Lie superalgebras. First, we go over some preliminaries on Lie superalgebras in Section 2.1, and then obtain a decomposition of the tensor superalgebra involving the free Lie superalgebra in Section 2.2. We then describe extensions of Theorem 1.2 and Theorem 1.3 in Section 2.3.

## 2.1 Lie superalgebras

A $\mathbb{Z}/2$-graded vector space $V = V_0 \oplus V_1$ is called a *super* vector space. A *superalgebra* $\widetilde{A} = \widetilde{A}_0 \oplus \widetilde{A}_1$ is a super vector space equipped with multiplication satisfying $\widetilde{A}_i \widetilde{A}_j \subseteq \widetilde{A}_{i+j}$, where the indices are taken modulo 2. A *Lie superalgebra* $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_0 \oplus \widetilde{\mathfrak{g}}_1$ is a super vector space equipped with a bilinear operation $[-,-]$ satisfying

S1. $[x, y] = -(-1)^{|x||y|}[y, x]$,

S2. $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|z||y|}[z, [x, y]] + (-1)^{|y||x|}[y, [z, x]] = 0$,

S3. $[\widetilde{\mathfrak{g}}_i, \widetilde{\mathfrak{g}}_j] \subseteq \widetilde{\mathfrak{g}}_{i+j}$,

where $|x| = i$ means that $x \in \widetilde{\mathfrak{g}}_i$.

The tensor algebra $\mathrm{T}(V) = \bigoplus_{n\geq 0}(V_0 \oplus V_1)^{\otimes n}$ of a super vector space $V$ admits a bigrading $\mathrm{T}(V) = \bigoplus_{n,m\geq 0} \mathrm{T}_{n,m}(V)$, where $\mathrm{T}_{n,m}(V)$ is spanned by tensors with $n$ terms from $V_0$ and $m$ terms from $V_1$. This bigrading in turn determines a $\mathbb{Z}/2$-grading on $\mathrm{T}(V)$, and the natural concatenation product on $\mathrm{T}(V) = \mathrm{T}(V)_0 \oplus \mathrm{T}(V)_1$ endows $\mathrm{T}(V)$ with the structure of a superalgebra, so we will write $\widetilde{\mathrm{T}}(V)$ to denote $\mathrm{T}(V)$ with its superalgebra structure.

## 2.2 A tensor superalgebra decomposition

We now construct a tensor superalgebra decomposition, closely following the set-up outlined in Section 1.1. Let $V = V_0 \oplus V_1 = \mathbb{C}^N \oplus \mathbb{C}^M$ be a super vector space. The tensor superalgebra $\widetilde{\mathrm{T}}(V)$ is naturally a $\mathrm{GL}(\mathbb{C}^N) \oplus \mathrm{GL}(\mathbb{C}^M)$-module.

The *free Lie superalgebra* $\widetilde{\mathcal{L}}(V)$ is the sub-Lie superalgebra of the tensor superalgebra $\widetilde{\mathrm{T}}(V)$ generated by $V$, with respect to the *super commutator* $[x, y] := x \otimes y - (-1)^{|x||y|} y \otimes x$. The free Lie superalgebra inherits a bigrading from $\widetilde{\mathrm{T}}(V)$ via

$$\widetilde{\mathcal{L}}_{n,m}(V) := \widetilde{\mathcal{L}}(V) \cap \widetilde{\mathrm{T}}_{n,m}(V),$$

which is compatible with the $\mathrm{GL}(\mathbb{C}^N) \oplus \mathrm{GL}(\mathbb{C}^M)$-action on $\widetilde{\mathrm{T}}(V)$, so that $\widetilde{\mathcal{L}}(V)$ is a bigraded $\mathrm{GL}(\mathbb{C}^N) \oplus \mathrm{GL}(\mathbb{C}^M)$-module.

The bigraded components $\widetilde{\mathcal{L}}_{n,m}(V)$ of $\widetilde{\mathcal{L}}(V)$ may now be used to define certain modules which will turn out to form a decomposition of $\widetilde{\mathrm{T}}(V)$.

**Definition 2.1** (Super higher Lie modules)**.** For a vector space $W$ and $j \in \mathbb{Z}_{\geq 0}$, let

$$\Gamma_j(W) = \begin{cases} S(W) & \text{if } j \text{ is even} \\ \bigwedge(W) & \text{if } j \text{ is odd.} \end{cases}$$

Given a super vector space $V$ and a $\mathbb{Z}_{\geq 0}$-valued matrix $A = (a_{i,j})_{i,j \geq 0}$ with finite support and $a_{0,0} = 0$, the *super Lie module* $\widetilde{\mathcal{L}}_A(V)$ is:

$$\widetilde{\mathcal{L}}_A(V) = \bigotimes_{i,j \geq 0} \Gamma_j^{a_{i,j}}(\widetilde{\mathcal{L}}_{i,j}(V)). \tag{2.1}$$

Note that for $m$ odd, the exterior power $\Gamma_m^a(W) = \bigwedge^a(W)$ is zero unless $a \leq \dim W$.

**Theorem 2.2** (Super Thrall decomposition)**.** *Let* $V = V_0 \oplus V_1 = \mathbb{C}^N \oplus \mathbb{C}^M$*. Then*

$$\widetilde{\mathrm{T}}(V) = \bigoplus_A \widetilde{\mathcal{L}}_A(V)$$

*as* $\mathrm{GL}(\mathbb{C}^N) \oplus \mathrm{GL}(\mathbb{C}^M)$*-modules, where the sum is over* $\mathbb{Z}_{\geq 0}$*-valued matrices* $A = (a_{i,j})_{i,j \geq 0}$ *with finite support and* $a_{0,0} = 0$*.*

*Proof.* (Sketch.) The universal enveloping superalgebra $\widetilde{U}(\widetilde{\mathcal{L}}(V))$ is canonically isomorphic to the tensor superalgebra $\widetilde{\mathrm{T}}(V)$. On the other hand, a super version of the Poincaré–Birkhoff–Witt theorem proves that $\widetilde{U}(\widetilde{\mathfrak{g}}) \cong S(\widetilde{\mathfrak{g}}_0) \otimes \bigwedge(\widetilde{\mathfrak{g}}_1)$ here. Thus combining these two results yields the following decomposition of $\widetilde{\mathrm{T}}(V)$:

$$\begin{aligned} \widetilde{\mathrm{T}}(V) &\cong S\left(\bigoplus_{n,m \geq 0} \widetilde{\mathcal{L}}_{n,2m}(V)\right) \otimes \bigwedge \left(\bigoplus_{n,m \geq 0} \widetilde{\mathcal{L}}_{n,2m+1}(V)\right) \cong \bigotimes_{i,j \geq 0} \Gamma_j\left(\widetilde{\mathcal{L}}_{i,j}(V)\right) \\ &\cong \bigoplus_{A=(a_{i,j} \geq 0)} \bigotimes_{i,j \geq 0} \Gamma_j^{a_{i,j}}\left(\widetilde{\mathcal{L}}_{i,j}(V)\right) = \bigoplus_{A=(a_{i,j} \geq 0)} \widetilde{\mathcal{L}}_A(V). \end{aligned}$$

□

Thus the free Lie superalgebra yields a $\mathrm{GL}(\mathbb{C}^N) \oplus \mathrm{GL}(\mathbb{C}^M)$-decomposition of $\widetilde{\mathrm{T}}(V)$. Now, restrict to the case $N = M$ so that $\widetilde{\mathcal{L}}(V)$ is a $\mathrm{GL}(\mathbb{C}^N)$-module under the diagonal inclusion $\mathrm{GL}(\mathbb{C}^N) \hookrightarrow \mathrm{GL}(\mathbb{C}^N) \oplus \mathrm{GL}(\mathbb{C}^N)$. It is then natural to ask for the irreducible $\mathrm{GL}(\mathbb{C}^N)$-decomposition of $\widetilde{\mathcal{L}}_A(V)$ as we did in the classical case.

**Problem 2.3** (Super Thrall's problem)**.** For $A = (a_{i,j} \geq 0)_{i,j}$, determine the coefficients $a_\lambda \in \mathbb{Z}_{\geq 0}$ in the Schur expansion of $\widetilde{\mathcal{L}}_A(V)$:

$$\mathrm{Ch}(\widetilde{\mathcal{L}}_A(V)) = \sum_\lambda a_\lambda s_\lambda(x_1, \ldots, x_N).$$

As before, we will let $N \to \infty$ and simply write $\widetilde{\mathcal{L}}(V) = \widetilde{\mathcal{L}}$ (resp. $\widetilde{\mathcal{L}}_{n,m}, \widetilde{\mathcal{L}}_A$) when working with characters.

### 2.3 Bigraded components of the free Lie superalgebra

We now restrict our attention to the bigraded components $\widetilde{\mathcal{L}}_{n,m}$ of the free Lie superalgebra. We first find an expression for $\mathrm{Ch}(\widetilde{\mathcal{L}}_{n,m})$ in the power-sum basis, generalizing Theorem 1.2.

**Theorem 2.4.** *The character of $\widetilde{\mathcal{L}}_{n,m}$ is given by*

$$\mathrm{Ch}(\widetilde{\mathcal{L}}_{n,m}) = \frac{1}{n+m} \sum_{d \mid \gcd(n,m)} (-1)^{m+\frac{m}{d}} \mu(d) \binom{\frac{n+m}{d}}{\frac{m}{d}} p_d(\mathbf{x})^{\frac{n+m}{d}}. \tag{2.2}$$

*Proof.* (Sketch.) Petrogradsky [15] found the Hilbert series $\mathrm{Hilb}(\widetilde{\mathcal{L}}; q, t)$ of the free Lie superalgebra, from which the graded character of $\widetilde{\mathcal{L}}$ is readily obtained. The result then follows by extracting the coefficient of $q^n t^m$. □

We also generalize Theorem 1.3 by constructing a $S_{n+m}$-module whose Frobenius characteristic agrees with $\mathrm{Ch}(\widetilde{\mathcal{L}}_{n,m})$, which identifies the Schur–Weyl dual of $\widetilde{\mathcal{L}}_{n,m}$. To describe this representation, we need a bit of notation.

Let $C_{n+m}$ denote the cyclic group generated by the long cycle $\pi_{n+m} = (1\ 2 \cdots n+m) \in S_{n+m}$, and let $\chi^1, \ldots, \chi^{n+m}$ denote its irreducible representations. The group $C_{n+m}$ acts on the set $\binom{[n+m]}{m}$ of $m$-subsets of $[n+m]$ by cyclic rotation, which determines a $C_{n+m}$-representation $\chi^{\mathrm{cyc}} : C_{n+m} \to \mathrm{GL}(\mathbb{C}^{\binom{n+m}{m}})$ whose trace is given by

$$\mathrm{tr}(\chi^{\mathrm{cyc}}(\pi_{n+m}^k)) = \left| \left\{ S \in \binom{[n+m]}{m} : \pi_{n+m}^k \cdot S = S \right\} \right|.$$

The following theorem then follows from a straightforward character computation.

**Theorem 2.5.** *We have*

$$\mathrm{Ch}(\widetilde{\mathcal{L}}_{n,m}) = \begin{cases} \mathrm{FrobCh}((\chi^{\mathrm{cyc}} \otimes \chi^1) \uparrow_{C_{n+m}}^{S_{n+m}}) & \text{if } m \text{ is odd} \\ \mathrm{FrobCh}((\chi^{\mathrm{cyc}} \otimes \chi^{m/2+1}) \uparrow_{C_{n+m}}^{S_{n+m}}) & \text{if } m \text{ is even.} \end{cases} \tag{2.3}$$

# 3 Irreducible decomposition of $\widetilde{\mathcal{L}}_{n,m}$

In this section we describe the following generalization of Kráskiewicz–Weyman's result, which determines the irreducible $\mathrm{GL}(\mathbb{C}^N)$-decomposition of the super Lie modules $\widetilde{\mathcal{L}}_{n,m}$:

**Theorem 3.1.** *For $n+m>0$, we have*

$$\mathrm{Ch}(\widetilde{\mathcal{L}}_{n,m}) = \sum_{\lambda \vdash n+m} a_\lambda s_\lambda(\mathbf{x}),$$

*where*

$$a_\lambda := |\{\mathcal{T} \in \mathrm{SYT}_\pm(\lambda) : \mathrm{smaj}(\mathcal{T}) \equiv_{n+m} 1, \mathrm{neg}(\mathcal{T}) = m\}|. \tag{3.1}$$

Here $\mathrm{SYT}_{\pm}(\lambda)$ denotes the set of *super tableaux* of shape $\lambda$, and smaj is a new major index statistic which we define on the set of super tableaux.

We define this new major index statistic in Section 3.1, and in Section 3.2 we show that the principal specializations of the super quasisymmetric function $\widetilde{Q}_{n,D}(\mathbf{x};\mathbf{y})$ and the super Schur function $\tilde{s}_{\lambda}(\mathbf{x};\mathbf{y})$ may be written as certain smaj-generating functions. The latter generating function corresponds to a $q,t$-hook formula of Macdonald, which is a crucial component in our proof of Theorem 3.1. We outline the proof in Section 3.3.

## 3.1 Super major index

We begin by defining super tableaux. Let $\mathcal{A} = \mathcal{A}_+ \sqcup \mathcal{A}_-$, where $\mathcal{A}_+ = \{1,2,\ldots\}$ and $\mathcal{A}_- = \{\overline{1},\overline{2},\ldots\}$. Endow $\mathcal{A}$ with the total order $\mathcal{A} = \{1 < \overline{1} < 2 < \overline{2} < \cdots\}$. We call the elements of $\mathcal{A}_+$ *positive* and the elements of $\mathcal{A}_-$ *negative*. We then have the following.

**Definition 3.2.** A *standard super tableau* of shape $\lambda \vdash n$ is a map $\mathcal{T} : \lambda \to \mathcal{A}$ that is strictly increasing along the rows and columns of $\lambda$, and contains exactly one of $i$ or $\overline{i}$ for each $i = 1,2,\ldots,n$. Let $\mathrm{SYT}_{\pm}(\lambda)$ denote the set of standard super tableaux of shape $\lambda$, so that $|\mathrm{SYT}_{\pm}(\lambda)| = 2^n|\mathrm{SYT}(\lambda)|$. For $\mathcal{T} \in \mathrm{SYT}_{\pm}(\lambda)$, we let $\mathrm{Neg}(\mathcal{T}) := \{i \in \mathcal{A}_+ : \overline{i} \in \mathcal{T}\}$ and $\mathrm{neg}(\mathcal{T}) := |\mathrm{Neg}(\mathcal{T})|$.

**Example 3.3.** The standard super tableau

$$\mathcal{T} = \begin{array}{|c|c|c|c|}\hline 1 & \overline{3} & 4 & 6 \\ \hline \overline{2} & 5 \\ \cline{1-2} \overline{7} \\ \cline{1-1}\end{array} \in \mathrm{SYT}_{\pm}(4,2,1)$$

has $\mathrm{Neg}(\mathcal{T}) = \{2,3,7\}$ and $\mathrm{neg}(\mathcal{T}) = 3$.

Recall that the *descent set* $\mathrm{Des}(T)$ of a standard tableau $T \in \mathrm{SYT}(\lambda)$ is the set of entries $i$ such that $i+1$ appears in a lower row of $T$ than $i$. We now define a generalization of the descent set for standard super tableaux.

**Definition 3.4.** For $\lambda \vdash n$ and $\mathcal{T} \in \mathrm{SYT}_{\pm}(\lambda)$, let $\mathcal{T}_+ \in \mathrm{SYT}(\lambda)$ denote the image of $\mathcal{T}$ under the natural projection $\mathcal{A} \to \mathcal{A}_+$. For $i = 1,\ldots,n-1$, we say that $i$ is a *super descent* of $\mathcal{T}$ if either

$$i \in \mathrm{Des}(\mathcal{T}_+) \text{ and } i+1 \notin \mathrm{Neg}(\mathcal{T}), \text{ or } i \notin \mathrm{Des}(\mathcal{T}_+) \text{ and } i \in \mathrm{Neg}(\mathcal{T}).$$

Define

$$\mathrm{sDes}(\mathcal{T}) = \{i : i \text{ is a super descent of } \mathcal{T}\} \subseteq [n-1].$$

**Example 3.5.** If

$$\mathcal{T} = \begin{array}{|c|c|c|c|}\hline 1 & \overline{3} & 4 & 6 \\ \hline \overline{2} & 5 \\ \cline{1-2} \overline{7} \\ \cline{1-1} \end{array} \in \mathrm{SYT}_{\pm}(4,2,1) \quad \text{then} \quad \mathcal{T}_{+} = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \in \mathrm{SYT}(4,2,1),$$

so $\mathrm{Des}(\mathcal{T}_{+}) = \{1,4,6\}$ and $\mathrm{Neg}(\mathcal{T}) = \{2,3,7\}$. Therefore $\mathrm{sDes}(\mathcal{T}) = \{2,3,4\}$.

Armed with the notion of super descents, we now define our new major index statistic on super tableaux.

**Definition 3.6.** For $D \subseteq [n-1]$ and $S \subseteq [n]$, define the *relative major index* and the *relative comajor index* respectively by

$$\mathrm{smaj}(D,S) := \sum_{\substack{1 \leq i \leq n-1, \\ i \in D, i+1 \notin S \\ \text{or } i \notin D, i \in S}} i, \qquad \mathrm{scomaj}(D,S) := \sum_{\substack{1 \leq i \leq n-1, \\ i \in D, i+1 \notin S \\ \text{or } i \notin D, i \in S}} (n-i).$$

For $\mathcal{T} \in \mathrm{SYT}_{\pm}(\lambda)$, we define the relative (co)major index by

$$\mathrm{smaj}(\mathcal{T}) := \sum_{i \in \mathrm{sDes}(\mathcal{T})} i, \quad \mathrm{scomaj}(\mathcal{T}) := \sum_{i \in \mathrm{sDes}(\mathcal{T})} (n-i).$$

**Example 3.7.** The super tableau in Example 3.5 has $\mathrm{smaj}(\mathcal{T}) = 2+3+4 = 9$.

Note that if $\mathcal{T} \in \mathrm{SYT}_{\pm}(\lambda)$ contains no negative entries, then $\mathrm{sDes}(\mathcal{T}), \mathrm{smaj}(\mathcal{T})$, and $\mathrm{scomaj}(\mathcal{T})$ agree with the classical notions of descent set, major index, and comajor index, respectively.

## 3.2 Specializations

In the classical case, the principal specializations of fundamental quasisymmmetric functions and Schur functions both admit elegant formulae in terms of the usual (co)major index statistic. Both of these functions admit supersymmetric analogs, and we prove that their principal specializations may be written in terms of our new major index statistic.

**Definition 3.8** ([11], Equation (23))**.** For $n \geq 2$ and $D \subseteq [n-1]$, the *super quasisymmetric function* $\widetilde{Q}_{n,D}(\mathbf{x};\mathbf{y})$ is given by

$$\widetilde{Q}_{n,D}(\mathbf{x};\mathbf{y}) = \sum_{\substack{a_1 \leq a_2 \leq \cdots \leq a_n, \\ a_i = a_{i+1} \in \mathcal{A}_{+} \Rightarrow i \notin D, \\ a_i = a_{i+1} \in \mathcal{A}_{-} \Rightarrow i \in D}} z_{a_1} z_{a_2} \cdots z_{a_n},$$

where $a_1 \leq \cdots \leq a_n$ is a weakly increasing sequence in $\mathcal{A}$, and $z_a = x_a$ for $a \in \mathcal{A}_{+}, z_b = y_b$ for $b \in \mathcal{A}_{-}$.

The fundamental quasisymmetric function $Q_{n,D}(\mathbf{x}) = \widetilde{Q}_{n,D}(\mathbf{x};\mathbf{0})$ has the following well-known principal specialization:

$$Q_{n,D}(1,q,q^2,\ldots) = \frac{1}{(q;q)_n} q^{\mathrm{comaj}(D)}, \tag{3.2}$$

where $\mathrm{comaj}(D) = \mathrm{scomaj}(D,\emptyset)$ denotes the usual comajor index statistic, and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ denotes the $q$-Pochhammer symbol. This can be generalized as follows.

**Proposition 3.9.** *For any $n \geq 2$ and $D \subseteq [n-1]$, the specialization of $\widetilde{Q}_{n,D}$ given by setting $x_i = q^{i-1}, y_i = tq^{i-1}$ is*

$$\widetilde{Q}_{n,D}(1,q,q^2,\ldots;t,tq,tq^2,\ldots) = \frac{1}{(q;q)_n} \sum_{S\subseteq[n]} q^{\mathrm{scomaj}(D,S)} t^{|S|}.$$

The proof closely follows the proof of (3.2) found in [17, Lemma 7.19.10].

**Definition 3.10** ([11], Proposition 2.4.2)**.** The *super Schur function* $\tilde{s}_\lambda(\mathbf{x};\mathbf{y})$ is given in terms of super quasisymmetric functions by

$$\tilde{s}_\lambda(\mathbf{x};\mathbf{y}) = \sum_{T\in\mathrm{SYT}(\lambda)} \widetilde{Q}_{|\lambda|,\mathrm{Des}(T)}(\mathbf{x};\mathbf{y}).$$

Macdonald found a formula for the principal specialization of a super Schur function:

**Theorem 3.11** ([14, page 27, Example 5 and page 45, Example 3])**.** *The specialization of $\tilde{s}_\lambda(\mathbf{x};\mathbf{y})$ given by setting $x_i = q^{i-1}$, $y_i = tq^{i-1}$ is given by*

$$\tilde{s}_\lambda(1,q,q^2,\ldots;t,tq,tq^2,\ldots) = \prod_{(r,c)\in\lambda} \frac{q^{r-1}+tq^{c-1}}{1-q^{h(r,c)}},$$

*where $h(r,c)$ denotes the* hook length *of the cell $(r,c)$.*

In the classical case, the principal specialization of the Schur function $s_\lambda(\mathbf{x})$ is given by

$$s_\lambda(1,q,q^2,\ldots) = \prod_{(r,c)\in\lambda} \frac{q^{r-1}}{1-q^{h(r,c)}} = \frac{1}{(q;q)_n} \sum_{T\in\mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)},$$

where $\mathrm{maj}(T)$ denotes the usual major index statistic. The first equality is a consequence of Stanley's hook-content formula, and the second follows by writing $s_\lambda$ in terms of quasisymmetric functions.

Theorem 3.11 lifts the first equality to the super setting, and we lift the second equality using our new major index statistic.

**Theorem 3.12.** *For $\lambda \vdash n$, we have*

$$\tilde{s}_\lambda(1, q, q^2, \ldots; t, tq, tq^2, \ldots) = \frac{1}{(q;q)_n} \sum_{\mathcal{T} \in \mathrm{SYT}_\pm(\lambda)} q^{\mathrm{scomaj}(T)} t^{\mathrm{neg}(T)}$$
$$= \frac{1}{(q;q)_n} \sum_{\mathcal{T} \in \mathrm{SYT}_\pm(\lambda)} q^{\mathrm{smaj}(T)} t^{\mathrm{neg}(T)}.$$

*Proof.* (Sketch.) The first equality follows from Proposition 3.9 by writing $\tilde{s}_\lambda$ in terms of super quasisymmetric functions. The second equality hinges on the following symmetry of the principal specialization of $\tilde{s}_\lambda$, which follows from Theorem 3.11:

$$\tilde{s}_\lambda(1, q, q^2, \ldots; t, tq, tq^2, \ldots) = t^n \tilde{s}_{\lambda'}(1, q, q^2, \ldots; t^{-1}, t^{-1}q, t^{-1}q^2, \ldots).$$

□

Theorem 1.5 follows by combining Theorem 3.11 and Theorem 3.12.

## 3.3 Proof of Theorem 3.1

We conclude by briefly outlining the proof of Theorem 3.1. It suffices to show that $\mathrm{Ch}(\widetilde{\mathcal{L}}_{n,m})$ can be obtained from

$$\varphi(\mathbf{x}; q, t) := \sum_{\lambda \vdash n} s_\lambda(\mathbf{x}) \sum_{\mathcal{T} \in \mathrm{SYT}_\pm(\lambda)} q^{\mathrm{smaj}(\mathcal{T})} t^{\mathrm{neg}(\mathcal{T})}$$

by extracting the coefficients of $q^{1+\ell(n+m)} t^m$ for all $\ell \in \mathbb{Z}$. By Theorem 3.12, we have

$$\varphi(\mathbf{x}; q, t) = (q;q)_n \sum_{\lambda \vdash n} s_\lambda(\mathbf{x}) \tilde{s}_\lambda(1, q, q^2, \ldots; t, tq, tq^2, \ldots),$$

and we then obtain a power-sum expansion of $\varphi(\mathbf{x}; q, t)$ using a super version of the Cauchy identity from [3, Corollary 10(a)]. We then extract the coefficients of $q^{(1+\ell(n+m)} t^m$ and show that the resulting expression agrees with the formula for $\mathrm{Ch}(\widetilde{\mathcal{L}}_{n,m})$ found in Theorem 2.4.

# 4 Future directions

A corollary of Stanley's $q$-hook formula is that the coefficients of the generating function

$$f^\lambda(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)}$$

are symmetric, in the sense that $f^\lambda(q) = q^{\deg(f^\lambda(q))} f^\lambda(q^{-1})$. However, to the authors' knowledge, there is no explicit combinatorial proof of this symmetry. A further consequence of the $q$-hook formula is that there exist unique tableaux $T_{\min}, T_{\max} \in \mathrm{SYT}(\lambda)$ with minimal and maximal major index, respectively. These tableaux were identified explicitly by Billey–Konvalinka–Swanson in [5] by greedily tiling $\lambda$ with horizontal (resp. vertical) strips.

**Example 4.1.** For $\lambda = (5, 3, 1)$,

$$T_{\min} = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 8 & 9 \\ \hline 2 & 6 & 7 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array}, \quad T_{\max} = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 7 \\ \hline 4 & 6 & 8 \\ \cline{1-3} 9 \\ \cline{1-1} \end{array}.$$

Our proof of Theorem 3.1 involves the following $q, t$-generalization of $f^\lambda(q)$:

$$f^\lambda(q, t) := \sum_{\mathcal{T} \in \mathrm{SYT}_\pm(\lambda)} q^{\mathrm{smaj}(\mathcal{T})} t^{\mathrm{neg}(\mathcal{T})}.$$

Theorem 3.11 proves that for any $m \geq 0$, the coefficients of $[t^m] f^\lambda(q, t)$ are symmetric in the above sense. We leave the following generalization of Billey–Konvalinka–Swanson's result as an open problem.

**Problem 4.2.** For fixed $\lambda$ and $m \geq 0$, identify the super tableaux (not necessarily unique) of shape $\lambda$ containing exactly $m$ negative entries with minimal and maximal smaj. Furthermore, give a bijective proof of the symmetry of $[t^m] f^\lambda(q, t)$.

Note that when $t = 0$ or $t = |\lambda|$, these tableaux are necessarily unique. When $t = 0$ they are given precisely by $T_{\min}, T_{\max} \in \mathrm{SYT}(\lambda)$. When $t = |\lambda|$, the super tableau $\mathcal{T}_{\min}$ with minimal smaj is given by making all of the entries in $T_{\max}$ negative, while $\mathcal{T}_{\max}$ is given by making all of the entries in $T_{\min}$ negative.

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# References

[1] C. Ahlbach and J. P. Swanson. "Cyclic sieving, necklaces, and branching rules related to Thrall's problem". *Electron. J. Combin.* **25**.4 (2018), Paper No. 4.42, 38. DOI.

[2] S. Armon and J. P. Swanson. "Super major index and Thrall's problem". 2024. arXiv: 2411.04302.

[3] A. Berele and J. B. Remmel. “Hook flag characters and their combinatorics”. *J. Pure Appl. Algebra* **35**.3 (1985), pp. 225–245. DOI.

[4] F. Bergeron, N. Bergeron, and A. M. Garsia. “Idempotents for the free Lie algebra and $q$-enumeration”. *Invariant theory and tableaux (Minneapolis, MN, 1988)*. Vol. 19. IMA Vol. Math. Appl. Springer, New York, 1990, pp. 166–190.

[5] S. C. Billey, M. Konvalinka, and J. P. Swanson. “Tableau posets and the fake degrees of coinvariant algebras”. *Adv. Math.* **371** (2020), pp. 107252, 46. DOI.

[6] S. C. Billey and J. P. Swanson. “The metric space of limit laws for $q$-hook formulas”. *Comb. Theory* **2**.2 (2022), Paper No. 5, 58. DOI.

[7] A. Brandt. “The free Lie ring and Lie representations of the full linear group”. *Trans. Amer. Math. Soc.* **56** (1944), pp. 528–536. DOI.

[8] W. Fulton. *Young tableaux*. Vol. 35. London Mathematical Society Student Texts. With applications to representation theory and geometry. Cambridge University Press, Cambridge, 1997, pp. x+260.

[9] A. M. Garsia. “Combinatorics of the free Lie algebra and the symmetric group”. *Analysis, et cetera*. Academic Press, Boston, MA, 1990, pp. 309–382.

[10] I. M. Gessel and C. Reutenauer. “Counting permutations with given cycle structure and descent set”. *J. Combin. Theory Ser. A* **64**.2 (1993), pp. 189–215. DOI.

[11] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov. “A combinatorial formula for the character of the diagonal coinvariants”. *Duke Math. J.* **126**.2 (2005), pp. 195–232. DOI.

[12] A. A. Klyachko. “Lie elements in a tensor algebra”. *Sibirsk. Mat. Ž.* **15** (1974), pp. 1296–1304, 1430.

[13] W. Kraśkiewicz and J. Weyman. “Algebra of coinvariants and the action of a Coxeter element”. *Bayreuth. Math. Schr.* 63 (2001), pp. 265–284.

[14] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Second. Oxford Mathematical Monographs. With contributions by A. Zelevinsky, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995, pp. x+475.

[15] V. M. Petrogradsky. “On Witt’s formula and invariants for free Lie superalgebras”. *Formal power series and algebraic combinatorics (Moscow, 2000)*. Springer, Berlin, 2000, pp. 543–551.

[16] M. Schocker. “Multiplicities of higher Lie characters”. *J. Aust. Math. Soc.* **75**.1 (2003), pp. 9–21. DOI.

[17] R. P. Stanley. *Enumerative combinatorics. Vol. 2*. Vol. 62. Cambridge Studies in Advanced Mathematics. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. Cambridge University Press, Cambridge, 1999, pp. xii+581. DOI.

[18] S. Sundaram. “The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice”. *Adv. Math.* **104**.2 (1994), pp. 225–296. DOI.

[19] R. M. Thrall. “On symmetrized Kronecker powers and the structure of the free Lie ring”. *Amer. J. Math.* **64** (1942), pp. 371–388. DOI.