Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #40, 12 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

Poset Permutahedra

Alexander E. Black *[1], Sophie Rehberg †[2], and Raman Sanyal[3]

[1] *Institute for Mathematical Research, ETH Zürich*
[2] *Department of Mathematics, Freie Universität Berlin*
[3] *Department of Mathematics, Goethe-Universität Frankfurt*

Abstract. Poset permutahedra are an amalgamation of order polytopes and permutahedra. We show that poset permutahedra give a unifying perspective on several recent classes of polytopes that occurred, for example, in connection with colorful subdivisions of polygons and Hessenberg varieties. As with order polytopes, the geometry and the combinatorics of poset permutahedra can be completely described in terms of the underlying poset. As applications of our results, we give a combinatorial description of the h-vectors of the partitioned permutahedra of Horiguchi et al. and poset generalizations of Landau's score sequences of tournaments. To prove our results, we show that poset permutahedra arise from order polytopes via the fiber polytope construction of Billera and Sturmfels.

Keywords: Order Polytopes, Fiber Polytopes, Monotone Path Polytopes, Score Sequences

1 Introduction

Order polytopes [24] provide a powerful link between polyhedral geometry and finite posets. Harnessing this connection resulted in many important results including the computation of order polynomials, the fundamental result that computing the volume of a polytope is ♯P-Hard [9], and that certain statistics on linear extensions are log-concave [23]. Since the foundational work of Stanley, other poset polytopes have been introduced including marked poset polytopes [2], double poset polytopes [10], and poset associahedra [14]. In this extended abstract we introduce another class of poset polytopes that provided a unified perspective of polytopes that have been studied recently.

Let $\mathcal{P} = ([n], \preceq)$ be a finite poset. The **order polytope** $\mathsf{O}(\mathcal{P})$ is the intersection of the $0/1$-hypercube $[0,1]^n$ with the **order cone** $\mathsf{C}_{\mathcal{P}} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}_a \prec \mathbf{x}_b \text{ for all } a \preceq b\}$. This is a polytope with vertices in $\{0,1\}^n$ with remarkable combinatorics that will be recalled later. For our construction, recall that the standard **permutahedron** $\Pi_n \subset \mathbb{R}^n$ is the

*alexander.black@math.ethz.ch. Alexander E. Black was partially supported by the NSF GRFP

†s.rehberg@fu-berlin.de. Sophie Rehberg was partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID: 390685689).

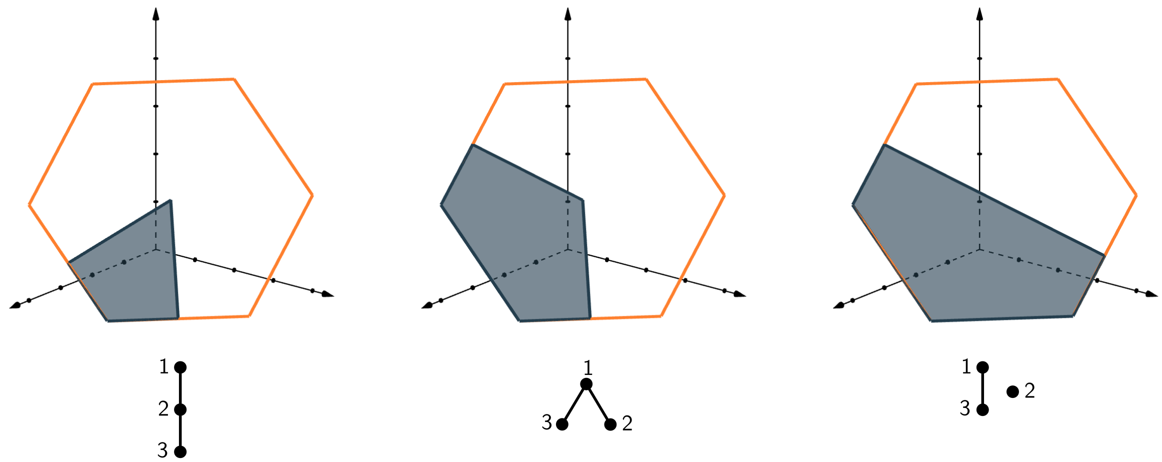

Figure 1: Pictured are each of the two dimensional poset permutahedra other than the permutahedron itself up to symmetry with their corresponding posets.

convex hull of all $n!$ permutations of $(1, 2, \ldots, n)$. We define the **poset permutahedron** of $\mathcal{P}$ by

$$\Pi_{\mathcal{P}} := \Pi_n \cap \mathsf{C}_{\mathcal{P}}. \tag{1.1}$$

This is an $(n-1)$-dimensional polytope with half-integral vertices. See Figure 1 for examples.

Poset permutahedra provide a unified construction principle for polytopes that have occurred in disparate areas. What follows is a non-exhaustive list:

- If $\mathcal{P}$ is the antichain, then $\Pi_{\mathcal{P}} = \Pi_n$.
- If $\mathcal{P}$ is a chain, then $2\Pi_{\mathcal{P}}$ is unimodularly equivalent to the Newton polytope of the discriminant [15, Section III.12.2]. The lattice points in $\Pi_{\mathcal{P}} - \mathbf{1}$ are the well-studied score sequences introduced by Landau [19]; see below for more.
- If $\mathcal{P}$ arises from the antichain by adjoining a maximial element, then $\Pi_{\mathcal{P}}$ is combinatorially equivalent to the stellahedron [22, Section 10.4].
- If $\mathcal{P}$ is the disjoint union of two chains of length m and n, respectively, then the face lattice of $\Pi_{\mathcal{P}}$ is ismorphic to the poset of colorful subdivisions of an $(m+n+2)$-gon with bicolored vertices (cf. [1]) and extends the combinatorial description of the Newton polytope of the classical resultant [15, Chapter 12].
- If $\mathcal{P}$ is the disjoint union of k chains of lengths $m_1, m_2, \ldots, m_k$, then $\Pi_{\mathcal{P}}$ is the type-A partitioned permutahedron introduced and studied in [17, 16] in the context of Hessenberg varieties and representation theory; see Section 5 for more.

Monotone Path Polytopes. Our key observation is that $\Pi_{\mathcal{P}}$ is a fiber polytope in the sense of Billera–Sturmfels [6]. For a polytope P and a non-constant linear function φ on P, the notion of **cellular strings** generalizes that of maximal φ-monotone paths in the graph of P oriented by φ. The collection of cellular strings ordered by inclusion (i.e., one cellular string is contained within another if the union of cells of one is a subset of the union of cells of the other) is the **Baues poset** of (P, φ) from algebraic topology; see [5]. While Baues posets can be rather wild, the subposet of **coherent** cellular strings is isomorphic to the face poset of the **monotone path polytope** $\Sigma_\varphi(\mathsf{P})$.

Theorem 1.1. *Let $\mathcal{P} = ([n], \preceq)$ be a poset. Then $\Pi_\mathcal{P} = \Sigma_\mathbf{1}(\mathsf{O}(\mathcal{P})) + \frac{1}{2}\mathbf{1}$ with respect to the linear function $\mathbf{1}(\mathbf{x}) = x_1 + \cdots + x_n$.*

It is typically nontrivial to determine if a cellular string is coherent. In our situation, however, it turns out that *all* cellular strings are coherent.

Theorem 1.2. *Let $\mathcal{P} = ([n], \preceq)$ and φ a linear function that is positive on $\mathbb{R}^n_{\geq 0}$. Then all cellular strings are coherent. In particular the face lattice of $\Pi_\mathcal{P}$ is isomorphic to the Baues poset of $(\mathsf{O}(\mathcal{P}), \mathbf{1})$.*

Theorem 1.2 allows us to prove the following results about poset permutahedra:

(1) Vertices of $\Pi_\mathcal{P}$ are in bijection to chains of filters $\emptyset = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k = \mathcal{P}$ such that the poset $\mathcal{F}_i \setminus \mathcal{F}_{i-1}$ is connected for all $i = 1, \ldots, k$ (Theorem 3.1).
(2) Theorem 3.2 gives a combinatorial characterization of edges and the corresponding edge directions.
(3) Facets of $\Pi_\mathcal{P}$ are in bijection to the set of proper filters $\mathfrak{F}(\mathcal{P})$ and the cover relations of $\mathcal{P}$ (Corollary 3.4).
(4) Corollary 3.5 yields the vertex-facet-incidences of $\Pi_\mathcal{P}$, and Theorem 3.6 shows that $\Pi_\mathcal{P}$ is simple if and only if the undirected Hasse diagram of $\mathcal{P}$ is a forest.
(5) Analogous to order polytopes, $\Pi_\mathcal{P}$ is subdivided by the set of all Π_L, where L ranges over the linear extensions $\mathfrak{L}(\mathcal{P})$ (Theorem 4.1). This allows us to compute the volume of $\Pi_\mathcal{P}$ as $|\mathfrak{L}(\mathcal{P})|\frac{n^{n-2}}{n!}$ (Corollary 4.2).
(6) For every poset $\mathcal{P}$, $2 \cdot \Pi_\mathcal{P}$ is a lattice polytope that has the **integer decomposition property**, that is, if $p \in m \cdot 2\Pi_\mathcal{P}$ is a lattice point, then there are lattice points $p_1, \ldots, p_m \in 2\Pi_\mathcal{P}$ with $p = p_1 + \cdots + p_m$; see Section 4.

Partitioned Permutahedra and h-vectors. In [17] a toric orbifold is associated to a Weyl group W and a choice of a parabolic subgroup W_K. In type A, they call the associated moment polytope a *partitioned permutahedron* $\Pi_n(K)$. It is shown partitioned permutahedra are simple and their h-vectors were determined by using the cohomology of regular Hessenberg varieties.

Theorem 1.3 ([17, Proposition 7.4])**.** *The h-polynomial of the partitioned permutahedron for $K \subseteq [n-1]$ is given by*

$$h_{\Pi_n(K)}(x) = \sum_{\sigma \in W(K)} x^{des(\sigma)},$$

where $W(K)$ is the set of permutations σ such that $\sigma^{-1}(i) - \sigma^{-1}(i+1) \leq 1$ for all $i \in K$.

In Section 5, we show that $\Pi_n(K)$ is the poset permutahedron of a disjoint union of chains, which implies simplicity by Theorem 3.6. We provide a direct bijective proof of Theorem 1.3.

$\mathcal{P}$-score sequences. Consider a tournament with teams $1, 2, \ldots, n$. Any two teams play against each other and during each match m points are distributed between the two teams. This gives rise to a score sequence $s = (s_1, s_2, \ldots, s_n)$. We call s a (strict) $(\mathcal{P}, m)$-score sequence if $s_i \leq s_j$ (respectively $s_i < s_j$) if $i \preceq j$. Thus strict $(\mathcal{P}, m)$-score sequences ensure a relative ranking of the team given by $\mathcal{P}$.

Theorem 1.4. *The $\mathcal{P}$-score sequences for m points are precisely the lattice points in $m \cdot (\Pi_{\mathcal{P}} - \mathbf{1})$.*

Corollary 3.4 gives a simple characterization of $(\mathcal{P}, m)$-score sequences. If $\mathcal{P}$ is a chain, then this characterization is classical and originally due to Landau [19]. While the question of the number of score sequence for $m = 1$ and varying number of teams n has received considerable attention (cf. [11] and entries A000571, A007747, A047729-A047731, and A047733-A047737 in OEIS [21]), we are not aware of results pertaining to the number of score sequences with fixed number of teams n and varying the number of points m.

All-coherent and Connectivity of flip graphs. We derive Theorem 1.2 from the more general Lemma 2.1, that gives a necessary condition when all cellular strings of $(\mathsf{P}, \mathbf{1})$ are coherent for 0/1-polytopes P. Little is known about polytopes for which all cellular strings are coherent for some choice of φ. Our results add to this list, which includes simplices and hypercubes [6], (poly)matroid independence polytopes [8], and certain zonotopes [12]. Furthermore, in very recent work [13] expanding on [3], it is shown that a certain graph called the **flip graph** of monotone paths on d-dimensional polytopes is $(d-1)$-connected assuming the polytope satisfies a condition called being **directionally simple** and that the graph is a Hasse diagram of a lattice. In their paper, they note "it is an interesting question for future study to determine exactly how much further this result can be pushed." In our case, since all cellular strings are coherent, this graph is precisely the one-skeleton of the corresponding poset permutahedron. By Balinski's theorem, this graph is always $(n-1)$-connected. However, order polytopes are not directionally simple for the orientation induced by the linear functional $\mathbf{1}(\mathbf{x}) = x_1 + \cdots + x_n$, so this family of polytopes pushes their results further.

2 Monotone Path Polytopes and Order Polytopes

In this section we show that poset permutahedra arise as monotone path polytopes of order polytopes. Let $\mathsf{P} \subset \mathbb{R}^n$ be a polytope and $\varphi : \mathbb{R}^n \to \mathbb{R}$ a linear function that is not constant on P. A **monotone path** on P is the sequence of edges $e_1, \ldots, e_m$ along a strictly φ-increasing path in the graph of P from a minimizer to a maximizer of φ. Cellular strings generalize monotone paths to faces of higher dimensions. A **cellular string** is a sequence $(\mathsf{F}_1, \mathsf{F}_2, \ldots, \mathsf{F}_k)$ of faces of P such that F_1 and F_k contain a φ-minimizer and maximizer respectively, and $\mathsf{F}_i \cap \mathsf{F}_{i+1}$ is the φ-maximal face of F_i and the φ-minimal face of F_{i+1} for all $1 \leq i \leq k-1$. Cellular strings are partially ordered by refinement. The resulting poset is the **Baues poset** of (P, φ), whose minimal elements are precisely the monotone paths; see [5]. Let ψ be a linear function linearly independent of φ. The ordered sequence of faces of P that maximize the linear functions $\psi + t\varphi$ as t ranges from $-\infty$ to $+\infty$ yields a cellular string, called a **coherent** cellular string. Billera and Sturmfels [6] showed that the subposet of coherent cellular strings is the face poset of a polytope, the **monotone path polytope** $\Sigma_\varphi(\mathsf{P})$. They show that the monotone path polytope is the Minkowski integral

$$\Sigma_\varphi(\mathsf{P}) \;=\; \int_{\mathbb{R}} P \cap \varphi^{-1}(s)\, ds\,.$$

Note that our definition differs from that in [6] by a scaling factor. For $\mathsf{P} = [0,1]^n$ and the linear function $\mathbf{1}(\mathbf{x}) = x_1 + \cdots + x_n$, we get $\Sigma_{\mathbf{1}}([0,1]^n) = \Pi_n - \frac{1}{2}\mathbf{1}$, as in [6].

Proof sketch of Theorem 1.1. For the proof, we verify

$$\begin{aligned}\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P})) \;&=\; \int_{\mathbb{R}} \mathsf{C}_{\mathcal{P}} \cap [0,1]^n \cap \mathbf{1}^{-1}(s)\, ds \\ &=\; \mathsf{C}_{\mathcal{P}} \cap \int_{\mathbb{R}} [0,1]^n \cap \mathbf{1}^{-1}(s)\, ds \;=\; \mathsf{C}_{\mathcal{P}} \cap \left(\Pi_n - \tfrac{1}{2}\mathbf{1}\right)\,.\end{aligned}$$

Since $-\frac{1}{2}\mathbf{1} + \mathsf{C}_{\mathcal{P}} = \mathsf{C}_{\mathcal{P}}$, this proves the claim. Note that proving this sequence of equalities is nontrivial, since in general one cannot pull an intersection with a cone outside of a Minkowski integral. □

The result yields that the face poset of $\Pi_{\mathcal{P}}$ is a subposet of the Baues poset. The following lemma, which is of interest in its own right, implies that all cellular strings of $(\mathsf{O}(\mathcal{P}), \mathbf{1})$ are coherent. A version of this statement had also appeared as Corollary 3.4.2 in the PhD Thesis [7].

Lemma 2.1. *Let P be a $0/1$-polytope. If for all edges $[\mathbf{u}, \mathbf{v}] \subset \mathsf{P}$ the nonzero entries $\mathbf{u} - \mathbf{v}$ are of the same sign, then all cellular strings of $(\mathsf{P}, \mathbf{1})$ are coherent.*

Recall from [24] that $\mathsf{O}(\mathcal{P})$ is a $0/1$-polytope with vertices corresponding to indicator vectors $\mathbf{e}_{\mathcal{F}} \in \{0,1\}^n$ of **filters** $\mathcal{F}$ of $\mathcal{P}$. The collection of filters of $\mathcal{P}$ ordered by inclusion is the **Birkhoff lattice** $\mathfrak{F}(\mathcal{P})$ and $\mathbf{1}(\mathbf{e}_{\mathcal{F}}) = |\mathcal{F}|$ is the rank of $\mathcal{F}$ in $\mathfrak{F}(\mathcal{P})$. Two filters $\mathcal{F}, \mathcal{F}'$ correspond to the endpoints of an edge of $\mathsf{O}(\mathcal{P})$ if and only if, say, $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{F}' \setminus \mathcal{F}$ is a connected poset. Hence all cellular strings of $(\mathsf{O}(\mathcal{P}), \mathbf{1})$ are coherent. In particular, the order complex of the face lattice of $\Sigma_{\mathbf{1}}(\mathsf{O}(\mathcal{P}))$ is a simplicial sphere. Theorem 2.1 of [12] implies that all cellular strings of $(\mathsf{O}(\mathcal{P}), \varphi)$ are coherent, provided φ induces the same orientation on the graph of $\mathsf{O}(\mathcal{P})$ as $\mathbf{1}$. This is the case if φ is positive on $\mathbb{R}^n_{\geq 0}$. This then yields Theorem 1.2.

3 Faces of Poset Permutahedra

It follows from Theorem 1.2 that the vertices of $\Pi_{\mathcal{P}}$ correspond to **1**-monotone paths on $\mathsf{O}(\mathcal{P})$, which in turn correspond to chains

$$\emptyset \;=\; \mathcal{F}_0 \;\subset\; \mathcal{F}_1 \;\subset\; \cdots \;\subset\; \mathcal{F}_k \;=\; \mathcal{P}$$

of filters of $\mathcal{P}$ such that the induced undirected subgraph of the Hasse diagram of $\mathcal{P}$ given by restricting to $\mathcal{F}_i \setminus \mathcal{F}_{i-1}$ is connected for all $i \in [k]$, that we denote by $\mathcal{F}_\bullet$. We denote the collection of such chains by $\mathfrak{C}(\mathcal{P})$ and call these **connected chains**. This is an accessible set system within the order complex of $\mathfrak{F}(\mathcal{P})$ but not a simplicial subcomplex. From [6, Theorem 5.3] we deduce the vertex set of $\Pi_{\mathcal{P}}$. For $A \subseteq [n]$, we write $A^c = [n] \setminus A$ for the complement.

Theorem 3.1. *The vertices of $\Pi_{\mathcal{P}}$ are in bijection to connected chains of filters $\mathcal{F}_\bullet$ in $\mathcal{P}$. For $\mathcal{F}_\bullet$, the corresponding vertex is*

$$\Psi(\mathcal{F}_\bullet) \;=\; \frac{1}{2}\sum_{i=1}^{l} \left(|\mathcal{F}_i^c| + |\mathcal{F}_{i-1}^c| + 1\right)\, \mathbf{e}_{\mathcal{F}_i \setminus \mathcal{F}_{i-1}}\,.$$

Two dimensional faces of 0/1-polytopes are either triangles or quadrilaterales. For order polytopes $\mathsf{O}(\mathcal{P})$, they correspond to filters $\mathcal{F} \subset \mathcal{F}'$ of $\mathcal{P}$ such that $\mathcal{F}' \setminus \mathcal{F}$ has at most two connected components. From the monotone path polytope perspective, edges of $\Pi_{\mathcal{P}}$ correspond to cellular strings $\mathsf{F}_1, \ldots, \mathsf{F}_k$ in $\mathsf{O}(\mathcal{P})$ such that F_i is a 2-face for precisely one $i \in \{1, \ldots, k\}$ and F_j are edges for all $j \neq i$.

Theorem 3.2. *Let $\Pi_{\mathcal{P}}$ be a poset permutahedron and $\mathcal{F}_\bullet, \mathcal{F}'_\bullet \in \mathfrak{C}(\mathcal{P})$ be a pair of connected chains of filters with the length of $\mathcal{F}_\bullet$ at least the length of $\mathcal{F}'_\bullet$. Then $\Psi(\mathcal{F}_\bullet)$ and $\Psi(\mathcal{F}'_\bullet)$ are adjacent if and only if there is an $1 \leq i < k$ such that:*

1. *$\mathcal{F}_{i+1} \setminus \mathcal{F}_{i-1}$ is connected and $\mathcal{F}'_\bullet$ is* ***(Coarsening edges)***

$$\mathcal{F}'_\bullet = \mathcal{F}_0 \;\subset\; \cdots \;\subset\; \mathcal{F}_{i-1} \;\subset\; \mathcal{F}_{i+1} \;\subset\; \cdots \;\subset\; \mathcal{F}_k\,.$$

Then we have

$$\Psi(\mathcal{F}'_\bullet) - \Psi(\mathcal{F}_\bullet) = \tfrac{1}{2}\left(\left(|\mathcal{F}_i| - |\mathcal{F}_{i+1}|\right)\mathbf{e}_{\mathcal{F}_{i-1} \setminus \mathcal{F}_i} + \left(|\mathcal{F}_{i-1}| - |\mathcal{F}_i|\right)\mathbf{e}_{\mathcal{F}_{i+1} \setminus \mathcal{F}_i}\right)\,.$$

2. *$\mathcal{F}_{i+1} \setminus \mathcal{F}_{i-1}$ consists of two connected components and $\mathcal{F}'_\bullet$ is* ***(Swapping edges)***

$$\mathcal{F}'_\bullet = \mathcal{F}_0 \;\subset\; \cdots \;\subset\; \mathcal{F}_{i-1} \;\subset\; \left(\mathcal{F}_{i-1} \cup (\mathcal{F}_{i+1} \setminus \mathcal{F}_i)\right) \;\subset\; \mathcal{F}_{i+1} \;\subset\; \cdots \;\subset\; \mathcal{F}_k\,.$$

Then we have

$$\Psi(\mathcal{F}'_\bullet) - \Psi(\mathcal{F}_\bullet) = \left(|\mathcal{F}_i| - |\mathcal{F}_{i+1}|\right)\mathbf{e}_{\mathcal{F}_i \setminus \mathcal{F}_{i-1}} + \left(|\mathcal{F}_i| - |\mathcal{F}_{i-1}|\right)\mathbf{e}_{\mathcal{F}_{i+1} \setminus \mathcal{F}_i},\,.$$

To determine the facets of $\Pi_{\mathcal{P}}$ it suffices by Theorems 1.1 and 1.2 to determine the coarsest, nontrivial cellular strings. We denote the cover relations of a poset $\mathcal{P}$ by $a \lessdot b$.

Theorem 3.3. *The coarsest nontrivial cellular strings of $(\mathsf{O}(\mathcal{P}), \mathbf{1})$ are of the following two forms:*

1. *(F_1), where F_1 is a facet of the order polytope $\mathsf{O}(\mathcal{P})$ corresponding to a cover relation $a \lessdot b$.*
2. *$(\mathsf{F}_1, \mathsf{F}_2)$, where F_1, F_2 are faces of $\mathsf{O}(\mathcal{P})$ and there exists a filter $\mathcal{F}$ in $\mathcal{P}$ such that all vertices in F_1 correspond to filters contained in $\mathcal{F}$ and and all vertices in F_2 correspond to filters containing $\mathcal{F}$.*

The proof of Theorem 3.3 makes use of the following two properties of order polytopes. First, for any nonempty face $\mathsf{F} \subseteq \mathsf{O}(\mathcal{P})$, there is a unique vertex that maximizes $\mathbf{1}(\mathbf{x})$. Second, for any two filters $\mathcal{F} \subseteq \mathcal{F}'$ there is a unique face $\mathsf{F} \subseteq \mathsf{O}(\mathcal{P})$ whose vertices are in bijection to the interval $[\mathcal{F}, \mathcal{F}']$ in the Birkhoff lattice of $\mathcal{P}$.

For $n \geq 1$ define the submodular set function f_n by $f_n(S) := \binom{n+1}{2} - \binom{n-|S|+1}{2}$ for $S \subseteq [n]$. Moreover, for $S \subseteq [n]$, let $\mathbf{1}_S$ be the linear function given by $\mathbf{1}_S(\mathbf{x}) := \sum_{i \in S} x_i$.

Corollary 3.4. *Let $\mathcal{P} = ([n], \preceq)$ be a poset. A point $\mathbf{x} \in \mathbb{R}^n$ is contain in $\Pi_{\mathcal{P}}$ if and only if*

$$\begin{aligned} x_a &\leq x_b && \text{for all cover relations } a \lessdot b \\ \mathbf{1}_{\mathcal{F}}(x) &\leq f_n(\mathcal{F}) && \text{for all proper, nonempty filters } \mathcal{F} \subset \mathcal{P} \\ \mathbf{1}(x) &= f_n([n]) = \tbinom{n+1}{2}. \end{aligned}$$

The inequality description is irredundant.

From Theorem 3.3 and Corollary 3.4 we can also derive explicit vertex-facet incidences for the poset permutahedron $\Pi_{\mathcal{P}}$.

Corollary 3.5. *A vertex given by the connected chain of filters $\emptyset = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k = \mathcal{P}$ is contained in the facets corresponding to*

1. *the proper nonempty filters $\mathcal{F}_1, \ldots, \mathcal{F}_{k-1}$ and*
2. *cover relations $a \lessdot b$ such that $a, b \in \mathcal{F}_i \setminus \mathcal{F}_{i-1}$ for some $i \in [k]$ with.*

Applying this characterization together with a counting argument enables us to characterize the simple poset permutahedra.

Theorem 3.6. *A poset permutahedron $\Pi_{\mathcal{P}}$ is a simple polytope if and only if the undirected Hasse diagram of $\mathcal{P}$ is a (not necessarily rooted) forest.*

4 Subdivision, Volumes and Integer Points

We view a linear extension L of $\mathcal{P}$ as a refinement of $\preceq$ to a total order $[n]$ and we collect the linear extensions in $\mathfrak{L}(\mathcal{P})$. For $L \in \mathfrak{L}(\mathcal{P})$ it is well known that $\mathsf{O}(L)$ is a unimodular simplex contained in $\mathsf{O}(\mathcal{P})$ and that $\{\mathsf{O}(L) : L \in \mathfrak{L}(\mathcal{P})\}$ is a triangulation of $\mathsf{O}(\mathcal{P})$. This generalizes to poset permutahedra.

Theorem 4.1. *For any poset $\mathcal{P}$, the set $\{\Pi_L : L \in \mathfrak{L}(\mathcal{P})\}$ is a subdivision of $\Pi_{\mathcal{P}}$.*

Note that this is immediate from the initial definition we give for the poset permutahedron in Equation (1.1) but is nonobvious from its equivalent formulation as a fiber polytope. If $\mathcal{P}$ is the antichain on n elements, then the symmetric group acts simply transitively on $\mathfrak{L}(\mathcal{P})$ and shows that any two Π_L are isometric. In particular $\operatorname{vol}(\Pi_L) = \frac{1}{n!} \operatorname{vol} \Pi_n$. The volume of Π_n is famously known to be the number of spanning trees of the complete graph on n nodes. This yields the following:

Corollary 4.2. *For a poset $\mathcal{P}$ on $[n]$, the volume of $\Pi_{\mathcal{P}}$ is $\frac{|\mathfrak{L}(\mathcal{P})|}{n!} n^{n-2}$. In particular, the probability that a random point of Π_n is in $\Pi_{\mathcal{P}}$ is precisely the probability that a random permutation is a linear extension of $\mathcal{P}$.*

A **score sequence** is an integer sequence $0 \le s_1 \le \cdots \le s_n \le n-1$ that is a possible result of an n-person round-robin tournament in which a single point is awarded to the winner of a match. Equivalently, a score sequence is a reordering of the **indegree sequence** of a tournament, that is, a directed complete graph on n nodes. For $n = 2$ the only score sequence is $0 \le 1$ and for $n = 3$ we have two score sequences: $0 \le 1 \le 2$ and $1 \le 1 \le 1$. A **score vector** is a tuple of integers $\mathbf{t} = (t_1, \ldots, t_n)$, where t_i records the number of points that team i wins during the tournament. For $n = 2$ there are two score vectors: $(0,1)$ and $(1,0)$; for $n = 3$ we have 7 score vectors: $(1,1,1)$ and the six permutations of $(0,1,2)$.

If we define $\alpha_{ij} \in \{-1,1\}$ for $1 \le i < j \le n$ by $\alpha_{ij} = 1$ if j wins the match between i and j and -1 otherwise, then the score vector $\mathbf{t}$ is

$$\mathbf{t} \;=\; \frac{1}{2}\left((n-1)\mathbf{1} + \sum_{i<j} \alpha_{ij}(\mathbf{e}_j - \mathbf{e}_i)\right) .$$

Thus, the collection of score vectors are the integer points in the projection of $\{-1,1\}^{\binom{n}{2}}$ with respect to the projection that sends the standard basis vector $\mathbf{e}_{ij}$ to $\mathbf{e}_j - \mathbf{e}_i$. This is the permutahedron Π_n up to the translation by $\mathbf{1}$. The characterization of the defining facets of the convex hull of score vectors is due to Landau [19]. The score sequences are the lattice points in $\Pi_{\mathcal{C}} - \mathbf{1}$, where $\mathcal{C}$ is a chain poset. By Theorem 4.1, for general posets $\mathcal{P}$ we have that the $\mathcal{P}$-score vectors are the integer points in the poset permutahedron $\Pi_{\mathcal{P}} - \mathbf{1}$.

Theorem 1.4 for chain posets follows from [20], Theorem 4.1 implies the result for every poset permutahedron. Combinatorially we can get this result by considering complete directed graphs with m directed edges between any pair of nodes.

Decomposing every such multi-tournament for $2m$ into m tournaments with two directed edges between any pair of nodes, shows the integer decomposition property (6) from the introduction. Note that for this to work in general we need the integer point in the second dilate of the poset permutahedron: for the chain poset on two elements we only have one integer point $(0,1)$ in the first dilate.

5 Partitioned Permutahedra

In [17], Horiguchi, Masuda, Shareshian, and Song defined **partitioned permutahedra** with motivation coming from combinatorial algebraic geometry. To define these polyhedra, they define a linear halfspace for each $k \in [n-1]$ given by

$$H(k) \ := \ \{\mathbf{x} \in \mathbb{R}^n : x_k \le x_{k+1}\}\,.$$

For each subset $K \subseteq [n-1]$, they define the **partitioned permutahedron** via

$$\Pi_n(K) = \Pi_n \cap \bigcap_{k \in K} H(k).$$

For every such subset $K \subseteq [n-1]$, we associate the poset $\mathcal{P}_K = ([n], \preceq)$ with cover relations are given by $i \lessdot i+1$ if $i \in K$. The following follows directly from our definition of poset permutahedra.

Proposition 5.1. *For every $K \subseteq [n-1]$, we have $\Pi_n(K) = \Pi_{\mathcal{P}_K}$.*

Geometrically $\mathsf{O}(\mathcal{P}_K)$ is a product of simplices for which cellular strings are easy to describe. In particular, Theorem 3.1, Corollary 3.4, and Theorem 3.3 recover the results on vertices, facets, and incidences in [17].

Corollary 5.2. *Partitioned permutahedra are simple polytopes.*

This follows directly from the observation that $\mathcal{P}_K$ is a disjoint union of chains and Theorem 3.6. This follows also as a consequence of Proposition 4.5 in [17] .

Proposition 7.4 in [17] shows that the h-vector of $\Pi(W)$ is the descent statistic restricted to the permutations σ of $[n]$ with $\sigma^{-1}(i) - \sigma^{-1}(i+1) \le 1$ for all $i \in K$. The set of these permutations is denoted by $W(K)$. This is shown by applying results on the cohomology of Hessenberg varieties. We sketch a simple combinatorial proof using the geometry of poset permutahedra.

Proof sketch of Theorem 1.3. Recall that we can compute the h-polynomial of a simple polytope the following way (see, e.g., [4, Chapter VI.6]): We first choose an edge-generic linear functional $\omega \colon \mathbb{R}^n \to \mathbb{R}$, which induces an acyclic orientation on the graph of the polytope P. If $\operatorname{outdeg}(\mathbf{v})$ denotes the out-degree of the vertex $\mathbf{v}$, i.e., the number ω-improving neighbors, then

$$h_{\mathsf{P}}(x) \ = \ \sum_{\mathbf{v}} x^{\operatorname{outdeg}(\mathbf{v})}\,.$$

In order to prove the stated expression for the h-polynomial, we define a bijection g between connected chains of filters in the poset $\mathcal{P} = [1,k_1] \uplus [k_1+1,k_2] \uplus \cdots \uplus [k_\ell+1,n]$ and $W(K)$. For each $\sigma \in W(K)$, we build $g(\sigma)$ via the following algorithm. Initialize the

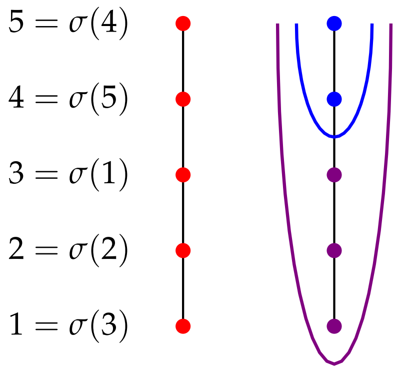

Figure 2: Pictured is an illustration of the bijection g used in the proof sketch of Theorem 1.3 for the permutation 32154. First consider $\sigma(5) = 4$. Then the smallest filter containing it is $\{4,5\}$. The largest element whose image is not covered is 3. Since $\sigma(3) = 1$, and the smallest remaining filter containing 1 is the whole chain. This yields the resulting chain of filters $\{4,5\} \subseteq \{1,2,3,4,5\}$. The descents of σ are $\{(1,2),(2,3),(4,5)\}$ and correspond exactly to refining edges, which are all $\omega = (1,2,4,8,16)$-improving.

chain of filters $\mathcal{F}_\bullet = \{\mathcal{F}_0 = \emptyset\}$. We write $\bigcup \mathcal{F}_\bullet = \bigcup_{\mathcal{F} \in \mathcal{F}_\bullet} \mathcal{F}$. While $\bigcup \mathcal{F}_\bullet \neq [n]$, let k be maximal such that $\sigma(k) \notin \bigcup \mathcal{F}_\bullet$. Let $\mathcal{F}'$ be the unique smallest filter containing $\sigma(k) \cup \bigcup \mathcal{F}_\bullet$ and add $\mathcal{F}'$ to $\mathcal{F}_\bullet$. When the process terminates, the result will be a connected chain of filters. This is a bijection. See Figure 2 for an illustration.

Let $\omega(\mathbf{x}) = \sum_{i=1}^{n} 2^i \mathbf{x}_i$. Then to prove the theorem, it suffices to show that the size of the descent set of σ is precisely the ω-out-degree of the corresponding vertex $g(\sigma)$. One can do this directly by applying our characterization of edge directions derived from Theorem 3.2. □

We also offer an alternative characterization of the vertices that makes them efficient to count as a sum of multinomials or via memoization and the recurrence

$$a_{m_1,m_2,\ldots,m_k} = \sum_{j=1}^{k} \sum_{i=0}^{m_j-1} a_{m_1,m_2,\ldots,m_{j-1},i,m_{j+1},\ldots,m_k}.$$

To do this, we require the notion of a high dimensional rook walk as found in [18]. Namely, a rook walk in a $m \times n$ grid is any sequence of moves a rook could take to move from $(0,0)$ to (m,n), where the rook is only allowed to move to the right or up. In high dimensions, one can consider any sequence from $(0,0,\ldots,0)$ to $(m_1,m_2,\ldots,m_n)$ such that at each step one can only increase in a single coordinate at a time.

Proposition 5.3. *For $K \subseteq [n-1]$, the following sets have the same cardinality:*

i) *The set of vertices of the partitioned permutahedron $\Pi_n(K)$.*
ii) *The set $W(K) = \{\sigma \in S_n : \sigma^{-1}(i) - \sigma^{-1}(i+1) \leq 1$ for all $i \in K\}$.*
iii) *The set of high dimensional rook walks from $(0,0,\ldots,0)$ to $(k_1, k_2 - k_1, k_3 - k_2, \ldots, n - k_\ell)$, where $\{k_1 < k_2 < \cdots < k_\ell < n\} = [n] \setminus K$.*

In [18], they studied asymptotics and recurrences for the sequence of high dimensional rook walks from $(0,0,\ldots,0)$ to $(n,n,\ldots,n)$, and our results give a new combinatorial perspective for arbitrary endpoints. Finally, we note as a corollary of Corollary 4.2, we can compute the volumes of the partitioned permutahedra:

Corollary 5.4. *The volume of the partitioned permutahedron $\Pi_n(K)$ for $K \subseteq [n-1]$ is given*

$$\frac{n^{n-2}}{n!}\binom{n}{k_1, k_2 - k_1, k_3 - k_2, \ldots, n - k_\ell}$$

for $\{k_1 < k_2 < \cdots < k_\ell < n\} = [n] \setminus K$.

Acknowledgements

We would like to thank Christian Haase for helpful discussions and the anonymous reviewers for useful comments on a previous version.

References

[1] R. Acharya, T. Mütze, and F. Verciani. "Flips in colorful triangulations". *J. Comput. Geom.* **16**.1 (2025), pp. 295–332. DOI.

[2] F. Ardila, T. Bliem, and D. Salazar. "Gelfand-Tsetlin polytopes and Feigin-Fourier-Littelmann-Vinberg polytopes as marked poset polytopes". *(FPSAC 2011)*. Vol. AO. Discrete Math. Theor. Comput. Sci. Proc. 2011, pp. 27–37.

[3] C. Athanasiadis, P. Edelman, and V. Reiner. "Monotone paths on polytopes". *Math. Z.* **235**.2 (2000), pp. 315–334. DOI.

[4] A. Barvinok. *A course in convexity*. Vol. 54. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2002, x+366 pp. DOI.

[5] L. Billera, M. Kapranov, and B. Sturmfels. "Cellular strings on polytopes". *Proc. Amer. Math. Soc.* **122**.2 (1994), pp. 549–555. DOI.

[6] L. Billera and B. Sturmfels. "Fiber polytopes". *Ann. of Math. (2)* **135**.3 (1992), pp. 527–549. DOI.

[7] A. Black. "Monotone Paths on Polytopes: Combinatorics and Optimization". PhD thesis. University of California, Davis, 2024.

[8] A. E. Black and R. Sanyal. "Underlying flag polymatroids". *Adv. Math.* **453** (2024), Paper No. 109835, 42 pp. DOI.

[9] G. Brightwell and P. Winkler. "Counting linear extensions is #P-complete". *STOC '91: Proceedings of the twenty-third annual ACM symposium on Theory of computing*. 1991, pp. 175–181. DOI.

[10] T. Chappell, T. Friedl, and R. Sanyal. "Two double poset polytopes". *SIAM J. Discrete Math.* **31**.4 (2017), pp. 2378–2413. DOI.

[11] A. Claesson, M. Dukes, A. Franklín, and S. Stefánsson. "Counting tournament score sequences". *Proc. Amer. Math. Soc.* **151**.09 (2023), pp. 3691–3704. DOI.

[12] R. Edman, P. Jiradilok, G. Liu, and T. McConville. "Zonotopes whose cellular strings are all coherent". *European J. Combin.* **96** (2021), Paper No. 103352, 18 pp. DOI.

[13] C. Gaetz and P. Hersh. "Poset topology, moves, and Bruhat interval polytope lattices". 2024. arXiv:2410.08076.

[14] P. Galashin. "P-associahedra". *Selecta Math. (N.S.)* **30**.1 (2024), Paper No. 6, 34 pp. DOI.

[15] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants and multidimensional determinants*. Modern Birkhäuser Classics. Reprint of the 1994 edition. Birkhäuser Boston, Inc., Boston, MA, 2008, x+523 pp.

[16] T. Horiguchi, M. Masuda, T. Sato, J. Shareshian, and J. Song. "The representation of $\mathfrak{S}_n$ on the cohomology of the permutohedral variety and gamma vectors of partitioned permutohedra". 2024. arXiv:2405.09242.

[17] T. Horiguchi, M. Masuda, J. Shareshian, and J. Song. "Toric orbifolds associated with partitioned weight polytopes in classical types". *Selecta Math. (N.S.)* **30**.5 (2024), Paper No. 84, 37 pp. DOI.

[18] M. Kauers and D. Zeilberger. "The computational challenge of enumerating high-dimensional rook walks". *Adv. in Appl. Math.* **47**.4 (2011), pp. 813–819. DOI.

[19] H. G. Landau. "On dominance relations and the structure of animal societies. III. The condition for a score structure". *Bull. Math. Biophys.* **15** (1953), pp. 143–148. DOI.

[20] J. Moon. "An extension of Landau's theorem on tournaments". *Pacific J. Math.* **13**.4 (1963), pp. 1343–1345.

[21] OEIS Foundation Inc. (2024). "The On-Line Encyclopedia of Integer Sequences (OEIS)". Published electronically at `https://oeis.org`.

[22] A. Postnikov, V. Reiner, and L. Williams. "Faces of generalized permutohedra". *Doc. Math.* **13** (2008), pp. 207–273. DOI.

[23] R. Stanley. "Two combinatorial applications of the Aleksandrov-Fenchel inequalities". *J. Combin. Theory Ser. A* **31**.1 (1981), pp. 56–65. DOI.

[24] R. Stanley. "Two poset polytopes". *Discrete Comput. Geom.* **1**.1 (1986), pp. 9–23. DOI.