Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #41, 12 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

Variations of the (α, t)-Eulerian polynomials and gamma-positivity

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Abstract. We define a multivariable generalization of the Eulerian polynomials using linear and descent based statistics of permutations and establish the connection with the (α, t)-Eulerian polynomials based on cyclic and excedance based statistics of permutations. As applications of this connection, we obtain the exponential generating function for the multivariable Eulerian polynomials and γ-positive formulas of two variants of Eulerian polynomials. We also show that enumerating the cycle André permutations with respect to the number of drops, fixed points and cycles gives rise to the normalised γ-vectors of the (α, t)-Eulerian polynomials. Our result generalizes and unifies several recent results in the literature.

Keywords: (α, t)-Eulerlian polynomials, left-to-right maxima, cycle André permutations, cyclic valley-hopping, gamma-positivity

1 Introduction

The Eulerian polynomials have a long and rich history, some of which is given in [7, 16, 8, 14]. For any positive integer n, we denote the symmetric group of $[n] := \{1, 2, \ldots, n\}$ by $\mathfrak{S}_n$. For $\sigma \in \mathfrak{S}_n$, the integer $i \in [n-1]$ is called a *descent* (**des**) if $\sigma(i) > \sigma(i+1)$; an *ascent* (**asc**) if $\sigma(i) < \sigma(i+1)$; an *excedance* (**exc**) if $i < \sigma(i)$. It is well-known that the Eulerian polynomials $A_n(x)$ have the following combinatorial interpretations:

$$A_n(x) := \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{asc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{exc}(\sigma)}. \tag{1.1}$$

Let $\mathcal{M}_n$ be the set of permutations $\sigma \in \mathfrak{S}_n$ such that the first descent (if any) of σ appears at $\sigma^{-1}(n)$. The *binomial-Eulerian polynomials* were introduced by Postnikov, Reiner, and Williams [17, Section 10.4] as the h-polynomials of stellohedrons, and can also be defined as in the following

$$\widetilde{A}_n(x) := \sum_{\sigma \in \mathcal{M}_{n+1}} x^{\mathrm{des}(\sigma)} = 1 + x \sum_{m=1}^{n} \binom{n}{m} A_m(x). \tag{1.2}$$

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It is well-known [8, 15] that the Eulerian polynomials $A_n(x)$ have the following γ-positive expansion

$$A_{n+1}(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_{n,j}\, x^j (1+x)^{n-2j} \tag{1.3}$$

$$= \sum_{j=0}^{\lfloor n/2 \rfloor} 2^j d_{n,j}\, x^j (1+x)^{n-2j}, \tag{1.4}$$

where $\gamma_{n,j}$ is the number of permutations without double descents having j descents in $\mathfrak{S}_{n+1}$ and $d_{n,j}$ is the number of André permutations with j descents in $\mathfrak{S}_{n+1}$. It is also known [17, Section 10.4] that the polynomials $\widetilde{A}_n(x)$ have the following gamma positive formula

$$\widetilde{A}_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \widetilde{\gamma}_{n,j}\, x^j (1+x)^{n-2j}, \tag{1.5}$$

where $\widetilde{\gamma}_{n,j}$ is the number of $\sigma \in \mathcal{M}_{n+1}$ such that σ has j descents and no double descents.

The coefficients of the polynomials $\widetilde{A}_n(x)$ can also be nicely expressed as sums of products of Eulerian numbers and binomial coefficients. For integers $a, b \geq 0$, Chung, Graham, and Knuth [5] found the following identity with three proofs:

$$\sum_{k\geq 0} \binom{a+b}{k} \left\langle {k \atop a-1} \right\rangle = \sum_{k\geq 0} \binom{a+b}{k} \left\langle {k \atop b-1} \right\rangle, \tag{1.6}$$

where $\left\langle {0 \atop 0} \right\rangle = 1$. Shareshian and Wachs [18] noticed that the above identity corresponds exactly to the palindromicity of the coefficients of $\widetilde{A}_n(x)$. Recently, Ji and Lin [13, Theorem 4.1] found an α-analogue of (1.6).

For $\sigma \in \mathfrak{S}_n$, an index $i \in [n]$ is a *drop* (**drop**) of σ if $i > \sigma(i)$; a *fixed point* (**fix**) of σ if $i = \sigma(i)$. We shall also consider a permutation $\sigma \in \mathfrak{S}_n$ as a word $\sigma = \sigma_1 \dots \sigma_n$ with $\sigma_i := \sigma(i)$ for $i \in [n]$. Say that a letter σ_i is a *left-to-right maximum* (**lrmax**) of σ if $\sigma_i > \sigma_j$ for every $j < i$; a *right-to-left maximum* (**rlmax**) of σ if $\sigma_i > \sigma_j$ for every $j > i$.

In the middle of 1970's Carlitz–Scoville considered several multivariate Eulerian polynomials, among which are the so-called (α, β)-Eulerian polynomials [3]

$$A_n(x, y \mid \alpha, \beta) := \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\mathrm{asc}(\sigma)} y^{\mathrm{des}(\sigma)} \alpha^{\mathrm{lmax}(\sigma)-1} \beta^{\mathrm{rmax}(\sigma)-1}, \tag{1.7a}$$

and the following ones [4], that we refer to (α, t)-**Eulerian polynomials**,

$$A_n^{\mathrm{cyc}}(x, y, t \mid \alpha) := \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{exc}(\sigma)} y^{\mathrm{drop}(\sigma)} t^{\mathrm{fix}(\sigma)} \alpha^{\mathrm{cyc}(\sigma)}, \tag{1.7b}$$

where $\mathrm{cyc}(\sigma)$ denotes the number of cycles of σ. As $y^n A_n^{\mathrm{cyc}}(x/y, 1, t/y \mid \alpha)$ is equal to $A_n^{\mathrm{cyc}}(x, y, t \mid \alpha)$, polynomial $A_n^{\mathrm{cyc}}(x, y, t \mid \alpha)$ is the homogeneous version of $A_n^{\mathrm{cyc}}(x, 1, t \mid \alpha)$, which is studied in [8, Chapter 4].

For $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$ with the boundary condition $\mathbf{0} - \mathbf{0}$, i.e., $\boldsymbol{\sigma_0 = \sigma_{n+1} = 0}$, a letter $\sigma_i \in [n]$ is called a *valley* (**val**) of σ if $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$; *peak* (**pk**) of σ if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$; *double ascent* (**da**) of σ if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$; *double descent* (**dd**) of σ if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.

It is clear that the following identities hold

$$\mathrm{val} = \mathrm{pk} - 1, \quad \mathrm{asc} = \mathrm{val} + \mathrm{da}, \quad \mathrm{des} = \mathrm{val} + \mathrm{dd}\,. \tag{1.8}$$

Recently, refining the (α, β)-Eulerian polynomials $A_n(x, y \mid \alpha, \beta)$, Ji [12] considered a variation of Eulerian polynomials incorporating six statistics over permutations in $\mathfrak{S}_{n+1}$:

$$A_n(u_1, u_2, u_3, u_4 \mid \alpha, \beta) := \sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1 u_2)^{\mathrm{val}(\sigma)} u_3^{\mathrm{da}(\sigma)} u_4^{\mathrm{dd}(\sigma)} \alpha^{\mathrm{lmax}(\sigma)-1} \beta^{\mathrm{rmax}(\sigma)-1}. \tag{1.9}$$

In a follow-up Ji–Lin [13] considered a binomial analogue of Carlitz–Scoville's polynomial (1.7a) when $\alpha = \beta$.

This paper originally arose from the desire to provide an alternative approach to Ji's generating function [12, Theorem 1.4] via previous known results in the literature, which led us to the following connection formula: let $xy = u_1 u_2$ and $x + y = u_3 + u_4$, then

$$A_n(u_1, u_2, u_3, u_4 \mid \alpha, \beta) = A_n^{\mathrm{cyc}}\left(x, y, \frac{\alpha u_3 + \beta u_4}{\alpha + \beta} \mid \alpha + \beta\right). \tag{1.10}$$

By combining a refined version of (1.10) and known group actions, among our main results, we generalize Ji and Ji–Lin's polynomials and results in [12, 13].

For a permutation $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$, we say that an index $i \in [n]$ is a *cycle peak* (**cpk**) of σ if $\sigma^{-1}(i) < i > \sigma(i)$; *cycle valley* (**cval**) of σ if $\sigma^{-1}(i) > i < \sigma(i)$; *cycle double ascent* (**cda**) of σ if $\sigma^{-1}(i) < i < \sigma(i)$; *cycle double descent* (**cdd**) of σ if $\sigma^{-1}(i) > i > \sigma(i)$. Note that $\mathrm{cpk}(\sigma) = \mathrm{cval}(\sigma)$. The following is our first main result.

Theorem 1.1. *If $xy = u_1 u_2$ and $x + y = u_3 + u_4$, then*

$$A_n^{\mathrm{cyc}}(x, y, t \mid \alpha) = \sum_{\sigma \in \mathfrak{S}_n} (u_1 u_2)^{\mathrm{cpk}(\sigma)} u_3^{\mathrm{cda}(\sigma)} u_4^{\mathrm{cdd}(\sigma)} t^{\mathrm{fix}(\sigma)} \alpha^{\mathrm{cyc}(\sigma)}. \tag{1.11}$$

Let $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$ with the boundary condition $\mathbf{0} - \mathbf{0}$. A letter $\sigma_i \in [n]$ is a

- *left-to-right-maximum-peak* (**lmaxpk**) if σ_i is a left-to-right maximum and also a peak;
- *right-to-left-maximum-peak* (**rmaxpk**) if σ_i is a right-to-left maximum and also a peak;
- *left-to-right-maximum-double-ascent* (**lmaxda**) if σ_i is a left-to-right maximum and also a double ascent;

- *right-to-left-maximum-double-descent* (**rmaxdd**) if σ_i is a right-to-left maximum and also a double descent.

Let $\mathbf{u} = (u_1, u_2, u_3, u_4)$ and define the generalized Eulerian polynomial

$$A_n(\mathbf{u}, f, g, t \mid \alpha, \beta) = \sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1 u_2)^{\mathrm{val}(\sigma)} u_3^{\mathrm{da}(\sigma)} u_4^{\mathrm{dd}(\sigma)} f^{\mathrm{lmaxpk}(\sigma)-1} g^{\mathrm{rmaxpk}(\sigma)-1} \times t^{\mathrm{lmaxda}(\sigma)+\mathrm{rmaxdd}(\sigma)} \alpha^{\mathrm{lmax}(\sigma)-1} \beta^{\mathrm{rmax}(\sigma)-1}. \tag{1.12}$$

The following is our second main result, which generalizes (1.10).

Theorem 1.2. *If $xy = u_1 u_2$ and $x + y = u_3 + u_4$, then*

$$A_n(\boldsymbol{u}, f, g, t \mid \alpha, \beta) = A_n^{\mathrm{cyc}}\left(x, y, \frac{\alpha u_3 + \beta u_4}{\alpha f + \beta g}\, t \mid \alpha f + \beta g\right). \tag{1.13}$$

In the next section, we present some consequences of the main theorems, and outline the proofs of Theorem 1.1 and 1.2 in Section 3.

2 Applications of the main theorems

2.1 Exponential generating functions

The exponential generating function of polynomials $A_n^{\mathrm{cyc}}(x, y, t \mid \alpha)$ is well-known [8, 4, 2] and reads as follows

$$\sum_{n \geq 0} A_n^{\mathrm{cyc}}(x, y, t \mid \alpha) \frac{z^n}{n!} = \left(\frac{(x-y)e^{tz}}{xe^{yz} - ye^{xz}}\right)^{\alpha}. \tag{2.1}$$

Combining Theorem 1.2 with (2.1), we derive immediately the exponential generating function of $A_n(\mathbf{u}, f, g, t \mid \alpha, \beta)$ in (1.12), namely

Theorem 2.1. *Let $xy = u_1 u_2$ and $x + y = u_3 + u_4$. We have*

$$\sum_{n \geq 0} A_n(\boldsymbol{u}, f, g, t \mid \alpha, \beta) \frac{z^n}{n!} = e^{(\alpha u_3 + \beta u_4)tz} \left(\frac{x-y}{xe^{yz} - ye^{xz}}\right)^{\alpha f + \beta g}. \tag{2.2}$$

From the above theorem we derive plainly the generating functions for Ji's generalized Eulerian polynomials $A_n(\mathbf{u}, 1, 1, 1 \mid \alpha, \beta)$ and Ji–Lin's binomial-Stirling Eulerian polynomials $A_n(\mathbf{u}, 0, 1, 1 \mid \alpha, \beta)$ [13, Theorem 1.5].

Furthermore, combining Theorem 1.2 and the continued fraction expansion of the ordinary generating function of $A_n^{\mathrm{cyc}}(x, y, t \mid \alpha)$ in [21, Théorème 3] we obtain the following continued fraction formula for the ordinary generating function of $A_n(\mathbf{u}, f, g, t \mid \alpha, \beta)$.

Theorem 2.2. *We have*

$$\sum_{n=0}^{\infty} A_n(\boldsymbol{u}, f, g, t \,|\, \alpha, \beta) z^n = \cfrac{1}{1 - b_0 z - \cfrac{\lambda_1 z^2}{1 - b_1 z - \cfrac{\lambda_2 z^2}{1 - b_2 z - \cdots}}}, \tag{2.3a}$$

where

$$b_k = k(u_3 + u_4) + (\alpha u_3 + \beta u_4)t, \tag{2.3b}$$

$$\lambda_{k+1} = (k + \alpha f + \beta g)(k+1) u_1 u_2 \quad (k \geq 0). \tag{2.3c}$$

In particular, the polynomials $A_n(\mathbf{u}, f, g, t \,|\, \alpha, \beta)$ encompass the moment sequences of the orthogonal Sheffer polynomials [22].

2.2 (α, t)-Eulerian and binomial-Eulerian polynomials

Define the (α, t)-Eulerian polynomials $A_n(x, y, t \,|\, \alpha)$ by

$$A_n\,(x, y, t \,|\, \alpha) := \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\mathrm{asc}(\sigma)} y^{\mathrm{des}(\sigma)} t^{\mathrm{lmaxda}(\sigma) + \mathrm{rmaxdd}(\sigma)} \alpha^{\mathrm{lmax}(\sigma) + \mathrm{rmax}(\sigma) - 2}, \tag{2.4a}$$

which is equal to $A_n(x, y, x, y, 1, 1, t \,|\, \alpha, \alpha)$. By Theorem 1.2 we have

$$A_n\,(x, y, t \,|\, \alpha) = A_n^{\mathrm{cyc}}\left(x, y, \frac{x+y}{2} t \,|\, 2\alpha\right). \tag{2.4b}$$

Combining (2.4b) and Theorem 1.2 with $f = g = 1$, $\alpha = \beta$, we obtain the following t-analogue of Ji-Lin's Theorem 1.6 [13].

Theorem 2.3. *If $xy = u_1 u_2$ and $x + y = u_3 + u_4$, then*

$$A_n\,(x, y, t \,|\, \alpha) = \sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1 u_2)^{\mathrm{val}(\sigma)} u_3^{\mathrm{da}(\sigma)} u_4^{\mathrm{dd}(\sigma)} t^{\mathrm{lmaxda}(\sigma) + \mathrm{rmaxdd}(\sigma)} \alpha^{\mathrm{lmax}(\sigma) + \mathrm{rmax}(\sigma) - 2}.$$

We define the (α, t)-binomial-Eulerian polynomials $\widetilde{A}_n(x, y, t \,|\, \alpha)$ by

$$\widetilde{A}_n(x, y, t \,|\, \alpha) = \sum_{\sigma \in \mathcal{M}_{n+1}} x^{\mathrm{asc}(\sigma)} y^{\mathrm{des}(\sigma)} t^{\mathrm{lmaxda}(\sigma) + \mathrm{rmaxdd}(\sigma)} \alpha^{\mathrm{lmax}(\sigma) + \mathrm{rmax}(\sigma) - 2}, \tag{2.5a}$$

which is equal to $A_n(x, y, x, y, 0, 1, t \,|\, \alpha, \alpha)$ because a permutation $\sigma \in \mathfrak{S}_n$ is an element of $\mathcal{M}_n$ if and only if $\mathrm{lmaxpk}(\sigma) = 1$. By Theorem 1.2 we have

$$\widetilde{A}_n(x, y, t \,|\, \alpha) = A_n^{\mathrm{cyc}}(x, y, (x+y)t \,|\, \alpha). \tag{2.5b}$$

Combining (2.5b) and Theorem 1.2 with $f = 0$, $g = 1$, and $\alpha = \beta$, we obtain the following t-analogue of Ji-Lin's Theorem 1.5 [13].

Theorem 2.4. *If $xy = u_1u_2$ and $x + y = u_3 + u_4$, then*

$$\widetilde{A}_n(x,y,t\,|\,\alpha) = \sum_{\sigma\in\mathcal{M}_{n+1}} (u_1u_2)^{\mathrm{val}(\sigma)} u_3^{\mathrm{da}(\sigma)} u_4^{\mathrm{dd}(\sigma)} t^{\mathrm{lmaxda}(\sigma)+\mathrm{rmaxdd}(\sigma)} \alpha^{\mathrm{lmax}(\sigma)+\mathrm{rmax}(\sigma)-2}.$$

2.3 A symmetric (α, t)-Eulerian identity

Define two kinds of (α, t)-*Eulerian numbers* as follows:

$$\left\langle {n \atop k} \right\rangle_{\alpha,t}^{\mathrm{exc}} := \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \mathrm{exc}(\sigma)=n-k}} \alpha^{\mathrm{cyc}(\sigma)} t^{\mathrm{fix}(\sigma)} \qquad (1 \le k \le n), \tag{2.6a}$$

and

$$\left\langle {n \atop k} \right\rangle_{\alpha,t}^{\mathrm{asc}} := \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \mathrm{asc}(\sigma)=n-k}} \alpha^{\mathrm{rmax}(\sigma)} t^{\mathrm{rmaxdd}(\sigma)} \qquad (1 \le k \le n). \tag{2.6b}$$

It is easy to see that $A_n^{\mathrm{cyc}}(x, y, t(x + y)\,|\,\alpha)$ is symmetric in x and y because the involution $\vartheta : \sigma \mapsto \sigma^{-1}$ for $\sigma \in \mathfrak{S}_n$ satisfies $(\mathrm{exc}, \mathrm{drop}, \mathrm{fix})\,\sigma = (\mathrm{drop}, \mathrm{exc}, \mathrm{fix})\,\sigma^{-1}$. We have the following symmetric (α, t)-Eulerian identity.

Theorem 2.5. *For integers $a, b \ge 0$, we have*

$$\left\langle {n \atop k} \right\rangle_{\alpha,t} := \left\langle {n \atop k} \right\rangle_{\alpha,t}^{\mathrm{exc}} = \left\langle {n \atop k} \right\rangle_{\alpha,t}^{\mathrm{asc}}, \tag{2.7a}$$

and

$$\sum_{k\ge 0} (\alpha t)^{a+b-k} \binom{a+b}{k} \left\langle {k \atop a} \right\rangle_{\alpha,t} = \sum_{k\ge 0} (\alpha t)^{a+b-k} \binom{a+b}{k} \left\langle {k \atop b} \right\rangle_{\alpha,t}, \tag{2.7b}$$

where $\left\langle {0 \atop k} \right\rangle_{\alpha,t} = \left\langle {k \atop 0} \right\rangle_{\alpha,t} = \delta_{k,0}$.

Remark 1. *When $\alpha = t = 1$ and $t = 1$, identity (2.7) reduces to (1.6) and [13, Theorem 4.1], respectively.*

2.4 γ-positivity of (α, t)-Eulerian polynomials

The following lemma can be derived from Theorem 1.1 and 1.2 with suitable substitutions [20, Lemma 2.2].

Lemma 2.1. *For any variable f, we have*

$$\begin{aligned} &A_n(x, y, 0, x + y, f, 1, t\,|\,\alpha, \alpha)\\ &= \sum_{\substack{\sigma\in\mathfrak{S}_{n+1}\\ \mathrm{da}(\sigma)=0}} (xy)^{\mathrm{asc}(\sigma)} (x + y)^{n-2\,\mathrm{asc}(\sigma)} f^{\mathrm{lmaxpk}(\sigma)-1} t^{\mathrm{rmaxdd}(\sigma)} \alpha^{\mathrm{lmax}(\sigma)+\mathrm{rmax}(\sigma)-2} && (2.8a)\\ &= \sum_{\substack{\sigma\in\mathfrak{S}_{n}\\ \mathrm{cda}(\sigma)=0}} (xy)^{\mathrm{exc}(\sigma)} (x + y)^{n-2\,\mathrm{exc}(\sigma)} t^{\mathrm{fix}(\sigma)} \alpha^{\mathrm{cyc}(\sigma)} (f + 1)^{\mathrm{cyc}(\sigma)-\mathrm{fix}(\sigma)}. && (2.8b) \end{aligned}$$

We define three types of subsets of $\mathfrak{S}_n$:

$$\mathfrak{S}^{\mathrm{cda}=0}_{n,\mathrm{exc}=j} := \{\sigma \in \mathfrak{S}_n : \mathrm{cda}(\sigma) = 0 \text{ and } \mathrm{exc}(\sigma) = j\}; \tag{2.9a}$$

$$\mathfrak{S}^{\mathrm{da}=0}_{n,\mathrm{asc}=j} := \{\sigma \in \mathfrak{S}_n : \mathrm{da}(\sigma) = 0 \text{ and } \mathrm{asc}(\sigma) = j\}; \tag{2.9b}$$

$$\mathcal{M}^{\mathrm{da}=0}_{n,\mathrm{asc}=j} := \{\sigma \in \mathcal{M}_n : \mathrm{da}(\sigma) = 0 \text{ and } \mathrm{asc}(\sigma) = j\}. \tag{2.9c}$$

From Lemma 2.1, we derive the following combinatorial interpretations of the coefficients in the γ-expansion of $A_n(x, y, t \mid \alpha)$ and $\widetilde{A}_n(x, y, t \mid \alpha)$.

Theorem 2.6. *For $0 \le j \le \lfloor n/2 \rfloor$, we have*

$$A_n(x, y, t \mid \alpha) = \sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_{n,j}(\alpha, t)(xy)^j (x + y)^{n-2j}, \tag{2.10}$$

where

$$\gamma_{n,j}(\alpha, t) = \sum_{\sigma \in \mathfrak{S}^{\mathrm{da}=0}_{n+1,\mathrm{asc}=j}} \alpha^{\mathrm{lmax}(\sigma)+\mathrm{rmax}(\sigma)-2} t^{\mathrm{rmaxdd}(\sigma)} \tag{2.11a}$$

$$= \sum_{\sigma \in \mathfrak{S}^{\mathrm{cda}=0}_{n,\mathrm{exc}=j}} 2^{\mathrm{cyc}(\sigma)-\mathrm{fix}(\sigma)} \alpha^{\mathrm{cyc}(\sigma)} t^{\mathrm{fix}(\sigma)}; \tag{2.11b}$$

Remark 2. *When $t = 1$, Carlitz and Scoville [3] studied the above γ-coefficients but did not give any combinatorial interpretation. By complement operation $\sigma \mapsto \sigma^c$, Equation (2.11a) with $t = 1$ can be obtained in Ji-Lin's [13, Theorem 1.6]. When $\alpha = t = 1$, Equation (2.11a) is equivalent to (1.3).*

Theorem 2.7. *For $n \ge 1$, we have*

$$\widetilde{A}_n(x, y, t \mid \alpha) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \widetilde{\gamma}_{n,j}(\alpha, t)(xy)^j (x + y)^{n-2j}, \tag{2.12}$$

where

$$\widetilde{\gamma}_{n,j}(\alpha, t) = \sum_{\sigma \in \mathcal{M}^{\mathrm{da}=0}_{n+1,\mathrm{asc}=j}} \alpha^{\mathrm{rmax}(\sigma)-1} t^{\mathrm{rmaxdd}(\sigma)} = \sum_{\sigma \in \mathfrak{S}^{\mathrm{cda}=0}_{n,\mathrm{exc}=j}} \alpha^{\mathrm{cyc}\,(\sigma)} t^{\mathrm{fix}(\sigma)}. \tag{2.13}$$

When $t = 1$, by complement operation $\sigma \mapsto \sigma^c$, the first combinatorial interpretations in (2.13) appeared in [13, Theorem 2.1]. When $\alpha = t = 1$, Equation (2.13) is equivalent to (1.5). More combinatorial interpretations of the γ-coefficients in (2.10) and (2.12) are given in [20, Theorem 2.5 and 2.6].

From Theorem 2.1, we can derive the exponential generating function for these two γ-coefficients, respectively.

Theorem 2.8. *Let $u = \sqrt{1-4x}$. We have*

$$1 + \sum_{n\geq 1} \sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_{n,j}(\alpha,t) x^j \frac{z^n}{n!} = \left(\frac{u\, e^{\frac{1}{2}(t-1)z}}{u\cosh(uz/2) - \sinh(uz/2)} \right)^{2\alpha}, \tag{2.14}$$

$$1 + \sum_{n\geq 1} \sum_{j=0}^{\lfloor n/2 \rfloor} \widetilde{\gamma}_{n,j}(\alpha,t) x^j \frac{z^n}{n!} = \left(\frac{u\, e^{(t-\frac{1}{2})z}}{u\cosh(uz/2) - \sinh(uz/2)} \right)^{\alpha}. \tag{2.15}$$

2.5 γ-vector of (α, t)-Eulerian polynomials and cycle André permutations

For $0 \leq j \leq \lfloor n/2 \rfloor$, let $d_{n,j}(\alpha,t) = \gamma_{n,j}(\alpha,t)/2^j$, then, Equation (2.10) reads

$$A_n(x,y,t \,|\, \alpha) = \sum_{j=0}^{\lfloor n/2 \rfloor} 2^j d_{n,j}(\alpha,t)(xy)^j (x+y)^{n-2j}. \tag{2.16}$$

From Theorem 2.2, we derive the continued fraction

$$\sum_{n=0}^{\infty} \sum_{j=0}^{\lfloor n/2 \rfloor} d_{n,j}(\alpha,t) x^j z^n = \cfrac{1}{1 - b_0 z - \cfrac{\lambda_1 z^2}{1 - b_1 z - \cfrac{\lambda_2 z^2}{1 - b_2 z - \cdots}}}, \tag{2.17}$$

where

$$b_k = k + \alpha t, \quad \lambda_{k+1} = \binom{k+1}{2} x + \alpha(k+1)x \quad (k \geq 0). \tag{2.18}$$

It follows that $d_{n,j}(\alpha,t)$ are polynomials in $\mathbb{N}[\alpha,t]$. The aim of this section is to provide three combinatorial interpretations in terms of André (resp. cycle André) permutations.

Definition 1 ([10]). *For a fixed $x \in [n]$, let $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$. Say that σ is an* ***André permutation of the first kind (resp. second kind)*** *if σ has no double descents, i.e., $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$, and each factorisation $u\,\lambda(x)\,x\,\rho(x)\,v$ of σ has property*

- *$\lambda(x) = \emptyset$ if $\rho(x) = \emptyset$,*
- *$\max(\lambda(x)) < \max(\rho(x))$ (resp. $\min(\rho(x)) < \min(\lambda(x))$) if $\lambda(x) \neq \emptyset$,*

where $\lambda(x)$ and $\rho(x)$ are the the maximal contiguous subword immediately to the left (resp. right) of x whose letters are all greater than x.

Let $\mathcal{A}_n^1$ (resp. $\mathcal{A}_n^2$) be the set of André permutations of the first (resp. second) kind in $\mathfrak{S}_n$. It is known [8] that the cardinality of $\mathcal{A}_n^1$ (resp. $\mathcal{A}_n^2$) is the *Euler number* E_n.

Let $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$. A right-to-left minimum (**rmin**) of σ is an element σ_i such that $\sigma_j > \sigma_i$ if $j > i$. A letter $\sigma_i \in [n]$ is a right-to-left-minimum-da (**rminda**) of σ if it is a double ascent and σ_i is a rmin. Let $A := \{a_1, \dots, a_k\}$ be a set of k positive integers. Let $C = (a_1, \dots, a_k)$ be a cycle (cyclic permutation) of A with $a_1 = \min\{a_1, \dots, a_k\}$. Then, cycle C is called an ***André cycle*** if the word $a_2 \dots a_k$ is an André permutation of the first kind. We say that a permutation is a **cycle André permutation** if it is a product of disjoint André cycles. Let $\mathcal{CA}_n$ be the set of cycle André permutations of $[n]$. Note that Hwang et al. [11] used cycle André permutations to characterise the so-called *Web permutations* to provide a combinatorial interpretation for entries of the transition matrix between the Specht and SL_2-web bases.

Theorem 2.9. *For $0 \le j \le \lfloor n/2 \rfloor$, we have*

$$d_{n,j}(\alpha, t) = \sum_{\substack{\sigma \in \mathcal{CA}_n \\ \mathrm{drop}(\sigma) = j}} t^{\mathrm{fix}(\sigma)} \alpha^{\mathrm{cyc}(\sigma)}, \tag{2.19a}$$

$$d_{n,j}(\alpha, t) = \sum_{\substack{\sigma \in \mathcal{A}_{n+1}^{(i)} \\ \mathrm{des}(\sigma) = j}} t^{\mathrm{rminda}(\sigma)} \alpha^{\mathrm{rmin}(\sigma) - 1}, \quad (i = 1, 2). \tag{2.19b}$$

We prove (2.19) by computing the exponential generating functions of both sides, and derive the $i = 1$ case of (2.19b) from (2.19a) by a bijection from $\mathcal{CA}_n$ to $\mathcal{A}_{n+1}^1$, and the $i = 2$ case by constructing another bijection from $\mathcal{A}_{n+1}^1$ to $\mathcal{A}_{n+1}^2$ via André trees, see [20, Section 4.3–4.5].

3 Proof outlines of Theorem 1.1 and 1.2

3.1 Preliminaries

Recall two variants θ_1 and θ_2 of Foata's fundamental transformation **FFT**. For $\sigma \in \mathfrak{S}_n$, the mapping $\theta_1 : \sigma \mapsto \theta_1(\sigma)$ (resp. $\theta_2 : \sigma \mapsto \theta_2(\sigma)$) goes as follows: (a) Factorize σ as product of disjoint cycles with the largest letter in the last (resp. first) position of each cycle; (b) Order the cycles from left to right in decreasing (resp. increasing) order of their largest letters, then erase the parentheses to obtain $\theta_1(\sigma)$ (resp. $\theta_2(\sigma)$). See [19, p. 30].

Fix a letter $x \in [n]$ and a permutation $\sigma \in \mathfrak{S}_n$, the *x-factorization* of σ is defined as the concatenation $\sigma = w_1 w_2\, x\, w_4 w_5$, where w_2 (resp. w_4) is the maximal contiguous subword immediately to the left (resp. right) of x whose letters are all smaller than x. Define the involution $\xi_x(\sigma) : \mathfrak{S}_n \to \mathfrak{S}_n$ by

$$\xi_x(\sigma) := \begin{cases} w_1 w_4\, x\, w_2 w_5, & \text{if } x \text{ is a double ascent or a double descent of } \sigma; \\ \sigma, & \text{if } x \text{ is a valley or a peak of } \sigma. \end{cases}$$

Here, we use the convention $\sigma(0) = \sigma(n+1) = \infty$. The involution $\xi_S := \prod_{x\in S}\xi_x$ ($S \subseteq [n]$) defines a $\mathcal{Z}_2^n$ action on $\mathfrak{S}_n$ [1], called the *modified Foata and Strehl action* or *MFS* action, see [9]. Cooper et al. [6, Propositions 3 and 4] defined the *cyclic valley hopping* $\psi_x(\sigma) : \mathfrak{S}_n \to \mathfrak{S}_n$ by

$$\psi_x(\sigma) := \begin{cases} \theta_2^{-1} \circ \xi_x \circ \theta_2(\sigma), & \text{if } x \text{ is not a fixed point of } \sigma; \\ \sigma, & \text{if } x \text{ is a fixed point of } \sigma, \end{cases}$$

where we treat the 0-th letter of $\theta_2(\sigma)$ as 0 and the (n+1)-th letter as ∞. Define the involution $\psi_S := \prod_{x\in S}\psi_x (S \subseteq [n])$.

3.2 Proof of Theorem 1.1

For any permutation $\sigma \in \mathfrak{S}_n$, denote the orbit of σ under cyclic valley-hopping by $\mathrm{Orb}(\sigma) := \{\psi_S(\sigma) | S \subseteq [n]\}$, which has a unique permutation $\bar{\sigma}$ without cyclic double descents. By [6, Propositions 3 and 4], we have

$$\sum_{\pi\in\mathrm{Orb}(\sigma)} (u_1u_2)^{\mathrm{cpk}(\pi)} u_3^{\mathrm{cda}(\pi)} u_4^{\mathrm{cdd}(\pi)} t^{\mathrm{fix}(\pi)} \alpha^{\mathrm{cyc}(\pi)} = (u_1u_2)^{\mathrm{cpk}(\bar{\sigma})} (u_3+u_4)^{\mathrm{cda}(\bar{\sigma})} t^{\mathrm{fix}(\bar{\sigma})} \alpha^{\mathrm{cyc}(\bar{\sigma})}. \tag{3.1a}$$

By definition, it is clear that for $\sigma \in \mathfrak{S}_n$,

$$\mathrm{exc}(\sigma) = \mathrm{cda}(\sigma) + \mathrm{cpk}(\sigma) \quad \text{and} \quad \mathrm{drop}(\sigma) = \mathrm{cdd}(\sigma) + \mathrm{cval}(\sigma). \tag{3.1b}$$

Setting $u_1 = u_3 = x$ and $u_2 = u_4 = y$ in (3.1a) yields

$$\sum_{\pi\in\mathrm{Orb}(\sigma)} x^{\mathrm{exc}(\pi)} y^{\mathrm{drop}(\pi)} t^{\mathrm{fix}(\pi)} \alpha^{\mathrm{cyc}(\pi)} = (xy)^{\mathrm{cpk}(\bar{\sigma})} (x+y)^{\mathrm{cda}(\bar{\sigma})} t^{\mathrm{fix}(\bar{\sigma})} \alpha^{\mathrm{cyc}(\bar{\sigma})}. \tag{3.1c}$$

Thus, if $u_1u_2 = xy$ and $u_3 + u_4 = x + y$, combining (3.1a) and (3.1c) we have

$$\sum_{\pi\in\mathrm{Orb}(\sigma)} (u_1u_2)^{\mathrm{cpk}(\pi)} u_3^{\mathrm{cda}(\pi)} u_4^{\mathrm{cdd}(\pi)} t^{\mathrm{fix}(\pi)} \alpha^{\mathrm{cyc}(\pi)} = \sum_{\pi\in\mathrm{Orb}(\sigma)} x^{\mathrm{exc}(\pi)} y^{\mathrm{drop}(\pi)} t^{\mathrm{fix}(\pi)} \alpha^{\mathrm{cyc}(\pi)}.$$

Then summing over all the orbits of $\mathfrak{S}_n$ gives Equation (1.11). □

3.3 Proof of Theorem 1.2

For a finite set of positive integers E, we denote by $\mathfrak{S}_E$ the set of permutations of E. For $\sigma \in \mathfrak{S}_E$ define the weight function

$$w(\sigma; \alpha, a, b) = (u_1u_2)^{\mathrm{cpk}(\sigma)} u_3^{\mathrm{cda}(\sigma)} u_4^{\mathrm{cdd}(\sigma)} a^{\mathrm{fix}(\sigma)} b^{\mathrm{cyc}(\sigma)-\mathrm{fix}(\sigma)} \alpha^{\mathrm{cyc}(\sigma)}, \tag{3.2a}$$

$$w_1(\sigma; \alpha, a, b, t) = w(\sigma; \alpha, a, b)\, t^{\mathrm{lmaxda}(\sigma)}, \tag{3.2b}$$

$$w_2(\tau; \alpha, a, b, t) = w(\sigma; \alpha, a, b)\, t^{\mathrm{rmaxdd}(\tau)}. \tag{3.2c}$$

Let $\mathfrak{S}_n^\star$ be the set of permutations in $\mathfrak{S}_n$ of which each cycle has a color in $\{Red, Blue\}$. An element of $\mathfrak{S}_n^\star$ is called a *cycle decorated permutation* (CDP). Hence, a CDP $\pi \in \mathfrak{S}_n^\star$ is in bijection with a pair of permutations $(\sigma, \tau) \in \mathfrak{S}_A \times \mathfrak{S}_B$ such that (A, B) is an ordered set partition of $[n]$ and $\pi = \sigma\tau$, namely, the permutation σ consists of all red cycles and τ the blue ones. By Theorem 1.1, under the assumption that $xy = u_1u_2$ and $x + y = u_3 + u_4$, the left-hand side of equality (1.13) can be written as

$$\begin{aligned} &A_n^{\mathrm{cyc}}\left(x, y, t(\alpha u_3 + \beta u_4)(\alpha f + \beta g)^{-1} \,|\, \alpha f + \beta g\right) \\ &\quad = \sum_{\pi \in \mathfrak{S}_n} (u_1u_2)^{\mathrm{cpk}(\pi)} u_3^{\mathrm{cda}(\pi)} u_4^{\mathrm{cdd}(\pi)} (t(\alpha u_3 + \beta u_4))^{\mathrm{fix}(\pi)} (\alpha f + \beta g)^{\mathrm{cyc}(\pi) - \mathrm{fix}(\pi)} \\ &\quad = \sum_{(\sigma,\tau) \in \mathfrak{S}_n^\star} w_1(\sigma; \alpha, u_3, f, t)\, w_2(\tau; \beta, u_4, g, t). \end{aligned} \tag{3.3}$$

We define a mapping $\rho : \mathfrak{S}_n^\star \to \mathfrak{S}_{n+1}$ as follows:

$$(\sigma, \tau) \mapsto \widetilde{\pi} := \theta_2(\sigma)\, x\, \theta_1(\tau) \quad \text{with } x = n + 1, \tag{3.4}$$

where θ_1 and θ_2 are the FFT, see Section 3.1. Clearly this is a bijection. It follows (1.13) from combining ρ and (3.3).

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