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Towards plethystic $\mathfrak{sl}_2$ crystals

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Abstract. To find crystals of $\mathfrak{sl}_2$ representations of the form $\Lambda^n \mathrm{Sym}^r \mathbb{C}^2$ it suffices to solve the combinatorial problem of decomposing Young's lattice into rank-symmetric chains. We review the literature on this latter problem, and present a strategy to solve it. For $n \leq 4$, the strategy recovers recently discovered solutions. We obtain (i) counting formulas for plethystic coefficients, (ii) new recursive formulas for plethysms of Schur functions, and (iii) formulas for the number of constituents of $\Lambda^n \mathrm{Sym}^r \mathbb{C}^2$.

Keywords: Plethysm, crystals, symmetric chain decompositions, Young's lattice

1 Introduction

Consider the Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$ with its natural action on $\mathbb{C}^2$. Two classic facts are (i) the finite dimensional irreducible representations of $\mathfrak{sl}_2$ are given by $\mathrm{Sym}^r\, \mathbb{C}^2$ for $r \in \mathbb{Z}_{\geq 0}$, and (ii) if V is a representation of $\mathfrak{sl}_2$ then so is $\Lambda^n\, V$ for all $n \in \mathbb{Z}_{\geq 0}$.

Problem A. *Decompose $\Lambda^n\, \mathrm{Sym}^r\, \mathbb{C}^2$ into irreducible representations of $\mathfrak{sl}_2$.*

That is, the problem asks to find the multiplicities $a^k_{1^n[r]}$ fitting into

$$\Lambda^n\, \mathrm{Sym}^r\, \mathbb{C}^2 = \bigoplus\nolimits_k (\mathrm{Sym}^k\, \mathbb{C}^2)^{\oplus a^k_{1^n[r]}}. \tag{1.1}$$

As is often the case in algebraic combinatorics, we ask for a solution that is *explicit and positive*, expressing $a^k_{1^n[r]}$ as the cardinality of a set given by quasipolynomial equations and inequalities —this solves Problem A in the sense of [13, 18].

This is one of the easiest cases of the problem of *plethysm*, and notoriously difficult to tackle [4, 18]. A solution to the deceptively similar problem of decomposing $\mathrm{Sym}^a\, \mathbb{C}^2 \otimes \mathrm{Sym}^b\, \mathbb{C}^2$ into irreducible representations of $\mathfrak{sl}_2$ goes back to Clebsch and Gordan in the XIX century [7]. The tensor product problem is nowadays best understood through crystal theory, and Kashiwara's tensor product rule [10].

For our purposes, an $\mathfrak{sl}_2$ *crystal* is a vertex-weighted directed graph attached to a representation of $\mathfrak{sl}_2$ (Figure 1a and Definition 2.1). If there is an arc $x \longrightarrow y$, then the weight of y satisfies $\mathrm{wt}(y) = \mathrm{wt}(x) - 1$. Decomposing a representation into irreducibles translates to decomposing its crystal into connected components. For $\mathfrak{sl}_2$, each irreducible representation has a crystal which is a *path graph* and *weight-symmetric*.

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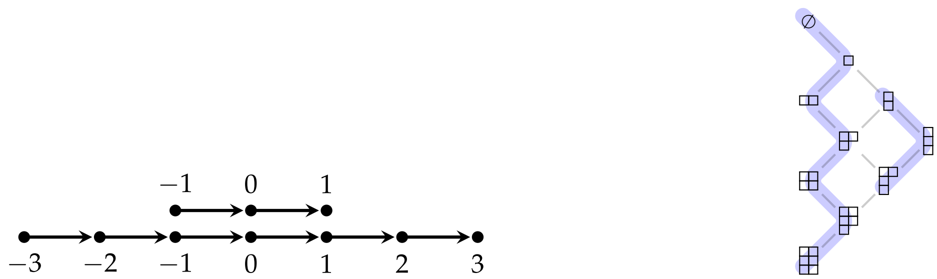


(a) Crystal of $\Lambda^2 \operatorname{Sym}^4 \mathbb{C}^2 \cong \operatorname{Sym}^2 \mathbb{C}^2 \oplus \operatorname{Sym}^6 \mathbb{C}^2$. (b) A decomposition of $L(2,3)$.

Figure 1: A solution to Problem B gives a solution to Problem A.

The set of vertices of the crystal of $\Lambda^n \operatorname{Sym}^r \mathbb{C}^2$ is in bijection with the set $L(n,m)$ of partitions whose Young diagram fits into an $n \times m$ rectangle, where $r+1 = n+m$. Endow $L(n,m)$ with a partial order, with the covering relation $\lambda \lessdot \mu$ if the Young diagram of λ is obtained from that of μ by removing one box: this is Young's lattice (Figure 1b).

With these ingredients, a natural way of tackling Problem A is to solve the following:

Problem B. *Decompose the poset $L(n,m)$ into rank-symmetric, disjoint (saturated) chains.*

See Figure 1. We reiterate that for the purposes of solving Problem A, we ask for an explicit solution to Problem B, in which the set of *highest weight elements* (see §2) of the decomposition is fully described by quasipolynomial equations and inequalities [12].

We begin by reviewing the literature for Problem B. The need for a review is apparent, since rediscoveries in the area are frequent. Our main contribution is to develop a crystal theoretic framework to tackle Problem B. For $0 \le n \le 4$, the framework produces explicit expressions for the crystal operators. These are reminiscent of Kashiwara's tensor product rule. We fully describe the highest weight elements, solving Problem A for $n \le 4$: this is state-of-the-art [4].

Our constructions for Problem B have properties in common with existing constructions in the literature (see the end of Section 1.1) —this suggests that we may be near to a canonical solution that will solve the cases for higher n. We retrieve counting formulas for the coefficients $a^k_{1^n[r]}$ involved in (1.1). After Theorem 5.2, these recover similar formulas found in [12] through a bijection. We obtain new recursive formulas for the plethysm of Schur functions $s_{(1^n)} \circ s_{(r)}(q^{\frac{1}{2}}, q^{-\frac{1}{2}})$ for $n = 3$ and 4. This plethysm is the q-binomial $\left[{r+1 \atop n}\right]$, which we prefer for a cleaner notation. For example, Figure 1 is lifting $\left[{5 \atop 2}\right] = [3] + [7]$, where $[k]$ is a q-integer. Our formulas express $\left[{r+1 \atop n}\right]$ in terms of $\left[{r \atop n}\right]$ and q-binomials of the form $\left[{* \atop n-2}\right]$. To state them, we introduce an operator on characters:

Definition 1.1 (Plus operator)**.** Given $f = \sum_i d_i \cdot [i]$ we define $f_{+j} = \sum_i d_i \cdot [i+j]$.

Theorem 1.2. *The character of $\Lambda^3 \operatorname{Sym}^r \mathbb{C}^2$ satisfies the following recursion, the sum ranging over $k \geq 0$ such that $4k < r - 1 - 2\delta_{r\ odd}$,*

$$\begin{bmatrix} r+1 \\ 3 \end{bmatrix} = \begin{bmatrix} r \\ 3 \end{bmatrix}_{+3} + \sum [r - 4k - 1].$$

Theorem 1.3. *The character of $\Lambda^4 \operatorname{Sym}^r \mathbb{C}^2$ satisfies the following recursion:*

$$\begin{bmatrix} r+1 \\ 4 \end{bmatrix} = \begin{bmatrix} r \\ 4 \end{bmatrix}_{+4} + \sum_{k \geq 0} \begin{bmatrix} r - 6k - 1 - 3\delta_{r\ even} \\ 2 \end{bmatrix} + \sum_{k \geq 0} \begin{bmatrix} r - 6k - 4 - 3\delta_{r\ odd} \\ 2 \end{bmatrix}_{+6}.$$

These formulas are non-trivial even as counting formulas for $q = 1$, and we do not know any other way of deriving them besides using the crystals constructed below. The formulas are in particular different than the ones found in [12, 19].

The number of constituents of $\Lambda^n \operatorname{Sym}^r \mathbb{C}^2$ is $\#\{\lambda \in L(n, m) : \lambda \vdash \lfloor mn/2 \rfloor\}$. In Corollary 5.1 we get that this number is $\lfloor (r+1)/2 \rfloor$ for $n = 2$ and $\lfloor (r+1)^2/8 \rfloor$ for $n = 3$. It is roughly $\lfloor (r+1)^3/36 \rfloor$ for $n = 4$. These formulas appear first in [1, page 69]; we obtain two new proofs for each formula: a combinatorial proof by counting tableaux and an algebraic proof as a corollary of the character formulas.

1.1 A literature review

Stanley [16] conjectures in 1980 that a solution to Problem B exists, observing that there is a solution for $n \leq 2$. Unbeknownst to him at the time, solutions for all $n \leq 4$ had been found by Rieß [14] two years earlier. This marks a precedent in the area that soon becomes a tradition. Solutions for $n = 3$ by Lindström [11] and $n = 4$ by West [22] appeared shortly after, only acknowledging [14] after the reviewing process. The constructions of Rieß do not coincide with those that came later; this too will be tradition.

In 1990, Greene [8] attributes to folklore that a greedy algorithm suffices to solve the problem, but shows that the approach is only successful for $n \leq 4$. Greene's paper is significant, as it is the first attempt to solve the problem *with a single construction*, in which n is nothing more than a parameter. Again, the constructions are new and distinct.

In 2004, Wen [20] finds new computer-generated solutions to the problem for $n = 3$ and 4 based on a modified greedy algorithm. In 2012, Dhand [6] creates a framework in the language of tropical geometry that produces solutions for $0 \leq n \leq 4$.

In 2017, David, Spink, and Tiba [5] develop a geometric framework fitting the solutions of [11, 20, 22]. Embed $L(n, m)$ into $\mathbb{R}^n$ by treating partitions as vectors. Then, dilate the embedding by $1/m$. For all m, the resulting set lies inside a fixed simplex Δ of $\mathbb{R}^n$, which is then divided into *regions*, each carrying a *direction*. A simultaneous solution for all $\{L(n, m) : m \geq 0\}$ is obtained from these regions, up to compatibility assumptions. In 2021, Xin and Zhong [23] study the applicability of a greedy algorithm to the problem, rediscovering the work of Greene.

In 2024 we see an explosion in interest for the problem. Most importantly, Wen [21] manages to find computer-generated solutions to the $n=5$ case: the first real progress after the conjecture was posed. However, the solution presented does not solve Problem A. Coggins, Donley, Gondal, and Krishna [3] reframe Lindström's construction in a diagrammatic way.

Orellana, Saliola, Schilling, and Zabrocki [12] solve the problem for $n=3$ and 4. They attribute the abundance of different solutions to Problem B being too unrestricted, and impose further *desirable properties* to the decompositions. They deduce counting formulas for $a^k_{n[m]} = a^k_{1^n[r]}$ and recursive formulas for $\left[\begin{smallmatrix} r+1 \\ n \end{smallmatrix}\right]$ for $n = 3$ and 4.

Our work [9] maintains the tradition of rediscovery in the area. The bulk of the research was done independently of the authors above, before some the mentioned works were released. Our framework has characteristics in common with several of the above:

1. The high-level strategy is independent of n, as in [6, 8, 23].

2. The framework is geometric in the sense of [5].

3. The constructions obtained are explicit in the sense of [12] and satisfy all of the additional desirable properties —some of which we do not impose a priori—, which allow us to show that the constructions are equivalent (Theorem 5.2). This is the first instance that we know of two essentially different methods arriving at equivalent constructions.

The constructions presented here have only been shown to solve Problem B for $n \leq 4$. The graphs have been implemented in SageMath [15] and the counting results (including the character formulas) have been checked for $r \leq 100$. The full article is available at [9].

2 Background

We assume familiarity with basic combinatorial objects of representation theory [17, Section 7]. The canonical basis of $\Lambda^n \operatorname{Sym}^r \mathbb{C}^2$ is labeled by the set $\mathrm{SSYT}_2(1^n[r])$ of semistandard Young tableaux of shape (1^n) with entries in $\mathrm{SSYT}_2(r)$. These are called *plethystic* $\mathfrak{sl}_2$ *tableaux*. We identify tableaux in $\mathrm{SSYT}_2(r)$ with bold integers $\mathbf{0}, \mathbf{1}, ..., \mathbf{r}$ by sending the tableau $\begin{array}{|c|c|c|c|c|c|}\hline 1&\cdots&1&2&\cdots&2\\\hline\end{array}$ with a twos to the number $\mathbf{a}$. Hence $\mathrm{SSYT}_2(1^n[r])$ is identified with the set $\mathrm{SSYT}_{[\mathbf{0},\mathbf{r}]}(1^n)$ of tableaux whose entries we write in bold. To save space, we write $\begin{array}{|c|c|c|}\hline \mathbf{c}&\mathbf{b}&\mathbf{a}\\\hline\end{array}'$ for a column tableau: for $n=2$ and $r=5$,

$$\begin{array}{|c|c|}\hline \mathbf{1}&\mathbf{3}\\\hline\end{array}' \quad \text{stands for} \quad \begin{array}{|c|}\hline \mathbf{1}\\\hline \mathbf{3}\\\hline\end{array} \quad \text{which stands for} \quad \begin{array}{|c|}\hline \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&2\\\hline\end{array}\\\hline \begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&2\\\hline\end{array}\\\hline\end{array}.$$

Sending a plethystic tableau $\begin{array}{|c|c|c|}\hline \mathbf{a_n}&\cdots&\mathbf{a_1}\\\hline\end{array}'$ to the partition $(n^{a_n} \dots i^{a_i - a_{i+1} - 1} \dots 1^{a_1 - a_2 - 1})$ is a bijection between $\mathrm{SSYT}_2(1^n[r])$ and $L(n, r+1-n) = L(n, m) = \{\lambda \,:\, \lambda_1 \leq n,\ \ell(\lambda) \leq m\}$.

We call this map Ψ. It is better understood via an example: let $n=3$ and $r=6$, then

$$\begin{array}{|c|}\hline \mathbf{0} \\ \hline \mathbf{3} \\ \hline \mathbf{5} \\ \hline \end{array} = \begin{array}{|c|}\hline \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&1\\ \hline\end{array} \\ \hline \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&2&2&2\\ \hline\end{array} \\ \hline \begin{array}{|c|c|c|c|c|c|}\hline 1&2&2&2&2&2\\ \hline\end{array} \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\ \hline 1&1&2&2\\ \hline 1&2&2&2\\ \hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline 2&2&1\\ \hline 2&2&1\\ \hline 2&1&1\\ \hline 1&1&1\\ \hline\end{array} \mapsto \begin{array}{|c|c|}\hline &\\ \hline &\\ \hline & \\ \cline{1-1}\end{array}. \tag{2.1}$$

Note $\Psi\, \boxed{\mathbf{0}\,|\,\mathbf{1}\,|\cdots|\,\mathbf{n-1}}' = \emptyset$. We introduce crystals for $\mathfrak{sl}_2$ only; otherwise we follow [10, 2].

Definition 2.1. A (seminormal $\mathfrak{sl}_2$) crystal is a set $\mathcal{B}$ together with maps $F : \mathcal{B} \to \mathcal{B} \cup \{0\}$ and $\mathrm{wt} : \mathcal{B} \to \mathbb{Z}$ such that:

(C0) F restricted to $F^{-1}(\mathcal{B})$ is injective,

(C1) $\varphi(b) = \varepsilon(b) + 2\,\mathrm{wt}(b)$ for all $b \in \mathcal{B}$, and

(C2) $\mathrm{wt}(F.\,b) = \mathrm{wt}(b) - 1$,

where $\varphi(b) = \max\{k \;:\; F^k.\,b \neq 0\}$, $\varepsilon(b) = \max\{k \;:\; E^k.\,b \neq 0\}$, and where $E : \mathcal{B} \to \mathcal{B} \cup \{0\}$ is the inverse of F whenever it exists, or otherwise 0. Here, F^k means $F \circ \underset{k}{\cdots} \circ F$.

We identify a crystal with the directed graph with vertex set $\mathcal{B}$ and an arc $x \longrightarrow y$ whenever $F.\,x = y$. Although not immediately clear, the weight wt can be recovered from the graph [2, Lemma 2.14]. By (C0) the graph is a disjoint union of paths, by (C1) the graph is weight-symmetric.

Let α be the simple root of $\mathfrak{sl}_2$, so that $\mathbb{Z}\frac{\alpha}{2}$ is its weight lattice. A *crystal of a representation V* is a crystal on the set of weights of V (with multiplicity) such that $\mathrm{wt}(k\frac{\alpha}{2}) = \frac{k}{2}$. An element of a crystal is *highest weight*, $b \in \mathrm{HW}$, if $E.b = 0$. The character of the representation is retrieved via $\sum_{b\in\mathcal{B}} q^{\mathrm{wt}(b)}$ [2, Section 2.6]. The character of $\mathrm{Sym}^r \mathbb{C}^2$ is $[r+1] = (q^{r+1} - q^{-(r+1)})/(q - q^{-1})$, and thus the character of any finite dimensional representation is $\sum_{b\in\mathrm{HW}}[2\,\mathrm{wt}(b)+1]$. The character of $\Lambda^n \mathrm{Sym}^r \mathbb{C}^2$ is

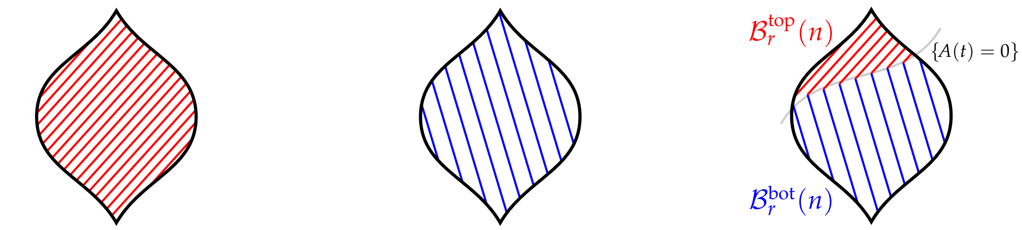
$$\begin{bmatrix} r+1 \\ n \end{bmatrix} = \frac{[r+1][r]\cdots[r-n+2]}{[n][n-1]\cdots[1]}.$$

3 Crystals of $\Lambda^n \mathrm{Sym}^r \mathbb{C}^2$

Set $\mathcal{B}_r(n) = \mathrm{SSYT}_2(1^n[r])$ and let $\mathcal{B}(n) = \bigcup_{r\geq n} \mathcal{B}_r(n)$. Define weight functions on each $\mathcal{B}_r(n)$ by $\mathrm{wt}_r\, \boxed{\mathbf{a_n}\,|\cdots|\,\mathbf{a_1}}' = nr - 2(a_1 + \cdots + a_n)$. We outline a program to find a map $F : \mathcal{B}(n) \to \mathcal{B}(n)$ for all n, such that restricting to each $\mathcal{B}_r(n)$ produces a crystal.

Conceptually, we look for two operators F^{top} and F^{bot}, each of which satisfies axioms (C0) and (C2) above when restricted to $\mathcal{B}_r(n)$, but not (C1). Assuming some compatibility properties (Problems 1, 2, 3 below), the two operators can be glued together into an operator F satisfying (C1) when restricted to each $\mathcal{B}_r(n)$. See Figure 2.

The operator F^{top} will be defined inductively using the operator F on $\mathcal{B}(n-2)$. The base cases of this induction are $F.\emptyset = 0$ on $\mathcal{B}(0)$ and $F.\boxed{\mathbf{a}} = \boxed{\mathbf{a+1}}$ on $\mathcal{B}(1)$.



(a) F^{top} in $\mathcal{B}_r(n)$. (b) F^{bot} in $\mathcal{B}_r(n)$. (c) F in $\mathcal{B}_r(n)$.

Figure 2: F is a function by parts giving a solution to Problem B.

3.1 Top operator

Let $t = \begin{array}{|c|c|c|}\hline \mathbf{a_n} & \cdots & \mathbf{a_1} \\ \hline \end{array}' \in \mathcal{B}_r(n)$. By removing the first and last entry, we get a tableau $t^{\downarrow}$ in $\mathcal{B}(n-2)$. Assuming by induction that we have a crystal structure on $\mathcal{B}_k(n-2)$ for all k, we can consider $F\,.\,t^{\downarrow}$. Adding back the removed entries, we obtain a new tableau in $\mathcal{B}_r(n)$, which we define to be $F^{\mathrm{top}}.\,t$. More precisely,

$$F^{\mathrm{top}}.\begin{array}{|c|}\hline \mathbf{a_n} \\ \hline \mathbf{a_{n-1}} \\ \hline \cdots \\ \hline \mathbf{a_2} \\ \hline \mathbf{a_1} \\ \hline \end{array} = \begin{array}{|c|}\hline \mathbf{a_n} \\ \hline \mathbf{b_{n-2}+(a_n+1)} \\ \hline \cdots \\ \hline \mathbf{b_1+(a_n+1)} \\ \hline \mathbf{a_1} \\ \hline \end{array}, \quad \text{where} \quad \begin{array}{|c|}\hline \mathbf{b_{n-2}} \\ \hline \cdots \\ \hline \mathbf{b_1} \\ \hline \end{array} = F.\begin{array}{|c|}\hline \mathbf{a_{n-1}-(a_n+1)} \\ \hline \cdots \\ \hline \mathbf{a_2-(a_n+1)} \\ \hline \end{array}.$$

If $b_1 + (a_n+1) = a_1$ then the resulting tableau is not strictly increasing: in this case we let $F^{\mathrm{top}}.\,t = 0$. This operator clearly satisfies (C0) and (C2) by induction on n.

3.2 Bottom operator

Embedding $\mathcal{B}(n)$ into $\mathbb{R}^n$ via $\begin{array}{|c|c|c|}\hline \mathbf{a_n} & \cdots & \mathbf{a_1} \\ \hline \end{array}' \mapsto (a_n, ..., a_1)'$ allows us to talk about directions. We look for a decomposition of $\mathcal{B}(n)$ into paths "parallel" to the vector $(1, ..., 1)'$. Each path is of the form $P(t_0, v_{t_0}) = \{t_k\}_{k\geq 0}$, where

$$t_k = t_0 + \mathrm{floor}\left(\frac{1}{n}v_{t_0} + \frac{k}{n}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}\right)$$

for some *initial tableau* t_0 and some *offset vector* $v_{t_0} \in \mathbb{S}_n.(0, 1, ..., n-1)'$. The floor is taken entry-wise. We think of $P(t_0, v_{t_0})$ as a discretisation of the line $t_0 + \langle(1, ..., 1)'\rangle$. We look for a set of pairs (t_0, v_{t_0}) that we call a *seed*.

Problem 1. *Find a seed S such that $\{P(t_0, v_{t_0}) : (t_0, v_{t_0}) \in S\}$ is a set partition of $\mathcal{B}(n)$.*

Assuming a solution to Problem 1, we can define F^{bot} via $F^{\mathrm{bot}}.\,t_k = t_{k+1}$ for each element t_k of a path $P(t_0, v_{t_0})$. Hence F^{bot} is defined on $\mathcal{B}(n)$. Automatically, for each r, the operator F^{bot} restricted to $\mathcal{B}_r(n)$ satisfies (C0) and (C2).

3.3 Gluing up

Define ε^{top} and φ_r^{bot} from the operators F^{top} and F^{bot} as in Definition 2.1. Recall that axiom (C1) says $\varphi - (\varepsilon + 2\,\text{wt}) = 0$ and is not in general satisfied by F^{bot} nor F^{top}. Let $A = \varphi_r^{\text{bot}} - \left(\varepsilon^{\text{top}} + 2\,\text{wt}_r\right)$. Although not immediate, A can be shown to not depend on r. As in Figure 2, we consider

$$\mathcal{B}^{\text{top}}(n) = \{t \in \mathcal{B}(n) \;:\; A(t) < 0\} \quad \text{and} \quad \mathcal{B}^{\text{bot}}(n) = \{t \in \mathcal{B}(n) \;:\; A(t) \geq 0\}.$$

Definition 3.1. Fix $n \in \mathbb{Z}_{\geq 0}$. Define an operator $\mathcal{B}(n) \to \mathcal{B}(n)$ by

$$F.\,t = \begin{cases} F^{\text{top}}.\,t & \text{if } t \in \mathcal{B}^{\text{top}}(n), \\ F^{\text{bot}}.\,t & \text{if } t \in \mathcal{B}^{\text{bot}}(n). \end{cases}$$

Define E, φ_r, and ε from F as in Definition 2.1. Restricted to each $\mathcal{B}_r(n)$, one can show that $F\,.\,t = 0$ implies $A(t) \geq 0$. We can similarly show that $E\,.\,t = 0$ implies $A(t) \leq 1$, but we need something stronger.

Problem 2. *Find a seed such that if $E\,.\,t = 0$ then $A(t) \leq 0$.*

Problem 3. *Find a seed such that if $A(t) < 0 \leq A(F.\,t)$ then $A(F.\,t) = 0 < A(F^k.\,t)$ for all $k \geq 2$.*

Theorem 3.2. *If a seed is a solution to Problems 1, 2, 3 then F defines a crystal on each $\mathcal{B}_r(n)$.*

Proof. We check axioms (C0), (C1), (C2) of Definition 2.1 are satisfied by the restriction of F to $\mathcal{B}_r(n)$. Axiom (C2) is clear by construction and axiom (C0) follows from the seed being a solution to Problem 3. We thus have a disjoint union of paths.

It suffices to check (C1) for one tableau on each path. Let t be such that $E\,.\,t = 0$, apply F repeatedly to obtain the sequence $\{F^k.\,t\}_{k\geq 0}$. By Problem 2, either $A(t) = 0$ or $t \in \mathcal{B}_r^{\text{top}}(n)$. Since $F^N.\,t = 0$ implies $A(F^N.\,t) \geq 0$, the path eventually enters $\mathcal{B}_r^{\text{bot}}(n)$, say at step k. The tableau $F^k.\,t$ satisfies

$$\varepsilon(F^k.\,t) = \varepsilon^{\text{top}}(F^k.\,t), \quad \text{and} \quad \varphi_r(F^k.\,t) = \varphi_r^{\text{bot}}(F^k.\,t).$$

By Problem 3 we deduce $A(F^k.\,t) = 0$, which for $F^k.\,t$ is precisely axiom (C1). $\square$

The theorem reduces Problems A and B to finding a suitable seed.

4 Explicit constructions for small n

To obtain a solution to Problem A for $\Lambda^n \operatorname{Sym}^r \mathbb{C}^2$, it suffices to find a set of "initial tableaux" and a seed $S(n) = \{(t, v_t) \;:\; t \text{ is an initial tableau}\}$. If the seed is a solution to Problems 1, 2, 3 then §3 gives a description of the crystal operators and we can compute the set $\text{HW}(n)$ of highest weight tableaux. A solution to Problem A is then

$$a^k_{1^n[r]} = \#\left\{ \boxed{\mathbf{a_n}}\boxed{\cdots}\boxed{\mathbf{a_1}}{}' \in \text{HW}(n) \;:\; \sum_i i \cdot a_i = k,\; a_1 \leq r \right\}.$$

4.1 Crystals of $\Lambda^2 \operatorname{Sym}^r \mathbb{C}^2$

Let $S(2) = \{ (\boxed{0}\boxed{a}', (0,1)') \ : \ a \text{ is odd} \}$ be a seed. We will hereafter write $a \equiv 1 \ (2)$, using notation from modular arithmetic.

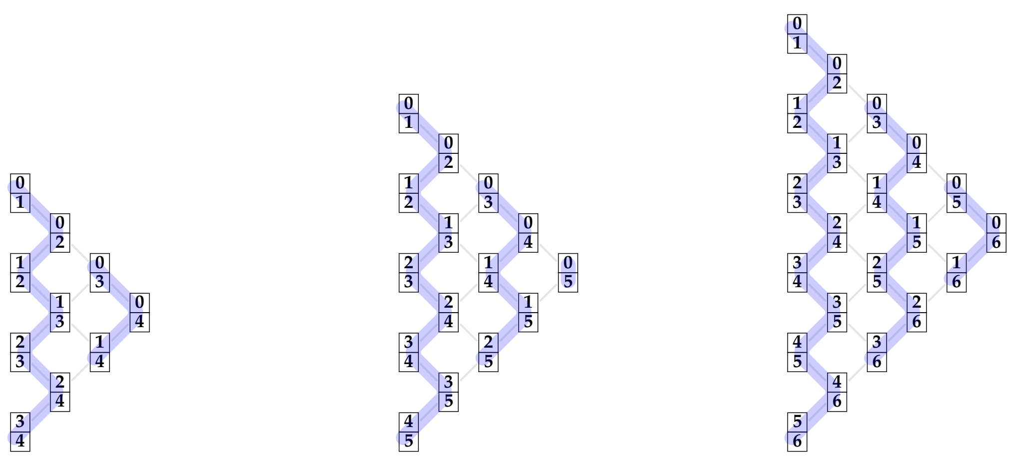
Proposition 4.1. *The seed $S(2)$ is a solution to Problems 1, 2, 3.*

Explicitly, $\mathcal{B}^{\text{top}}(2) = \varnothing$ and we obtain an operator $F = F^{\text{bot}} : \mathcal{B}(2) \to \mathcal{B}(2)$ by

$$F.\begin{array}{|c|}\hline b \\ \hline a \\ \hline\end{array} = \begin{cases} \boxed{b+1}\boxed{a}' & \text{if } a \equiv b \, (2), \\ \boxed{b}\boxed{a+1}' & \text{if } a \not\equiv b \, (2). \end{cases}$$

We illustrate some examples in Figure 3; note that the paths follow the $(1,1)'$ direction (South). The highest weight tableaux are $\mathrm{HW}(2) = \{ t \ : \ E\,.\,t = 0 \} = \{ \boxed{b}\boxed{a}' \ : \ {\scriptstyle b=0, \atop a \equiv 1(2)} \}$.

Note that $\mathcal{B}_{r-1}(2) \subseteq \mathcal{B}_r(2)$, and that the crystal of $\mathcal{B}_r(2)$ is obtained by extending each connected component of that of $\mathcal{B}_{r-1}(2)$, and then adding one extra component if r is odd. We have shown that the character of $\Lambda^2 \operatorname{Sym}^r \mathbb{C}^2$ satisfies $\left[{r+1 \atop 2}\right] = \left[{r \atop 2}\right]_{+2} + \delta_{r \equiv 1(2)} \cdot [1]$. Although easy, the $n = 2$ case perfectly illustrates the nature of our constructions.



(a) Crystal on $\mathcal{B}_4(2)$. (b) Crystal on $\mathcal{B}_5(2)$. (c) Crystal on $\mathcal{B}_6(2)$.

Figure 3: Crystal structures on $\mathcal{B}_r(2) = \mathrm{SSYT}_2(1^2[r])$ as embedded in $\mathbb{R}^2$.

4.2 Crystals of $\Lambda^3 \operatorname{Sym}^r \mathbb{C}^2$

Let $\{ \boxed{c}\boxed{b}\boxed{a}' \ : \ c = 0, \ b \text{ odd}, \ a - b \not\equiv 2 \ (3) \}$ be the set of initial tableaux. For an initial tableau $t = \boxed{c}\boxed{b}\boxed{a}'$ such that $a - b \equiv 1 \ (3)$, let $v_t = (0,1,2)'$; otherwise let $v_t = (0,2,1)'$.

Proposition 4.2. *The seed $S(3) = \{ (t, v_t) \ : \ t \text{ is initial} \}$ is a solution to Problems 1, 2, 3.*

Explicitly, one computes $\mathcal{B}^{\mathrm{top}}(3) = \{\boxed{c}\boxed{b}\boxed{a}' \ : \ {\scriptstyle a \geq 4c+2, \atop a \neq 4c+3}\}$ and

$$F^{\mathrm{bot}}.\begin{array}{|c|}\hline c\\\hline b\\\hline a\\\hline\end{array} = \begin{cases} \boxed{c+1}\boxed{b}\boxed{a}' & \text{if } b \equiv c \ (2), \text{ and } a-b \not\equiv 2 \ (3), \\ \boxed{c}\boxed{b+1}\boxed{a}' & \text{if } b \not\equiv c \ (2), \text{ and } a-b \not\equiv 1 \ (3), \\ \boxed{c}\boxed{b}\boxed{a+1}' & \text{otherwise,} \end{cases}$$

from which F can be obtained as in Definition 3.1.

Example 4.3. We follow an example, with the help of Figure 4. The crystal of $\Lambda^3 \operatorname{Sym}^6 \mathbb{C}^2$ has five connected components, each of which is a path. One such path is

$$\boxed{0}\boxed{1}\boxed{4}', \quad \boxed{0}\boxed{2}\boxed{4}', \quad \boxed{0}\boxed{3}\boxed{4}', \quad \boxed{0}\boxed{3}\boxed{5}', \quad \boxed{0}\boxed{4}\boxed{5}', \quad \boxed{1}\boxed{4}\boxed{5}', \quad \boxed{1}\boxed{4}\boxed{6}', \quad \boxed{1}\boxed{5}\boxed{6}', \quad \boxed{2}\boxed{5}\boxed{6}'.$$

The first two steps are governed by F^{top}. These are in the $(0,1,0)'$ direction and are illustrated in red. The remaining steps are governed by F^{bot} and follow the $(1,1,1)'$ direction. The direction changes because $A(\,\boxed{0}\boxed{3}\boxed{4}'\,) = 0$.

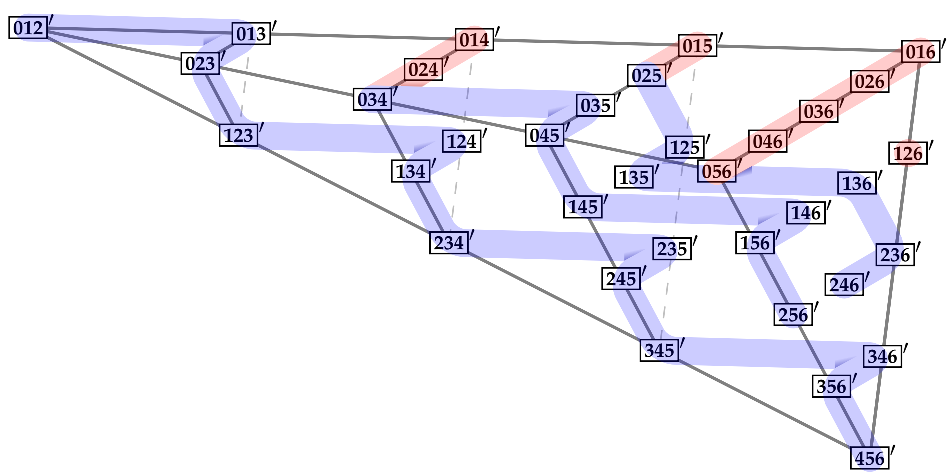


Figure 4: A crystal structure on $\mathcal{B}_6(3) = \mathrm{SSYT}_2(1^3[6])$ as embedded in $\mathbb{R}^3$.

Direct computation gives

$$\mathrm{HW}(3) = \left\{ \boxed{c}\boxed{b}\boxed{a}' \ : \ b=c+1, \ {\scriptstyle a \geq 4c+2, \atop a \neq 4c+3} \right\}.$$

The crystal of $\mathcal{B}_r(3)$ is obtained from that of $\mathcal{B}_{r-1}(3)$ by extending each connected component, and then adding some other components that are governed by F^{top} (which is given by the operator F on $\mathcal{B}(1)$). Following this argument shows Theorem 1.2.

4.3 Crystals of $\Lambda^4 \operatorname{Sym}^r \mathbb{C}^2$

Set $\{\boxed{\mathbf{d}|\mathbf{c}|\mathbf{b}|\mathbf{a}}' \ : \ d=0,\ a \text{ and } c \text{ odd}\}$ to be the set of initial tableaux. For an initial tableau $t = \boxed{\mathbf{d}|\mathbf{c}|\mathbf{b}|\mathbf{a}}'$ such that b is even, let $v_t = (0,1,2,3)'$; otherwise let $v_t = (0,3,2,1)'$.

Proposition 4.4. *The seed $S(4) = \{(t, v_t) \ : \ t \text{ is initial}\}$ is a solution to Problems 1, 2, 3.*

Explicitly, we obtain F as in Definition 3.1 with $\mathcal{B}^{\text{top}}(n) = \left\{\boxed{\mathbf{d}|\mathbf{c}|\mathbf{b}|\mathbf{a}}' \ : \ {\scriptstyle a \geq b+2d+1, \atop a \neq b+2d+2}\right\}$ and

$$F^{\text{bot}} \cdot \begin{array}{|c|}\hline \mathbf{d} \\ \hline \mathbf{c} \\ \hline \mathbf{b} \\ \hline \mathbf{a} \\ \hline \end{array} = \begin{cases} \boxed{\mathbf{d+1}|\mathbf{c}|\mathbf{b}|\mathbf{a}}' & \text{if } a \equiv c \equiv d\ (2), \\ \boxed{\mathbf{d}|\mathbf{c+1}|\mathbf{b}|\mathbf{a}}' & \text{if } b \equiv c \not\equiv d\ (2), \\ \boxed{\mathbf{d}|\mathbf{c}|\mathbf{b+1}|\mathbf{a}}' & \text{if } a \equiv b \not\equiv c\ (2), \\ \boxed{\mathbf{d}|\mathbf{c}|\mathbf{b}|\mathbf{a+1}}' & \text{if } a \not\equiv b \equiv d\ (2). \end{cases}$$

We deduce as before

$$\mathrm{HW}(4) = \left\{\boxed{\mathbf{d}|\mathbf{c}|\mathbf{b}|\mathbf{a}}' \ : \ {\scriptstyle c=d+1, \quad a \geq b+2d+1, \atop b \not\equiv c\ (2), \quad a \neq b+2d+2}\right\}.$$

Some components of $\mathcal{B}_r(4)$ are obtained by extending those of $\mathcal{B}_{r-1}(4)$; the rest are governed by F^{top}, which is given by the crystal operator on $\mathcal{B}(2)$. On the level of characters, we get Theorem 1.3.

One can only visualize our constructions for $\mathcal{B}(4)$ through its two- or three-dimensional slices. This requires some set up, so we leave it out of this abstract.

5 Other works and conclusion

The number of constituents of $\Lambda^n \operatorname{Sym}^r \mathbb{C}^2$ is the number of tableaux in $\mathrm{SSYT}_2(1^n[r])$ of weight 0 if $n+r$ is odd, or -1 if $n+r$ is even. Through the bijection Ψ of (2.1) this number is equal to $\#\{\lambda \in L(n,m) \ : \ \lambda \vdash \lfloor nm/2 \rfloor\}$, where $r+1 = n+m$.

Corollary 5.1. *The cardinality of $\{\lambda \in L(n,m) \ : \ \lambda \vdash \lfloor nm/2 \rfloor\}$ is*

1. *$\lfloor (r+1)/2 \rfloor$ for $n=2$, and*
2. *$\lfloor (r+1)^2/8 \rfloor$ for $n=3$.*

Proof. The number of components in our crystals is the number of the partitions in the center rank of $L(n,m)$. But we can similarly count the number of highest weight tableaux $\mathrm{HW}(n)$ intersecting $\mathcal{B}_r(n)$. Given the expression of $\mathrm{HW}(2)$ given in the previous section, the statement for $n=2$ is now clear; the set $\mathrm{HW}(3) \cap \mathcal{B}_r(3)$ is simply the set of integer points in a right triangle of $\mathbb{R}^3$ with base $r+1$ and height $(r+1)/4$. □

The set $\mathrm{HW}(4) \cap \mathcal{B}_r(4)$ is the integer points of a tetrahedron of volume $(r+1)^3/18$ *to which roughly half the points where removed* by imposing two coordinates to be of different parity. Thus the number of constituents is roughly $\lfloor (r+1)^3/36 \rfloor$ for $n=4$. The precise number is $\lfloor (2r^3 - 3r^2 + 6r + 27)/72 \rfloor$ [1, p. 69].

Transporting our operators F through the bijection Ψ of (2.1), we find symmetric chain decompositions of $L(n, r+1-n) = L(n,m)$ for $n \leq 4$.

Theorem 5.2. *For all $t \in \mathcal{B}(3)$, we have $\Psi(F.t) = f.\Psi(t)$, where f is given in [12, Theorem 5.8]. For all $t \in \mathcal{B}(4)$, we have $\Psi(F.t) = f.\Psi(t)$, where f is given in [12, Theorem 5.15].*

We skip the proof of this theorem, which is tedious but automatic. However, we remark once again that this is unexpected, since (i) both approaches do not coincide, and (ii) the "desirable properties" imposed in [12] are not all a priori required in our constructions. Moreover, we remind the reader that our decompositions for $n=3$ and 4 are examples of one unique construction, which is not the case in [12]. All these facts might be pointing to some uniqueness result, which we leave for future exploration.

On the other hand, we were not able to prove or disprove the applicability of our framework for $n=5$ and beyond. It seems like the number of choices needed to define seeds $S(n)$ grows quickly with n. Standardizing these choices might be the only thing keeping us away from a solution to Problems A and B in all generality.

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