Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #42, 12 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

3D permutations and triangle solitaire

Juliette Schabanel*[1]

[1]*LaBRI, Université de Bordeaux*

Abstract. We provide a bijection between a class of 3-dimensional pattern avoiding permutations and triangle bases, special sets of integer points arising from the theory of TEP subshifts. This answers a conjecture of Bonichon and Morel.

Keywords: d-permutations, tilings, bijection, patterns in permutation, subshifts

1 Introduction

In this paper, we build a bijection between objects that are seemingly unrelated: a class of pattern avoiding 3-dimensional permutations and objects arising from the theory of subshifts called triangle bases, that was conjectured by Bonichon and Morel in [6].

Permutations are a central object in combinatorics and their study has received a lot of attention. One topic of particular interest is pattern avoidance in permutations, which led to numerous enumerative and bijective results with various objects (see [10, 5] and references therein).

One can see a permutation σ as a (2-dimensional) diagram with points $(i, \sigma(i))$, which satisfies the property that in each row and column lies a unique point. With this definition comes a natural generalization to higher dimensions by defining a d-dimensional diagram, and the notion of pattern avoidance extends naturally [2, 6, 1].

The second object arises from the theory of *tilings and symbolic dynamics*. A tiling is a coloring of the grid $\mathbb{Z}^2$ with allowed subpatterns determined by a rule set $\mathcal{R}$. The set of tilings respecting a certain rule set is called a *subshift* [8]. Tilings have been widely studied during the last decades and many questions, such as the *tiling problem*, were proved undecidable [4]. However, for some classes with strong structure or combinatorial properties, such as cellular automata, some problems become decidable and a deeper study can be carried [9]. The class we consider in this paper, named TEP for *totally extremally permutive*, was introduced by Salo in [11] and generalizes bipermutive cellular automata. The language of any TEP subshift is decidable, and they possess bases, i.e. sets of cells whose content can be chosen freely and determine the values of a larger set. Our object of interest are the bases of triangles for the *Ledrappier subshift*.

In [6], Bonichon and Morel considered 3-permutations avoiding small patterns and enumerated the first terms of these classes. It turned out that several of those sequences

*juliette.schabanel@u-bordeaux.fr

matched existing sequences in the OEIS[1]. In particular, the sets of triangle bases of size n and $Av_n((12,12),(312,231))$ have the same cardinalities up to $n=8$, and they conjectured that these two sets are in bijection. Our main result is to prove their conjecture:

Theorem 1.1. *There is an explicit bijection between 3-permutations of size n avoiding patterns $(12,12)$ and $(312,231)$ and triangle bases of size n.*

This bijection makes a link between objects from two previously independent classes which are hard to enumerate, and it allows to transport structure properties and methods from one class to the other, such as a random sampling method relying on a Markov chain. In addition to that, the construction is quite simple and could be applied to other classes of pattern avoiding d-permutations (even for $d>3$).

Precise definitions are given Section 2 and the construction is explained Section 3. This bijection is quite simple in its construction and brings keys to understand better those two objects. In the process of the proof, we also provide a description of $Av_n((12,12),(312,231))$ through sums analogous to the one of separable permutations but with a shift. In Section 4, we describe a dynamical system on bases, the *solitaire*, which translates to permutations though the bijection and allows uniform sampling.

2 Definitions and Setting

2.1 3-Permutations avoiding a pattern

A permutation $\sigma=\sigma(1)\sigma(2)\ldots\sigma(n)\in\mathfrak{S}_n$ is a bijection from $[\![1,n]\!]=\{1,2,\ldots n\}$ to itself. The (2 dimensional) *diagram* of a permutation $\sigma\in\mathfrak{S}_n$ is the set of points $P_\sigma:=\{(1,\sigma(1)),\ldots,(n,\sigma(n))\}$. The diagrams of permutations of size n are exactly the point sets such that each row and column of $[\![1,n]\!]^2$ contains exactly one point.

A permutation σ *contains* a *pattern* $\pi\in\mathfrak{S}_k$ if there is a set of indices $i_1<i_2<\ldots<i_k$ such that $\sigma(i_1)\sigma(i_2)\ldots\sigma(i_k)=\pi$ (once standardized). Otherwise, we say that σ *avoids* π. Given a set of patterns $\pi_1,\ldots,\pi_k$, we denote by $Av_n(\pi_1,\ldots,\pi_k)$ the set of permutations of size n avoiding all the π_is.

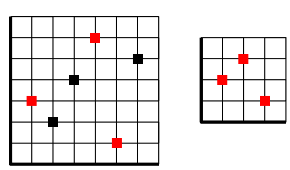

Figure 1: The permutation $\sigma=324615$ with an occurrence of the pattern 231 in red.

[1] such as $Av((12,12))$ matching the number of intervals of the weak Bruhat order, for which they were able to provide a bijection

Definition 2.1. *A d-*permutation *(or d-*dimensional permutation*) of size n, is a tuple $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_{d-1})$ of $d-1$ permutations of size n . We denote by $\mathfrak{S}_n^{d-1}$ their set. The* diagram *of a d-permutation is the set of points $P_{\boldsymbol{\sigma}} := \{(i, \sigma_1(i), \ldots, \sigma_{d-1}(i)) \mid 1 \leqslant i \leqslant n\} \in \mathbb{Z}^d$.*

The diagrams of d-permutations of size n are exactly the point sets such that every hyperplane $x_i = j$ with $i \in [\![1, d]\!]$ and $j \in [\![1, n]\!]$ contains exactly one point. Examples are given in Figure 2.

The definition of pattern avoidance extends naturally to d-permutations as follows. Let $\boldsymbol{\pi} \in \mathfrak{S}_k^{d-1}$ be a pattern, a d-permutation $\boldsymbol{\sigma} \in \mathfrak{S}_n^{d-1}$ *contains* $\boldsymbol{\pi}$ if there is a set of indices $I \subset [\![1, n]\!]$ such that $\boldsymbol{\sigma}_{|I} = \boldsymbol{\pi}$ (once standardized), otherwise it *avoids* it. Given a set of d-patterns $\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi_k}$ we denote by $Av_n(\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_k)$ the set of d-permutations of size n that avoid all the $\boldsymbol{\pi}_i$.

In what follows, we focus on 3-permutations avoiding two particular patterns : $(12, 12)$ and $(312, 231)$ which are depicted Figure 2.

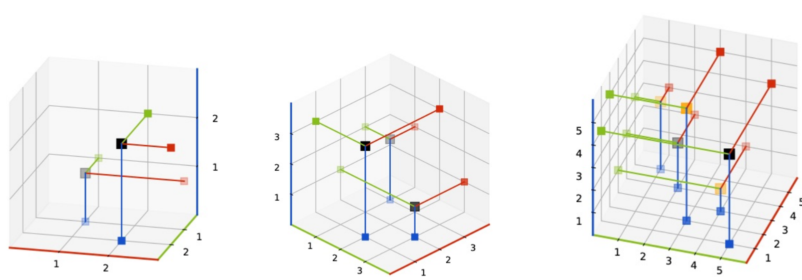

Figure 2: Left : The two forbidden patterns : $(12, 12)$ (left) and $(312, 231)$ (right). Right : a 3-permutation, $(54231, 32514)$, with an occurrence of $(312, 231)$ in orange

Remark 2.2. *Note that a* 3*-permutation avoiding a* 3*-pattern is not the same as a couple of permutations each avoiding a pattern as we require the occurrence to be on the same indices. For example* $(312, 231)$ *avoids* $(12, 12)$ *although both* 312 *and* 231 *contain* 12.

2.2 Tilings, configurations and bases

Our other class of objects, *triangle bases*, arises from the study of tilings which we will now introduce.

2.2.1 Tilings

Let A be a set of symbols and S be a finite subset of $\mathbb{Z}^2$. Let $\mathcal{R}$ be a subset of A^S. A *tiling* with *tile set* A and *allowed patterns* $\mathcal{R}$ is a coloring $c : \mathbb{Z}^2 \to A$ of the grid $\mathbb{Z}^2$ with symbols of A such that for each translation S' of S, $c_{|S'} \in \mathcal{R}$. In other words, we require that wherever one looks at the coloring through an S-shaped window, what one sees is

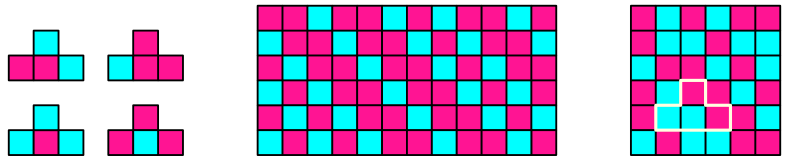

Figure 3: An example of a tiling. Left: the rule set $\mathcal{R}$. Middle: a valid tiling. Right: an invalid tiling with a forbidden pattern highlighted.

in $\mathcal{R}$. The elements of $S^A \setminus \mathcal{R}$ are called the *forbidden patterns* and the set of valid tilings for $\mathcal{R}$ the *subshift* (of finite type) with rule set $\mathcal{R}$.

In what follows, we consider a special kind of subshifts, called *TEP subshifts* (where TEP stands for Totally Extremally Permutive), which were introduced in [11]. A subshift is TEP if for all $x \in S$, for each partial coloring $c : S \setminus \{x\} \to A$, there is a unique $a \in A$ such that the extension of c with $c(x) = a$ is in $\mathcal{R}$. In other words, if one fills all cells of S but one with arbitrary symbols of A, there is a unique way to complete it so that this does not create a forbidden pattern. An example ,the XOR automaton tiling, is given Figure 4. In what follows, we only consider TEP subshifts with $S = \{(0,0),(1,0),(0,1)\}$.

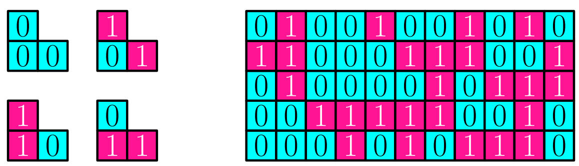

Figure 4: Left: The tiling rules of the XOR automaton. Right: An example of a valid tiling. Adding or removing a pattern from $\mathcal{R}$ breaks the TEP property.

2.2.2 Filling

When confronted to a TEP subshift, one natural question to ask is "given a set of cells P whose content are known, what other values can be deduced ?". The following definition aims at formalizing this notion.

Definition 2.3. *Let $P \subset \mathbb{Z}^2$ be a set of cells. Performing a* filling step *of P consists in choosing a position (x,y) such that $|P \cap \{(x,y),(x+1,y),(x,y+1)\}| = 2$ and adding the missing point to P. This process is confluent and converges to a limit set denoted $\varphi(P)$ called the* filling *of P.*

Intuitively, if we know the values of a tiling on P, then a filling step consists in using the tiling rules to deduce the value of a new cell and adding it to the known set. The filling of P is then the set of values which can always be deduced from P.

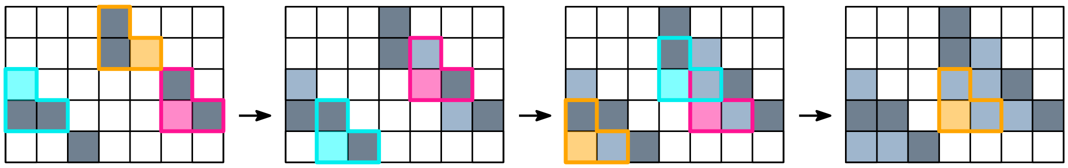

Figure 5: An example of application of the filing process. Dark cells are the original set, colored ones are added by the current step and light cells were filled earlier.

2.2.3 Independence and Bases

The other natural question is "given a set of cells P, can any choice of symbols for P be extended into a valid tiling of $\mathbb{Z}^2$?". When it is the case, we say that P is *independent*.

Denote $T_n = \{(x, y) \mid x, y \in \mathbb{N}, x + y < n\}$ the triangle of size n. A set of cells P is a *basis* (of T_n) if for any partial coloring $c : P \to A$, by iteratively deducing value using the TEP rules of $\mathcal{R}$ we always end up with a valid coloring of T_n (i.e. all of T_n is determined by P and there is no conflict). We denote by $\mathcal{B}_n$ the set of bases of T_n. We call *configuration* of size n a set of n points $C \subset T_n$, those are the candidates for bases.

Theorem 2.4 ([12])**.** *A configuration of size n is a triangle basis if and only if its filling is T_n.*

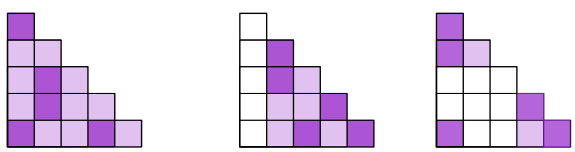

Figure 6: Configurations of size 5 (dark cells) with their filling (light cells).
Left : A basis of T_5. Middle : A non independent pattern. Right : A non filling pattern.

The idea is that n is the minimal amount of information needed to fill T_n, and if there is a conflict then one point is redundant and there is not enough left to fill all of T_n.

The number of triangle bases for n up to 8 were computed in [11] and are the following: 1, 3, 16, 122, 1188, 13844, 185448, 2781348, . . .

3 The bijection

In this section, we define a function that send 3-permutations to configurations and prove that it is a bijection between $Av_n((12, 12), (312, 231))$ and the set of triangle bases of size n. This function relies on the inversion sequence of the permutations, which we will now define.

3.1 Inversions

Let $\sigma \in \mathfrak{S}_n$ be a permutation. For $i \in [\![1, n]\!]$, we denote by $r_\sigma(i)$ (resp. $l_\sigma(i)$) the number of $i < j \leqslant n$ (resp. $1 \leqslant j < i$) such that $\sigma(i) > \sigma(j)$ (resp. $\sigma(i) < \sigma(j)$). These are called the number of right (resp. left) *inversions* of σ at i and the sequence $(r_\sigma(i))_{1\leqslant i\leqslant n}$ (resp. $(l_\sigma(i))_{1\leqslant i\leqslant n}$) the right (resp. left) *inversion sequence* of sigma.

Looking at the diagram of σ, for each i $r_\sigma(i)$ is the number of points to the bottom right of $(i, \sigma(i))$ and $l_\sigma(i)$ the number of points to the top left of it (see Figure 7). We denote respectively by $R_\sigma(i)$ and $L_\sigma(i)$ those sets of points.

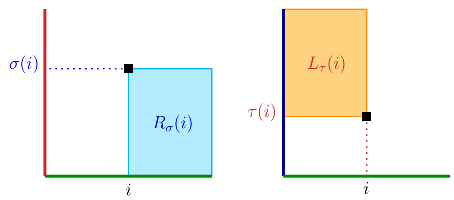

Figure 7: The set $R_\sigma(i)$ (resp. $L_\tau(i)$) is the points in the blue (resp. orange) area.

Proposition 3.1. *The mapings $\sigma \mapsto (r_\sigma(i))_{1\leqslant i\leqslant n}$ and $\sigma \mapsto (l_\sigma(i))_{1\leqslant i\leqslant n}$ are bijective.*

3.2 The function Γ

Let Γ be the function that maps a 3-permutation $(\sigma, \tau) \in \mathfrak{S}_n^2$ to the configuration composed of the points $(r_\sigma(i), l_\tau(i))$ for $1 \leqslant i \leqslant n$. When there is no ambiguity, we write $x_i = r_\sigma(i)$, $y_i = l_\tau(i)$ and denote p_i the point $(x_i, y_i) \in \Gamma(\sigma, \tau)$.

Example 3.2. *Let us consider the 3-permutation (254361, 624315). Its inversion sequences are (132110, 011241) and its image through Γ is depicted Figure 8.*

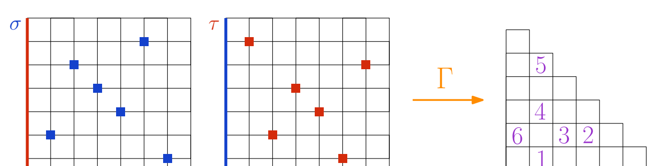

Figure 8: The diagram of the 3-permutation (254361, 624315) and its image through Γ.

Proposition 3.3. *For all $(\sigma, \tau) \in Av_n((12, 12))$, $\Gamma(\sigma, \tau)$ is a configuration of size n.*

Proof. If $(\sigma, \tau) \in \mathfrak{S}_n^2$ avoids $(12, 12)$ then for all $i < j$, either $\sigma(i) > \sigma(j)$ and $r_\sigma(i) > r_\sigma(j)$ or $\tau(i) > \tau(j)$ and $l_\tau(i) < l_\tau(j)$, hence all p_i's are distinct. $\square$

Remark 3.4. Γ *in fact sends* 3*-permutations to* labeled *configuration (by labeling point with the integer used to compute its coordinates). Observe that if points are labeled, we can recover the inversion sequences and so the permutations. However, our triangle bases are not labeled so we have to forget this labeling and find a way to recover it. Still, the label variant might be useful for some generalizations (see Section 5).*

In the rest of this section, we will prove that when $(\sigma, \tau) \in Av_n((12, 12), (312, 231))$, $\Gamma(\sigma, \tau)$ is a triangle basis. Intuitively, avoiding $(12, 12)$ ensures that the points are "not too close" and avoiding $(312, 231)$ that they are "not too far". In the two following subsections we formalize these notions and prove that this is true.

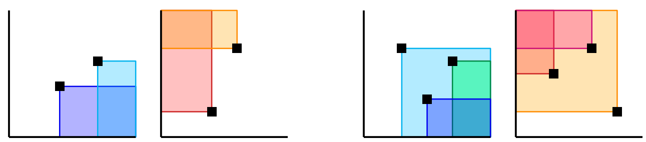

Figure 9: The positions forbidden by the patterns $(12, 12)$ (left) and $(312, 231)$ (right).

3.3 Sparsity

For integers a, b and k, we denote by $(a, b) + T_k$ the triangle $\{(x, y) \mid a \leqslant x, b \leqslant y \text{ and } x + y < a + b + k\}$. A configuration C is *sparse* if for all $1 \leqslant k < n$, there is no triangle $T = (a, b) + T_k$ such that $|C \cap T| > k$.

Proposition 3.5 ([12])**.** *All independent sets (and so all bases) are sparse.*

Intuitively, if a configuration is not sparse then we have too much information in one area and it creates a default of independence.

Theorem 3.6. *If* $(\sigma, \tau) \in \mathfrak{S}_n^2$ *avoids* $(12, 12)$ *then* $\Gamma(\sigma, \tau)$ *is sparse.*

We omit the proof of this theorem as it is not needed to prove that Γ is a bijection. Nonetheless, this theorem helps to understand the consequences of avoiding $(12, 12)$ on the image through Γ and can be useful for potential generalizations of this function to some other classes (see Section 5).

3.4 Cuts and shifted sums

To prove that Γ is a bijection, we will give a recursive definition of triangle bases, define an analogous decomposition for 3-permutations avoiding $(12, 12)$ and $(312, 231)$ and prove that Γ transports these decompositions.

Definition 3.7. *Let* C_1 *and* C_2 *be two configurations and* $h \in [\![0, |C_1|]\!]$*. The h-vertically* shifted sum *of* C_1 *and* C_2 *is* $C_1 \oplus_h C_2 = ((|C_2|, 0) + C_1) \cup ((0, h) + C_2)$*. It is a configuration of size*

$|C_1| + |C_2|$. *Similarly, we define the h-horizontally* shifted sum *by* $C_1 \ominus_h C_2 = ((0, |C_2|) + C_1) \cup ((h, 0) + C_2)$ *and the h-diagonally* shifted sum *by* $C_1 \oslash_h C_2 = C_1 \cup ((|C_1| - h, h) + C_2)$.

If C is a configuration, a couple of subconfigurations of C, (C_1, C_2)*, is a vertical (resp. horizontal, resp. diagonal)* cut *of C if C is a vertical (resp. horizontal, resp. diagonal) shifted-sum of* C_1 *and* C_2*. Its position is the coordinate of the line separating* C_1 *and* C_2*.*

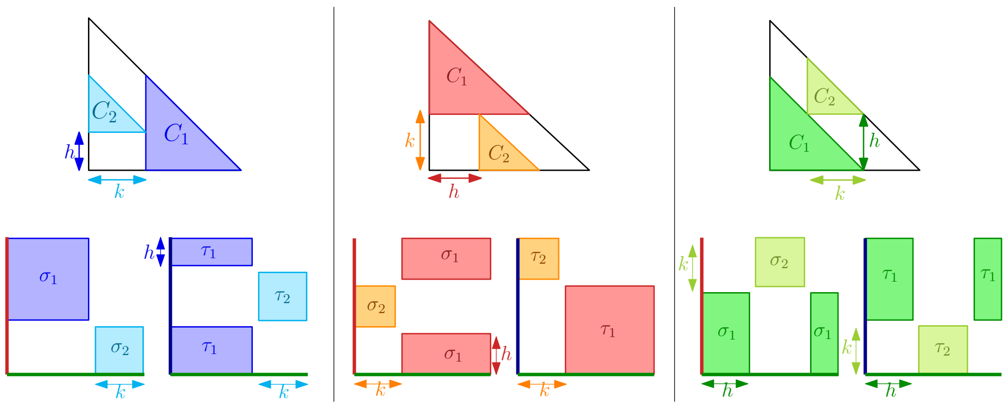

Figure 10: Shifted sums of configurations and 3-permutations. Left : Vertical sum. Middle : Horizontal. Right : Diagonal. Parameters k and h are preserved through Γ.

Proposition 3.8. *A configuration is a triangle basis if and only if it is either a single point or a shifted sum of two triangle bases.*

Proof. If $C = B_1 \odot B_2$, then each B_i fills a triangle of its size and those triangles touch, so C fills the smallest triangle containing them, which has size $|C|$. So C is also a basis. $\square$

We now define the shifted sums on 3-permutations and show that Γ transports the cuts.

Definition 3.9. *Let* $(\sigma_1, \tau_1) \in \mathfrak{S}^2_{k_1}$ *and* $(\sigma_2, \tau_2) \in \mathfrak{S}^2_{k_2}$ *be two* 3*-permutations and* $h \in [\![0, k_1]\!]$*. Their h-shifted sums are defined as the* 3*-permutations* (σ, τ) *of size* $k_1 + k_2$ *obtained by inserting the diagram* (σ_2, τ_2) *into the one of* (σ_1, τ_1) *as described in Figure 10. As for configuration, we say that a* 3*-permutation admits a cut if it is a shifted sum of two* 3*-permutations. We denote by* $\odot$ *a shifted sum of unknown direction and shift.*

Lemma 3.10. *Let* $(\sigma, \tau) \in Av_n((12, 12))$*. It admits a cut* $(\sigma, \tau) = (\sigma_1, \tau_1) \odot (\sigma_2, \tau_2)$ *in a given direction, at a position k with shift h if and only* $\Gamma(\sigma, \tau)$ *does. In that case,* $\Gamma((\sigma_1, \tau_1) \odot (\sigma_2, \tau_2)) = \Gamma(\sigma_1, \tau_1) \odot \Gamma(\sigma_2, \tau_2)$*.*

Proof. Let $(\sigma, \tau) \in Av_n((12, 12))$. We only consider the vertical cut case $(\sigma, \tau) = (\sigma_1, \tau_1) \obar_h (\sigma_2, \tau_2)$, the others can be obtained with similar arguments. Denote k the size of (σ_1, τ_1), for all $i \in [\![1, k]\!]$ $R_\sigma(i) = R_{\sigma_1}(i) \cup [\![k+1, n]\!]$ and $L_\tau(i) = L_{\tau_1}(i)$ so the first k

points of $\Gamma(\sigma, \tau)$ are $\Gamma(\sigma_1, \tau_1) + (n-k, 0)$. And for all $i \in [\![k+1, n]\!]$, $R_\sigma(i) = R_{\sigma_2}(i)$ and $L_\tau(i) = L_{\tau_1}(i) + h$ so the last $n-k$ points of $\Gamma(\sigma, \tau)$ are $\Gamma(\sigma_2, \tau_2) + (0, h)$. This is exactly $\Gamma((\sigma_1, \tau_1) \oplus_h (\sigma_2, \tau_2)) = \Gamma(\sigma_1, \tau_1) \odot \Gamma(\sigma_2, \tau_2)$.

Reciprocally, assume that $\Gamma(\sigma, \tau) = C_1 \oplus_h C_2$. All points in C_1 have abscissa at least $n-k$, so they must be $[\![1, k]\!]$ since for all i, $x_i < n - i$. We first need to prove that for all $i \in [\![1, k]\!]$, $[\![k+1, n]\!] \subset R_\sigma(i)$. Equality holds for $i = k$, and then for the others by induction: for $i \leqslant k$, either there is a $j > i$ such that $j \in R_\sigma(i)$ and so $[\![k+1, n]\!] \subset R_\sigma(j) \subset R_\sigma(i)$ or there is none and $R_\sigma(i) = [\![k+1, n]\!]$. Now consider the points M and m where τ reaches respectively it maximum and minimum on $[\![k+1, n]\!]$, $l_\tau(M) = h$ so there must be exactly h points in $[\![1, k]\!]$ above $(M, \tau(M))$. Now for m, all integers greater than k contribute to either x_m or y_m since (σ, τ) avoids $(12, 12)$ and $L_\tau(M) \subset L_\tau(m)$ since $L_\tau \subset [\![1, k]\!]$ and $\tau(m) \leqslant \tau(M)$. So $x_m + y_m = n - k - 1 + |L_\tau(m) \cap [\![1, k]\!]| \geqslant n - k - 1 + h$ but $x_m + y_m < n - k + h$ so equality must hold, which means $L_\tau(m) \cap [\![1, k]\!] = L_\tau(M)$. So (σ, τ) admits a vertical cut at index k with shift h. $\square$

All that remains to prove now is that when (σ, τ) avoids $(12, 12)$ and $(312, 231)$, it admits a cut. An induction will then give that $\Gamma(Av_n((12, 12), (312, 231))) = \mathcal{B}_n$.

Lemma 3.11. *For all $n \geqslant 2$, if $(\sigma, \tau) \in Av_n((12, 12), (312, 231))$ then (σ, τ) admits a cut.*

Proof. We proceed by induction. One can easily check that all permutations of $\mathfrak{S}_n^2$ avoiding $(12, 12)$ admit a cut (even two).

Let $n > 2$ and $(\sigma, \tau) \in Av_n((12, 12), (312, 231))$. Observe that if n is an extrema for either σ or τ, then isolating n is a valid cut : if $\sigma(n) = 1$ (resp $\tau(n) = 1$), there is a vertical (resp. horizontal) cut and if $\sigma(n) = n$ (resp $\tau(n) = n$) then $\tau(n) = 1$ (resp $\sigma(n) = 1$).

Now assume n is not an extrema. For $1 \leqslant i \leqslant n$, denote by (σ_i, τ_i) the 3-permutation obtained by deleting n. By induction hypothesis, it admits a shifted cut. We denote by k_i its position and h_i its shift. To simplify, we do not standardize (σ_i, τ_i). Consider the cut of (σ_n, τ_n) and assume that n is not compatible with the cut, otherwise we are done, and that the cut is vertical (the other directions can be treated similarly).

If n is in the bottom right sector of σ, then it must be under the right block of τ (or there would be a $(12, 12)$), but then one can take $i \leqslant k_n$ such that $\tau(n) < \tau(i) < h_n$ and j such that $\sigma(j) < \sigma(n)$ which is an occurrence of $(312, 231)$. Now assume n is in the top right sector of σ, n must still be under the right block of τ. Let $m = \sigma^{-1}(1) \neq n$ and consider the cut of (σ_m, τ_m).

1. (σ_m, τ_m) **admits a vertical cut.** Then n must be in the bottom right sector of σ so $k_m < k_n$ and so the right block of τ in this cut must contain the one of the first cut, thus m is compatible with the cut.

2. (σ_m, τ_m) **admits a horizontal cut.** Then since $h_n = 0$ (or we would have $\tau(n) = 1$) $k_m < k_n$ or there would be empty planes, so all the right block of τ_n must be in the bottom right sector of τ_m and therefore m is compatible with the cut.

3. (σ_m, τ_m) **admits a diagonal cut.** Denote I its middle interval. It is impossible to have $n \in I$ because that would require an empty column in σ. This means that $I \subset [\![1, k_n]\!]$ and so m is compatible with this cut.

□

Combining the two previous lemmas, we get the following :

Theorem 3.12. *A 3-permutation avoiding $(12, 12)$ and $(312, 231)$ is either $(1, 1)$ or a shifted sum of two permutations in $Av((12, 12),(312, 231))$.*

Corollary 3.13. *For all $(\sigma, \tau) \in Av_n((12, 12),(312, 231))$, $\Gamma(\sigma, \tau)$ is a triangle basis.*

Proof. This is true for $n = 1$. If $n \geqslant 2$, then (σ, τ) admits a cut. By induction, the image of each side of the cut is a basis and so $\Gamma(\sigma, \tau)$ is a shifted sum of two bases, so a basis. □

Theorem 3.14. *For $n \geqslant 1$, Γ is a bijection between $Av_n((12, 12),(312, 231))$ and bases of size n.*

Proof. The decomposition with shifted sums allows to inverse Γ as when a basis B is the sum of two others, we know exactly which labels should appear in which subconfiguration, so we can recover all labels this way. A pre-image of B by Γ can then be reconstruct from the inversion sequences.

As for injectivity, although the decomposition is not unique, we can choose a canonical cut, for instance prioritize vertical over horizontal over diagonal and minimize the size of C_1. Since a 3-permutation and its image throught Γ admit exactly the same cuts, this ensures that no two 3-permutations give the same basis which different labels. □

4 Solitaire

In this section, we introduce a dynamical system on configurations called the *triangle solitaire* (defined in [11]) and we extend it to 3-permutations. This dynamical system presents nice properties that makes it useful for random generation purposes, or to extend our bijection to other pattern avoiding classes.

Definition 4.1. *Let P and Q be sets of points. There is a* solitaire move *from P to Q, denoted $P \to Q$, if there is a position $(x, y) \in \mathbb{Z}^2$ such that their symmetric difference $P \Delta Q$ is a subset of size 2 of $\{(x, y),(x + 1, y),(x, y + 1)\}$. If P is a configuration, its* orbit*, denoted $\mathcal{O}(P)$ is the set of configurations reachable from it using solitaire moves.*

Intuitively, seeing $\mathbb{Z}^2$ as a playing grid and P and Q as marbles on this grid, when two marbles are adjacent a solitaire move allows to move one using the other as a “pivot”.

Theorem 4.2 ([12])**.** *For $n \geqslant 1$, the orbit of the line $[\![0, n - 1]\!] \times \{0\}$ is exactly the bases $\mathcal{B}_n$.*

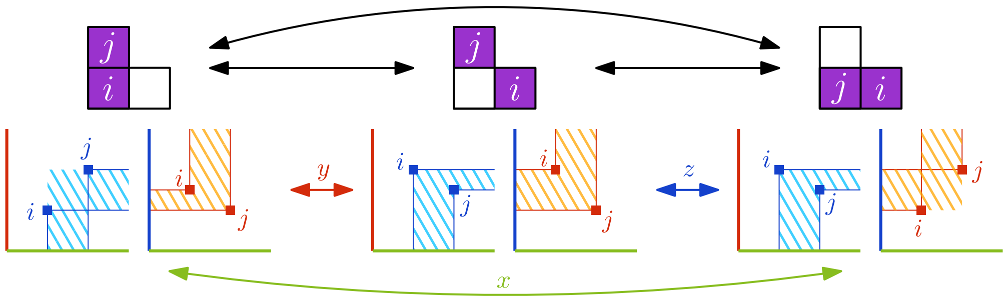

Figure 11: The solitaire on the grid and the corresponding situation on 3-permutations avoiding $(12, 12)$ with $i < j$. Hashed areas must be empty for the move to be allowed. One moves from one position to the other by exchanging the coordinates of points i and j along the indicated axis in the diagram.

The solitaire extends naturally to labeled configurations, it can then be pulled back through Γ to obtain a dynamical system on $Av((12, 12))$. See Figure 11.

Denote $\overline{\mathrm{id}}$ the permutation $n(n-1)(n-2)\dots 21$.

Theorem 4.3. *For all $n \geqslant 1$, $\mathcal{O}((\overline{id}, id)) = Av_n((12, 12), (312, 231))$.*

5 Discussion

Enumerative bounds: We now know that $Av_n((12, 12), (312, 231))$ and the set of triangle bases of size n have the same cardinality, but the question of its value remains open. The difficulty comes the lack of nice recursive decomposition of the objects.

In [12], we gave the following bounds on triangle bases, which transfer to the class $Av_n((12, 12), (312, 231))$ through our bijection. To our knowledge, those are the first bounds on this class.

$$3n! \leqslant |\mathcal{B}_n| \leqslant c \left(\frac{e}{2}\right)^n n^{n-\frac{5}{2}} \text{ with } c > 0.$$

Random sampling: The solitaire defines a Markov chain on triangle bases. The diameter of the reconfiguration graph is $O(n^3)$ [12], and we conjecture that its mixing time is also *polynomial* since the graph has strong connectivity properties. This would allow to sample uniformly triangle bases, and hence 3-permutations of $Av_n((12, 12), (312, 231))$ by our bijection.

Other pattern avoiding classes: Our function Γ is well defined and invertible on all of $Av((12, 12))$ if we consider labeled configurations, so it might be use to study bijectively other sub classes of $Av((12, 12))$. For this, studying the orbits of the solitaire might be relevant. Γ can also be generalized to d-permutations by using each permutation to compute a coordinate, the image set would then be configurations of dimension $d-1$.

Large random permutations: The combinatorial (and especially bijective) understanding of pattern avoiding permutations gives detailed information about their structure, which can be applied to study the properties of large random objects. For instance, the theory of *permutons* describes the scaling limit of many classes of pattern avoiding permutations (see e.g. [3]). One can ask the same questions about large random 3-permutations [7], and perhaps our bijection can help study the properties of large random permutations of $Av_n((12, 12), (312, 231))$.

References

[1] R. Aldred, M. Atkinson, H. van Ditmarsch, C. Handley, D. Holton, and D. McCaughan. "Permuting machines and priority queues". *Theoret. Comput. Sci.* **349**.3 (2005), pp. 309–317. DOI.

[2] A. Asinowski and T. Mansour. "Separable d-Permutations and Guillotine Partitions". *Ann. Com.* **14**.1 (2010), pp. 17–43. DOI.

[3] F. Bassino, M. Bouvel, V. Féray, L. Gerin, M. Maazoun, and A. Pierrot. "Universal limits of substitution-closed permutation classes". *J. Eur. Math. Soc.* **22**.11 (2020), pp. 3565–3639. DOI.

[4] R. L. Berger. "The undecidability of the domino problem". *Mem. Amer. Math. Soc.* **66** (1966), 72 pp.

[5] M. Bóna. *Combinatorics of permutations*. Discrete Mathematics and its Applications (Boca Raton). With a foreword by Richard Stanley. Chapman & Hall/CRC, Boca Raton, FL, 2004, xiv+383 pp. DOI.

[6] N. Bonichon and P.-J. Morel. "Baxter *d*-permutations and other pattern-avoiding classes". *J. Integer Seq.* **25**.8 (2022), Art. 22.9.3, 36 pp.

[7] J. Borga and A. Lin. "High-dimensional permutons: theory and applications". 2025. arXiv: 2412.19730.

[8] M. Hochman. "Multidimensional shifts of finite type and sofic shifts". *Combinatorics, words and symbolic dynamics*. Vol. 159. Encyclopedia Math. Appl. Cambridge Univ. Press, Cambridge, 2016, pp. 296–358. DOI.

[9] J. Kari. "Theory of cellular automata: A survey". *Theor. Comput. Sci.* **334**.1 (2005), pp. 3–33. DOI.

[10] S. Kitaev. *Patterns in permutations and words*. Monographs in Theoretical Computer Science. An EATCS Series. With a foreword by Jeffrey B. Remmel. Springer, Heidelberg, 2011, xxii+494 pp. DOI.

[11] V. Salo. "Cutting corners". *J. Comput. System Sci.* **128** (2022), pp. 35–70. DOI.

[12] V. Salo and J. Schabanel. "Triangle solitaire". *Cellular automata and discrete complex systems*. Vol. 14152. Lecture Notes in Comput. Sci. Springer, Cham, [2023] ©2023, pp. 123–136. DOI.