Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #43, 12 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

Equivariant Cohomology of Grassmannian Spanning Lines

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Abstract. For positive integers $d \leq k \leq n$, let $X_{n,k,d}$ be the moduli space of n-tuples $(\ell_1, \ldots, \ell_n)$ of lines in $\mathbb{C}^k$ such that $\ell_1 + \cdots + \ell_n$ has vector space dimension d. The space $X_{n,k,d}$ carries an action of the rank k torus $T = (\mathbb{C}^*)^k$, and we present the T-equivariant cohomology of $X_{n,k,d}$. This solves a problem of Pawlowski and Rhoades. Our methods feature the orbit harmonics technique of combinatorial deformation theory and suggest a relationship between orbit harmonics and equivariant cohomology.

Keywords: equivariant cohomology, orbit harmonics, grassmannian line configurations

1 Introduction

Let X be a topological space carrying an action of a complex torus T. The *T-equivariant cohomology ring* $H^*_T(X)$ is an enhancement of the ordinary cohomology $H^*(X)$ of X which accounts for the action of T.[1] The map $X \to \{\text{pt}\}$ endows $H^*_T(X)$ with the structure of a $\mathbb{C}[\mathfrak{t}]$-module, where $\mathfrak{t}$ is the Lie algebra of T. Under mild conditions, the ordinary cohomology $H^*(X)$ may be recovered from $H^*_T(X)$ by the relation

$$H^*(X) = \mathbb{C} \otimes_{\mathbb{C}[\mathfrak{t}]} H^*_T(X) \tag{1.1}$$

where the generators of the polynomial ring $\mathbb{C}[\mathfrak{t}]$ act by zero on $\mathbb{C}$.

A *line* in $\mathbb{C}^k$ is a 1-dimensional subspace $\ell \subseteq \mathbb{C}^k$. We compute the equivariant cohomology of the following moduli space of line configurations.

Definition 1.1. Let $d \leq k \leq n$ be positive integers. Let $X_{n,k,d}$ be the set of n-tuples $(\ell_1, \ldots, \ell_n)$ of lines in $\mathbb{C}^k$ such that the vector space sum $\ell_1 + \cdots + \ell_n$ has dimension d.

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[1]For simplicity, we use complex coefficients for cohomology throughout this extended abstract, but our main results hold over any coefficient ring.

Writing $\mathbb{P}^{k-1}$ for the complex projective space of lines in $\mathbb{C}^k$, we have a natural inclusion $X_{n,k,d} \subseteq (\mathbb{P}^{k-1})^n$ which gives $X_{n,k,d}$ the structure of an algebraic variety. The variety $X_{n,k,d}$ carries an action of the rank k torus $T = (\mathbb{C}^*)^k$ given by

$$t \cdot (\ell_1, \ldots, \ell_n) := (t \cdot \ell_1, \ldots, t \cdot \ell_n). \tag{1.2}$$

Special cases of $X_{n,k,d}$ have been considered before.

- If $n = k = d$, the variety $X_{n,n,n}$ consists of n-tuples $(\ell_1, \ldots, \ell_n)$ of lines in $\mathbb{C}^n$ which span $\mathbb{C}^n$. This space is homotopy equivalent to the type A_{n-1} complete flag variety Fl_n.[2] The geometry of Fl_n is governed by the combinatorics of permutations in the symmetric group $\mathfrak{S}_n$.
- If $k = d$, the variety $X_{n,k} := X_{n,k,k}$ consists of n-tuples $(\ell_1, \ldots, \ell_n)$ of lines in $\mathbb{C}^k$ which span $\mathbb{C}^k$. This variety of *spanning line configurations* was introduced by Pawlowski and Rhoades [13]. The geometry of $X_{n,k}$ is governed by the combinatorics of *Fubini words* in $\mathcal{W}_{n,k}$; these are surjective functions $w : [n] \twoheadrightarrow [k]$. Billey and Ryan [3] gave combinatorial descriptions of the corresponding Bruhat order(s) on $\mathcal{W}_{n,k}$.

Pawlowski and Rhoades proved [13] that the ordinary cohomology of $X_{n,k}$ is presented by the following quotient $R_{n,k}$ of $\mathbb{C}[\mathbf{x}_n]$:

$$H^*(X_{n,k}) = R_{n,k} := \mathbb{C}[\mathbf{x}_n]/(x_1^k, \ldots, x_n^k, e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \ldots, e_{n-k+1}(\mathbf{x}_n)). \tag{1.3}$$

Here $e_d(\mathbf{x}_n)$ is the degree d elementary symmetric polynomial in the variable set $\mathbf{x}_n := \{x_1, \ldots, x_n\}$. The ring $R_{n,k}$ was introduced by Haglund, Rhoades, and Shimozono [10] to give an analogue of the coinvariant algebra in the context of the Haglund–Remmel–Wilson *delta conjecture* [9]. Pawlowski and Rhoades asked [13, Problem 9.8] for a presentation of the T-equivariant cohomology $H_T^*(X_{n,k})$; we solve the more general problem of presenting $H_T^*(X_{n,k,d})$.

To state our main result we need some notation. Write $\mathrm{Gr}(d, \mathbb{C}^k)$ for the Grassmannian of d-dimensional subspaces $V \subseteq \mathbb{C}^k$. We have a surjection

$$p : X_{n,k,d} \twoheadrightarrow \mathrm{Gr}(d, \mathbb{C}^k) \tag{1.4}$$

which sends an n-tuple $(\ell_1, \ldots, \ell_n)$ of lines to $\ell_1 + \cdots + \ell_n$. The rank k torus $T = (\mathbb{C}^*)^k$ acts naturally on $\mathrm{Gr}(d, \mathbb{C}^k)$, and the map p is T-equivariant. We write $\mathcal{V}_d$ for the rank d tautological vector bundle over $\mathrm{Gr}(d, \mathbb{C}_k)$ whose fiber over a point $V \in \mathrm{Gr}(d, \mathbb{C}^k)$ is the vector space V. The pullback $p^*(\mathcal{V}_d)$ is a vector bundle over $X_{n,k,d}$.

[2]The homotopy equivalence is the natural projection $G/T \twoheadrightarrow G/B$. This map is a fiber bundle with contractible fiber $\cong U$.

In addition to the length n list $\mathbf{x}_n = \{x_1, \ldots, x_n\}$ of x-variables, we consider a length d list $\mathbf{y}_d = \{y_1, \ldots, y_d\}$ of y-variables and a length k list $\mathbf{t}_k = \{t_1, \ldots, t_k\}$ of t-variables. Let $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_d, \mathbf{t}_k]$ be the rank $n + d + k$ polynomial ring over all of these variables. The symmetric group $\mathfrak{S}_d$ acts on the middle set of variables in $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_d, \mathbf{t}_k]$; we write $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_d, \mathbf{t}_k]^{\mathfrak{S}_d}$ for the associated invariant subring. Our presentation of the T-equivariant cohomology of $X_{n,k,d}$ reads as follows, where h_i stands for the complete homogeneous symmetric polynomial of degree i.

Theorem 1.2. *For positive integers $n \geq k \geq d$, let $I_{n,k,d} \subseteq \mathbb{C}[\mathbf{x}_n, \mathbf{y}_d, \mathbf{t}_k]^{\mathfrak{S}_d}$ be the ideal generated by*

- $e_r(\mathbf{t}_k) - e_{r-1}(\mathbf{t}_k)h_1(\mathbf{y}_d) + \cdots + (-1)^r h_r(\mathbf{y}_d)$ *for* $r > k - d$,
- $e_r(\mathbf{x}_n) - e_{r-1}(\mathbf{x}_n)h_1(\mathbf{y}_d) + \cdots + (-1)^r h_r(\mathbf{y}_d)$ *for* $r > n - d$, *and*
- $x_i^d - x_i^{d-1}e_1(\mathbf{y}_d) + \cdots + (-1)^d e_d(\mathbf{y}_d)$ *for* $i = 1, \ldots, n$.

The T-equivariant cohomology of $X_{n,k,d}$ has presentation

$$H_T^*(X_{n,k,d}) = \mathbb{C}[\mathbf{x}_n, \mathbf{y}_d, \mathbf{t}_k]^{\mathfrak{S}_d} / I_{n,k,d} \tag{1.5}$$

where

- *x_i represents the first equivariant Chern class of the line bundle $\mathcal{L}_i$ over $X_{n,k,d}$ with fiber ℓ_i over $(\ell_1, \ldots, \ell_n)$,*
- *$y_1, \ldots, y_d$ represent equivariant Chern roots of $p^*(\mathcal{V}_d)$, and*
- *$t_1, \ldots, t_n$ are the images in $H_T^*(X_{n,k,d})$ of the standard generators of $H_T^*(\mathrm{pt})$.*

When $k = d$, the variety $X_{n,k,d}$ specializes to the variety $X_{n,k}$ of spanning line configurations in $\mathbb{C}^k$. Specializing Theorem 1.2 to $k = d$, our solution to the problem [13, Problem 9.8] of Pawlowski and Rhoades is as follows.

Corollary 1.3. *For positive integers $n \geq k$, the T-equivariant cohomology of $X_{n,k}$ has quotient presentation*

$$H_T^*(X_{n,k}) = \mathbb{C}[\mathbf{x}_n, \mathbf{y}_k] / I_{n,k} \tag{1.6}$$

where $I_{n,k} \subseteq \mathbb{C}[\mathbf{x}_n, \mathbf{y}_k]$ is the ideal generated by

- $e_r(\mathbf{x}_n) - e_{r-1}(\mathbf{x}_n)h_1(\mathbf{t}_k) + \cdots + (-1)^r h_r(\mathbf{t}_k)$ *for* $r > n - k$, *and*
- $x_i^k - x_i^{k-1}e_1(\mathbf{t}_k) + \cdots + (-1)^k e_k(\mathbf{t}_k)$ *for* $i = 1, \ldots, n$.

The proof of Theorem 1.2 has two main parts: geometric and algebraic. Geometrically, we show that the relations in $I_{n,k,d}$ hold in the ring $H_T^*(X_{n,k,d})$ using the Whitney Sum Formula and calculate the rank of $H_T^*(X_{n,k,d})$ as a free $\mathbb{C}[\mathbf{t}]$-module using a T-stable affine paving of $X_{n,k,d}$. Algebraically, we show that $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_d, \mathbf{t}_k]^{\mathfrak{S}_d}/I_{n,k,d}$ is a free $\mathbb{C}[\mathbf{t}_k]$-module of the appropriate rank. For this, we apply a technique in combinatorial deformation theory called *orbit harmonics*.

The remainder of this extended abstract is organized as follows. In Section 2 we give background on equivariant cohomology, affine pavings, and orbit harmonics. In Section 3 we describe our general approach to equivariant cohomology via orbit harmonics and study the permutohedral variety from this point of view. In Section 4 we sketch the proof of Theorem 1.2.

2 Background

2.1 Equivariant Cohomology

Let X be a smooth complex variety equipped with an action of the rank k torus $T = (\mathbb{C}^*)^k$. As mentioned in the introduction, the equivariant cohomology ring $H_T^*(X)$ is an enhancement of the ordinary cohomology $H^*(X)$. We describe the basic features of $H_T^*(X)$, referring the reader to Anderson and Fulton's book [1] for a detailed treatment.

Let ET be a contractible space with a free left action of T. We define $ET \times_T X := ET \times X/\sim$ where $(e, x) \sim (e \cdot t^{-1}, t \cdot x)$ for all $e \in ET$, $x \in X$, and $t \in T$. The *equivariant cohomology ring* of X is defined by

$$H_T^*(X) := H^*(ET \times_T X) \tag{2.1}$$

where $H^*(ET \times_T X)$ is the usual singular cohomology. For example, if $X = \{\mathrm{pt}\}$ is a single point, we may identify $ET \times_T \{\mathrm{pt}\} = (\mathbb{P}^\infty)^k$, where $\mathbb{P}^\infty$ is infinite-dimensional complex projective space. Since $H^*((\mathbb{P}^\infty)^k) = \mathbb{C}[\mathbf{t}_k]$, the map $X \to \{\mathrm{pt}\}$ gives the ring $H_T^*(X)$ the structure of a $\mathbb{C}[\mathbf{t}_k]$-module.

Let $Y \subseteq X$ be a closed subvariety which is closed under the T-action. If Y has codimension c in X, we have a class $[Y] \in H_T^*(X)$ of cohomological degree $2c$. An *affine paving* of X is a filtration

$$\varnothing = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_m = X \tag{2.2}$$

of X by closed subvarieties such that each difference $X_i \setminus X_{i-1}$ is isomorphic to a disjoint union of affine spaces (possibly of varying dimensions). These affine spaces are referred to as *cells*, and the affine paving is *T-invariant* if the cells are closed under the action of T. If X has a T-invariant affine paving, the equivariant cohomology ring $H_T^*(X)$ is a free $\mathbb{C}[\mathbf{t}_k]$-module with basis given by the classes $[\overline{C}]$ of the closures $\overline{C}$ of these cells.

Let $\mathcal{E} \twoheadrightarrow X$ be a T-equivariant vector bundle of rank r. For $1 \leq i \leq r$ we have the *equivariant Chern class* $c_i^T(\mathcal{E}) \in H_T^{2i}(X)$. The *total equivariant Chern class* is the sum $c^T(\mathcal{E}) := 1 + c_1^T(\mathcal{E}) + \cdots + c_r^T(\mathcal{E})$. If we have a short exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0 \tag{2.3}$$

of T-equivariant bundles, there holds the relation $c^T(\mathcal{E}) = c^T(\mathcal{E}') \cdot c^T(\mathcal{E}'')$. It follows that if $\mathcal{E}$ splits as a direct sum of equivariant line bundles $\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$, we have the factorization

$$c^T(\mathcal{E}) = c^T(\mathcal{L}_1) \cdots c^T(\mathcal{L}_r) = (1 + c_1^T(\mathcal{L}_1)) \cdots (1 + c_1^T(\mathcal{L}_r)). \tag{2.4}$$

Even if $\mathcal{E}$ does not split as a direct sum of equivariant line bundles, by choosing an appropriate flag extension $X' \twoheadrightarrow X$ we may still factor $c^T(\mathcal{E}) = (1 + \alpha_1) \cdots (1 + \alpha_r)$ where $\alpha_1, \ldots, \alpha_r \in H_T^2(X')$. The classes $\alpha_1, \ldots, \alpha_r$ are the *equivariant Chern roots* of $\mathcal{E}$; any symmetric polynomial in $\alpha_1, \ldots, \alpha_r$ lies in $H_T^*(X)$. See [1, Section 2.3] for more information on these facts.

2.2 Orbit Harmonics

Let $\mathcal{Z} \subseteq \mathbb{C}^n$ be a finite locus of points in affine n-space $\mathbb{C}^n$. We have the vanishing ideal

$$\mathbf{I}(\mathcal{Z}) := \{ f \in \mathbb{C}[\mathbf{x}_n] \,:\, f(\mathbf{z}) = 0 \text{ for all } \mathbf{z} \in \mathcal{Z} \} \subseteq \mathbb{C}[\mathbf{x}_n]. \tag{2.5}$$

If $I \subseteq \mathbb{C}[\mathbf{x}_n]$ is an ideal, recall that the *associated graded ideal* $\operatorname{gr} I \subseteq \mathbb{C}[\mathbf{x}_n]$ is the homogeneous ideal

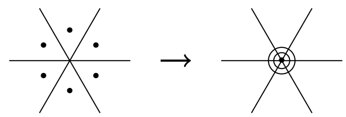
$$\operatorname{gr} I := (\tau(f) \,:\, f \in I,\, f \neq 0) \subseteq \mathbb{C}[\mathbf{x}_n] \tag{2.6}$$

where $\tau(f)$ is the top-degree homogeneous component of a nonzero polynomial f. The *orbit harmonics* deformation associates to $\mathcal{Z}$ the graded quotient ring $\mathbb{C}[\mathbf{x}_n]/\operatorname{gr} \mathbf{I}(\mathcal{Z})$ where $\operatorname{gr} \mathbf{I}(\mathcal{Z}) \subseteq \mathbb{C}[\mathbf{x}_n]$. We have a vector space isomorphism

$$\mathbb{C}[\mathcal{Z}] := \mathbb{C}[\mathbf{x}_n]/\mathbf{I}(\mathcal{Z}) \cong_{\mathbb{C}} \mathbb{C}[\mathbf{x}_n]/\operatorname{gr} \mathbf{I}(\mathcal{Z}) \tag{2.7}$$

where $\mathbb{C}[\mathbf{x}_n]/\operatorname{gr} \mathbf{I}(\mathcal{Z})$ is a graded vector space. If the locus $\mathcal{Z}$ is a stable under the action of a finite matrix group $G \subseteq GL_n(\mathbb{C})$, (2.7) is an isomorphism of G-modules, where $\mathbb{C}[\mathbf{x}_n]/\operatorname{gr} \mathbf{I}(\mathcal{Z})$ is a graded G-module.

In geometric terms, orbit harmonics is a flat family which linearly deforms the reduced locus $\mathcal{Z} \subseteq \mathbb{C}^n$ to a subscheme of degree $\#\mathcal{Z}$ supported at the origin. This deformation is shown schematically below in the case of a locus of size $|\mathcal{Z}| = 6$ in $\mathbb{C}^2$ carrying an action of $G \cong \mathfrak{S}_3$ via reflection in the three displayed lines. Orbit harmonics quotients $\mathbb{C}[\mathbf{x}_n]/\operatorname{gr} \mathbf{I}(\mathcal{Z})$ have been applied to Donaldson–Thomas theory [14], increasing subsequence combinatorics [17], Ehrhart theory [15], and (importantly for us) cohomology presentation.



3 Equivariant Cohomology and Orbit Harmonics

Let X be a variety equipped with an action of a rank k torus $T \cong (\mathbb{C}^*)^k$, and let $\mathfrak{t} \cong \mathbb{C}^k$ be the Lie algebra of T. In many combinatorially interesting situations [6, 7, 8, 10, 13, 16], there is a nonempty Zariski-open subset $U \subseteq \mathfrak{t}$ such that

- for each $\boldsymbol{\alpha} \in U$, we have a finite locus $\mathcal{Z}(\boldsymbol{\alpha}) \subseteq \mathbb{C}^n$ depending on $\boldsymbol{\alpha}$,
- the homogeneous ideal $\operatorname{gr} \mathbf{I}(\mathcal{Z}(\boldsymbol{\alpha})) \subseteq \mathbb{C}[\mathbf{x}_n]$ does not depend on $\boldsymbol{\alpha}$, and
- the ordinary cohomology ring $H^*(X)$ has presentation

$$H^*(X) = \mathbb{C}[\mathbf{x}_n]/\operatorname{gr} \mathbf{I}(\mathcal{Z}(\boldsymbol{\alpha})). \tag{3.1}$$

The loci $\mathcal{Z}(\alpha)$ may be put into a family $\mathcal{Z} \subseteq \mathbb{C}^n \times \mathfrak{t}$ given by

$$\mathcal{Z} := \text{Zariski closure of } \bigcup_{\boldsymbol{\alpha} \in U} \mathcal{Z}(\boldsymbol{\alpha}) \times \{\boldsymbol{\alpha}\} \text{ in } \mathbb{C}^n \times \mathfrak{t}. \tag{3.2}$$

We have the vanishing ideal $\mathbf{I}(\mathcal{Z}) \subseteq \mathbb{C}[\mathbb{C}^n \times \mathfrak{t}] \cong \mathbb{C}[\mathbf{x}_n, \mathbf{t}_k]$. It turns out that the equivariant cohomology of X often has quotient presentation

$$H_T^*(X) = \mathbb{C}[\mathbf{x}_n, \mathbf{t}_k]/\mathbf{I}(\mathcal{Z}) \tag{3.3}$$

where the t-variables come from the torus action. The formula $H^*(X) = \mathbb{C} \otimes_{\mathbb{C}[\mathfrak{t}]} H_T^*(X)$ may be interpreted in terms of (3.3) as taking the scheme-theoretic fiber $\pi^{-1}(0)$ where $\pi : \mathcal{Z} \to \mathfrak{t}$ is the natural projection. Going in the other direction, orbit harmonics can sometimes *predict* equivariant cohomology rings, a theme we explore. We give two examples of this phenomenon (one old and one new) involving previously studied varieties before turning to the new variety $X_{n,k,d}$.

3.1 Type A Springer Fibers

Recall that the complete flag variety Fl_n is the moduli space

$$\mathrm{Fl}_n = \{V_\bullet = (0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n) : \dim V_i = i\} \tag{3.4}$$

of maximal chains of nested subspaces of $\mathbb{C}^n$. The rank n torus $(\mathbb{C}^*)^n \subseteq GL_n(\mathbb{C})$ of diagonal matrices acts naturally on $\mathbb{C}^n$, and induces an action on Fl_n.

Let $\mu = (\mu_1, \ldots, \mu_k) \vdash n$ be a partition of n with k parts and let $X : \mathbb{C}^n \to \mathbb{C}^n$ be a nilpotent matrix of Jordan type μ. The *Springer fiber* is the closed subvariety

$$\mathcal{B}_\mu := \{V_\bullet \in \mathrm{Fl}_n \,:\, XV_i \subseteq V_i \text{ for all } i\} \tag{3.5}$$

of Fl_n. The Springer fiber $\mathcal{B}_\mu$ is stable under the action of the rank k subtorus $T \subseteq (\mathbb{C}^*)^n$ consisting of matrices of the form $\lambda_1 I_{\mu_1} \oplus \cdots \oplus \lambda_k I_{\mu_k}$ for $\lambda_1, \ldots, \lambda_k \in \mathbb{C}^*$. This gives rise to identifications $T = (\mathbb{C}^*)^k$ and $\mathfrak{t} = \mathbb{C}^k$.

Let $U \subseteq \mathfrak{t} = \mathbb{C}^k$ be the Zariski-open set

$$U := \{\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \mathbb{C}^k \,:\, \alpha_1, \ldots, \alpha_k \text{ are distinct}\}. \tag{3.6}$$

For any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k) \in U$, we have a locus $\mathcal{Z}(\boldsymbol{\alpha}) \subset \mathbb{C}^n$ given by

$$\mathcal{Z}(\boldsymbol{\alpha}) := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \,:\, \alpha_i \text{ appears } \mu_i \text{ times among } z_1, \ldots, z_n\}. \tag{3.7}$$

For example, if $\mu = (2,1)$ we have $\mathcal{Z}(\alpha) = \{(\alpha_1, \alpha_1, \alpha_2), (\alpha_1, \alpha_2, \alpha_1), (\alpha_2, \alpha_1, \alpha_1)\}$. Garsia and Procesi proved [6] that the ideal $\mathrm{gr}\, \mathbf{I}(\mathcal{Z}(\boldsymbol{\alpha})) \subseteq \mathbb{C}[\mathbf{x}_n]$ does not depend on $\boldsymbol{\alpha} \in U$. Combining their result with the presentation of $H^*(\mathcal{B}_\mu)$ given by Hotta and Springer [11], we arrive at the identification

$$H^*(\mathcal{B}_\mu) = \mathbb{C}[\mathbf{x}_n] / \mathrm{gr}\, \mathbf{I}(\mathcal{Z}(\boldsymbol{\alpha})) \text{ for any } \boldsymbol{\alpha} \in U. \tag{3.8}$$

Let $\mathcal{Z} \subseteq \mathbb{C}^n \times \mathfrak{t} = \mathbb{C}^{n+k}$ be the Zariski closure of $\bigcup_{\boldsymbol{\alpha} \in U} \mathcal{Z}(\boldsymbol{\alpha}) \times \{\boldsymbol{\alpha}\}$ inside $\mathbb{C}^{n+k}$. We have $\mathbf{I}(\mathcal{Z}) \subseteq \mathbb{C}[\mathbf{x}_n, \mathbf{t}_k]$. Kumar and Procesi derived [12] the presentation

$$H_T^*(\mathcal{B}_\mu) = \mathbb{C}[\mathbf{x}_n, \mathbf{t}_k] / \mathbf{I}(\mathcal{Z}) \tag{3.9}$$

of the equivariant cohomology of $\mathcal{B}_\mu$.

3.2 Permutohedral variety

Let $S : \mathbb{C}^n \to \mathbb{C}^n$ be a diagonal matrix with distinct entries. The *permutohedral variety* Perm_n is the closed subvariety of Fl_n given by

$$\mathrm{Perm}_n := \{V_\bullet \in \mathrm{Fl}_n \,:\, SV_i \subseteq V_{i+1} \text{ for } i = 1, 2, \ldots, n-1\}. \tag{3.10}$$

This is an important example of a *toric variety* and a *regular semisimple Hessenberg variety*. The variety Perm_n is stable under the action of the rank $n-1$ torus $T \subset GL_n(\mathbb{C})$ given by

$$T = \{\mathrm{diag}(\lambda_1, \ldots, \lambda_n) \,:\, \lambda_1 \cdots \lambda_n = 1\}. \tag{3.11}$$

We describe how known presentations of the (equivariant) cohomology of Perm_n may be interpreted via orbit harmonics.

Let $\mathcal{F}$ be the family of $N := 2^n - 2$ nonempty and proper subsets $I \subset [n]$ and let $\mathbb{C}^{\mathcal{F}}$ be the N-dimensional complex vector space with basis $\mathcal{F}$. Let $\mathbf{x}_{\mathcal{F}} := \{x_I : I \in \mathcal{F}\}$ be a family of variables indexed by $\mathcal{F}$ and let $\mathbb{C}[\mathbf{x}_{\mathcal{F}}] = \mathbb{C}[x_I : I \in \mathcal{F}]$ be the rank N polynomial ring over these variables.

The Lie algebra $\mathfrak{t}$ may be identified with the $(n-1)$-dimensional vector space

$$\mathfrak{t} = \{\mathrm{diag}(\alpha_1, \dots, \alpha_n) : \alpha_1 + \cdots + \alpha_n = 0\} \tag{3.12}$$

and we identify its coordinate ring as $\mathbb{C}[\mathfrak{t}] = \mathbb{C}[\mathbf{t}_n]/(t_1 + \cdots + t_n)$. Let $U \subset \mathfrak{t}$ be the Zariski-open subset of points $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ with distinct coordinates. For $\boldsymbol{\alpha} \in U$ and any permutation $w \in \mathfrak{S}_n$, we define a point

$$p(w, \boldsymbol{\alpha}) = (p(w, \boldsymbol{\alpha})_I)_{I \in \mathcal{F}} \in \mathbb{C}^{\mathcal{F}}$$

as follows. The permutation w determines a flag $\varnothing = I_0(w) \subset I_1(w) \subset \cdots \subset I_n(w) = [n]$ of subsets of $[n]$ where $I_j(w) := \{w(1), \dots, w(j)\}$. For a subset $I \in \mathcal{F}$, we define the I-th component $p(w, \boldsymbol{\alpha})_I$ of $p(w, \boldsymbol{\alpha})$ by

$$p(w, \boldsymbol{\alpha})_I := \begin{cases} \alpha_{w(j)} - \alpha_{w(j+1)} & \text{if } I = I_j(w) \text{ for some } 0 < j < n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

We let $\mathcal{Z}(\boldsymbol{\alpha}) \subseteq \mathbb{C}^{\mathcal{F}}$ be the locus

$$\mathcal{Z}(\boldsymbol{\alpha}) := \{p(w, \boldsymbol{\alpha}) : w \in \mathfrak{S}_n\} \tag{3.14}$$

so that $\mathbf{I}(\mathcal{Z}(\boldsymbol{\alpha})) \subseteq \mathbb{C}[\mathbb{C}^{\mathcal{F}}] = \mathbb{C}[\mathbf{x}_{\mathcal{F}}]$ for each $\boldsymbol{\alpha} \in U$. We also have the family

$$\mathcal{Z} := \text{Zariski closure of } \bigcup_{\boldsymbol{\alpha} \in U} \mathcal{Z}(\boldsymbol{\alpha}) \times \{\boldsymbol{\alpha}\} \text{ in } \mathbb{C}^{\mathcal{F}} \times \mathfrak{t}. \tag{3.15}$$

The ideal $\mathbf{I}(\mathcal{Z})$ is a subset of $\mathbb{C}[\mathbb{C}^{\mathcal{F}} \times \mathfrak{t}] = \mathbb{C}[\mathbf{x}_{\mathcal{F}}] \otimes \mathbb{C}[\mathbf{t}_n]/(t_1 + \cdots + t_n)$.

Theorem 3.1. *The associated graded ideal* $\mathrm{gr}\, \mathbf{I}(\mathcal{Z}(\boldsymbol{\alpha})) \subseteq \mathbb{C}[\mathbb{C}^{\mathcal{F}}]$ *does not depend on* $\boldsymbol{\alpha} \in U$. *For any* $\boldsymbol{\alpha} \in U$, *the ordinary cohomology of* Perm_n *has presentation*

$$H^*(\mathrm{Perm}_n) = \mathbb{C}[\mathbb{C}^{\mathcal{F}}]/\mathrm{gr}\, \mathbf{I}(\mathcal{Z}(\boldsymbol{\alpha})). \tag{3.16}$$

Furthermore, the T-equivariant cohomology of Perm_n *has presentation*

$$H_T^*(\mathrm{Perm}_n) = \mathbb{C}[\mathbb{C}^{\mathcal{F}} \times \mathfrak{t}]/\mathbf{I}(\mathcal{Z}). \tag{3.17}$$

Proof. (Sketch) Danilov proved [5] that the cohomology of Perm_n has presentation

$$H^*(\mathrm{Perm}_n) = \mathbb{C}[\mathbf{x}_{\mathcal{F}}]/J \tag{3.18}$$

where $J \subseteq \mathbb{C}[\mathbf{x}_{\mathcal{F}}]$ is the ideal generated by

- all products $x_I \cdot x_{I'}$ for $I, I' \in \mathcal{F}$ where $I \not\subseteq I'$ and $I' \not\subseteq I$, and
- all differences of the form
$$\sum_{i \in I} x_I - \sum_{i+1 \in I'} x_{I'}$$
for $i = 1, 2, \ldots, n-1$.

Let $\boldsymbol{\alpha} \in U$. To show $J \subseteq \operatorname{gr} \mathbf{I}(\mathcal{Z}(\boldsymbol{\alpha}))$, we prove that each generator of J is the top-degree homogeneous component of a polynomial in $\mathbb{C}[\mathbf{x}_{\mathcal{F}}]$ which vanishes on $\mathcal{Z}(\boldsymbol{\alpha})$. If $I, I' \in \mathcal{F}$ satisfy $I \not\subseteq I'$ and $I' \not\subseteq I$, the product $x_I \cdot x_{I'}$ vanishes on $\mathcal{Z}(\boldsymbol{\alpha})$ since for any $w \in \mathfrak{S}_n$ the nonzero components of $p(w, \boldsymbol{\alpha})$ are indexed by a flag of subsets of $[n]$. Furthermore, for each $i = 1, 2, \ldots, n-1$ it can be checked that the polynomial

$$\left(\sum_{i \in I} x_I - \sum_{i+1 \in I'} x_{I'} \right) - (\alpha_i - \alpha_{i+1}) \tag{3.19}$$

vanishes on $\mathcal{Z}(\boldsymbol{\alpha})$, and the top degree component of this polynomial is the generator $\sum_{i \in I} x_I - \sum_{i+1 \in I'} x_{I'}$ of J. This proves that $J \subseteq \operatorname{gr} \mathbf{I}(\mathcal{Z}(\boldsymbol{\alpha}))$. We have a canonical surjection

$$H^*(\mathrm{Perm}_n) = \mathbb{C}[\mathbf{x}_{\mathcal{F}}]/J \twoheadrightarrow \mathbb{C}[\mathbf{x}_{\mathcal{F}}]/\operatorname{gr} \mathbf{I}(\mathcal{Z}(\boldsymbol{\alpha})). \tag{3.20}$$

The orbit harmonics isomorphism (2.7) implies that $\mathbb{C}[\mathbf{x}_{\mathcal{F}}]/\operatorname{gr} \mathbf{I}(\mathcal{Z}(\boldsymbol{\alpha}))$ has dimension $\#\mathcal{Z}(\boldsymbol{\alpha}) = n!$. It is well-known that $H^*(\mathrm{Perm}_n)$ also has vector space dimension $n!$, which forces the surjection (3.20) to be an isomorphism and $\operatorname{gr} \mathbf{I}(\mathcal{Z}(\boldsymbol{\alpha})) = J$ for any $\boldsymbol{\alpha} \in U$.

Let $L \subseteq \mathbb{C}[\mathbf{x}_{\mathcal{F}}]$ be the ideal generated by all products $x_I \cdot x_{I'}$ for $I, I' \in \mathcal{F}$ for which $I \not\subseteq I'$ and $I' \not\subseteq I$. Bifet, De Concini, and Procesi proved [2] that the equivariant cohomology ring of Perm_n has presentation

$$H_T^*(\mathrm{Perm}_n) = \mathbb{C}[\mathbf{x}_{\mathcal{F}}]/L. \tag{3.21}$$

We have a $\mathbb{C}$-algebra homomorphism

$$\widetilde{\varphi} : \mathbb{C}[\mathbb{C}^{\mathcal{F}} \times \mathfrak{t}] = \mathbb{C}[\mathbf{x}_{\mathcal{F}}] \otimes \mathbb{C}[\mathbf{t}_n]/(t_1 + \cdots + t_n) \longrightarrow \mathbb{C}[\mathbf{x}_{\mathcal{F}}]/L \tag{3.22}$$

characterized by $\widetilde{\varphi} : x_I \mapsto x_I$ for $I \in \mathcal{F}$ and $\widetilde{\varphi} : t_i - t_{i+1} \mapsto \sum_{i \in I} x_I - \sum_{i+1 \in I'} x_I$ for $i = 1, 2, \ldots, n-1$. It can be checked that $\mathbf{I}(\mathcal{Z}) \subseteq \operatorname{Ker}(\widetilde{\varphi})$, so we have an induced homomorphism

$$\varphi : \mathbb{C}[\mathbb{C}^{\mathcal{F}} \times \mathfrak{t}]/\mathbf{I}(\mathcal{Z}) \longrightarrow \mathbb{C}[\mathbf{x}_{\mathcal{F}}]/L. \tag{3.23}$$

On the other hand, since $L \subseteq \mathbf{I}(\mathcal{Z})$ we have another $\mathbb{C}$-algebra homomorphism

$$\psi : \mathbb{C}[\mathbf{x}_{\mathcal{F}}]/L \longrightarrow \mathbb{C}[\mathbb{C}^{\mathcal{F}} \times \mathfrak{t}]/\mathbf{I}(\mathcal{Z}) \tag{3.24}$$

characterized by $\psi : x_I \mapsto x_I$ for $I \in \mathcal{F}$. It is not hard to see that φ and ψ are mutually inverse. □

4 Grassmannian Line Configurations

This section outlines the main ideas used to prove Theorem 1.2. Recall that we aim to show $H_T^*(X_{n,k,d}) = \mathbb{C}[\mathbf{x}_n, \mathbf{y}_d, \mathbf{t}_k]^{\mathfrak{S}_d}/I_{n,k,d}$ where $I_{n,k,d}$ has generators as described in that theorem. We define

$$\mathcal{W}_{n,k,d} := \{w : [n] \to [k] \,:\, \text{the image of } w \text{ has size } d\}. \tag{4.1}$$

It is not difficult to see that $\mathcal{W}_{n,k,d}$ is counted by $\#\mathcal{W}_{n,k,d} = \frac{k!}{(k-d)!} \cdot \mathrm{Stir}(n,d)$ where $\mathrm{Stir}(n,d)$ is the *Stirling number of the second kind* counting d-block set partitions of $[n]$.

Lemma 4.1. *The quotient ring $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_d, \mathbf{t}_k]^{\mathfrak{S}_d}/I_{n,k,d}$ is a free $\mathbb{C}[\mathbf{t}_k]$-module of rank equal to $\#\mathcal{W}_{n,k,d}$.*

Lemma 4.1 is established using orbit harmonics arguments related to those in Section 3; see [4] for details. With this result in hand, the cohomology presentation in Theorem 1.2 is established as follows.

Proof. (of Theorem 1.2, sketch) The map $p : X_{n,k,d} \to \mathrm{Gr}(d, \mathbb{C}^k)$ given by $p : (\ell_1, \ldots, \ell_n) \mapsto \ell_1 + \cdots + \ell_n$ is a fiber bundle with fiber isomorphic to $X_{n,d}$. The Leray–Hirsch Theorem and results of Pawlowski–Rhoades [13] may be used to show that $H_T^*(X_{n,k,d})$ is generated by

- the equivariant Chern classes $c_1^T(\mathcal{L}_i)$ where $\mathcal{L}_i \twoheadrightarrow X_{n,k,d}$ has fiber ℓ_i over $(\ell_1, \ldots, \ell_n)$,
- the equivariant Chern classes $c_i^T(p^*(\mathcal{V}_d))$ where $\mathcal{V}_d \twoheadrightarrow \mathrm{Gr}(d, \mathbb{C}^k)$ is the vector bundle with fiber V over $V \in \mathrm{Gr}(d, \mathbb{C}^k)$ and $i = 1, 2, \ldots, d$, and
- the standard generators $t_1, \ldots, t_k$ of $H_T^*(\mathrm{pt})$.

We therefore have a surjective $\mathbb{C}[\mathbf{t}_k]$-algebra homomorphism

$$\widetilde{\varphi} : \mathbb{C}[\mathbf{x}_n, \mathbf{y}_d, \mathbf{t}_k]^{\mathfrak{S}_d} \twoheadrightarrow H_T^*(X_{n,k,d}) \tag{4.2}$$

characterized by $\widetilde{\varphi} : x_i \mapsto c_1^T(\mathcal{L}_i)$ and $\widetilde{\varphi} : e_i(\mathbf{y}_d) \mapsto c_i^T(p^*(\mathcal{V}_d))$.

We show that each generator of $I_{n,k,d}$ lies in the kernel of $\widetilde{\varphi}$. Let $\mathbb{C}^k$ be the trivial rank k bundle over a variety with the standard action of $T = (\mathbb{C}^*)^k$ on each fiber. The bundle $\mathbb{C}^k$ has total Chern class $c^T(\mathbb{C}^k) = (1 + t_1) \cdots (1 + t_k)$ over any variety. We have a short exact sequence

$$0 \to \mathcal{V}_d \to \mathbb{C}^k \to \mathbb{C}^k/\mathcal{V}_d \to 0 \tag{4.3}$$

of bundles over $\mathrm{Gr}(d, \mathbb{C}^k)$. Since $\mathbb{C}^k/\mathcal{V}_d$ has rank $k - d$, we have $c_r^T(\mathbb{C}^k/\mathcal{V}_d) = 0$ for $r > k - d$. Since $c^T(\mathbb{C}^k) = c^T(\mathcal{V}_d) \cdot c^T(\mathbb{C}^k/\mathcal{V}_d)$ we have

$$\textstyle\sum_{a+b=r} (-1)^a e_a(\mathbf{t}_k) h_b(\mathbf{y}_d) \in \mathrm{Ker}(\widetilde{\varphi}) \text{ whenever } r > k - d.$$

Similarly, we have a short exact sequence

$$0 \to \mathcal{K}_{n-d} \to \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n \xrightarrow{\psi} p^*(\mathcal{V}_d) \to 0 \tag{4.4}$$

of bundles over $X_{n,k,d}$ where ψ is induced by vector addition $(v_1, \ldots, v_n) \mapsto v_1 + \cdots + v_n$ and $\mathcal{K}_{n-d}$ is the kernel of ψ. Since $\mathcal{K}_{n-d}$ has rank $n-d$, we have $c_r^T(\mathcal{K}_{n-d}) = 0$ whenever $r > n-d$. The relation $c^T(\mathcal{L}_1) \cdots c^T(\mathcal{L}_n) = c^T(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n) = c^T(\mathcal{K}_{n-d}) \cdot c^T(p^*(\mathcal{V}_d))$ implies

$$\textstyle\sum_{a+b=r} (-1)^a e_a(\mathbf{x}_n) h_b(\mathbf{y}_d) \in \mathrm{Ker}(\widetilde{\varphi}) \text{ whenever } r > n-d.$$

Finally, for each $i = 1, 2, \ldots, n$ we have a short exact sequence

$$0 \to \mathcal{L}_i \to \mathbb{C}^k \to \mathbb{C}^k/\mathcal{L}_i \to 0 \tag{4.5}$$

of vector bundles over $X_{n,k,d}$. Since $\mathbb{C}^k/\mathcal{L}_i$ has rank $k-1$, we have $c_k^T(\mathbb{C}^k/\mathcal{L}_i) = 0$. The relation $c^T(\mathbb{C}^k) = c^T(\mathcal{L}_i) \cdot c^T(\mathbb{C}^k/\mathcal{L}_i)$ implies

$$x_i^k - x_i^{k-1} e_1(\mathbf{t}_k) + \cdots + (-1)^k e_k(\mathbf{t}_k) \in \mathrm{Ker}(\widetilde{\varphi}) \text{ for } i = 1, 2, \ldots, n.$$

We conclude that $I_{n,k,d} \subseteq \mathrm{Ker}(\widetilde{\varphi})$, so we have an induced surjection of $\mathbb{C}[\mathbf{t}_k]$-algebras

$$\varphi : \mathbb{C}[\mathbf{x}_n, \mathbf{y}_d, \mathbf{t}_k]^{\mathfrak{S}_d} / I_{n,k,d} \twoheadrightarrow H_T^*(X_{n,k,d}). \tag{4.6}$$

We want to prove that the surjection φ is in fact an isomorphism. The fiber bundle

$$X_{n,d} \hookrightarrow X_{n,k,d} \xrightarrow{p} \mathrm{Gr}(d, \mathbb{C}^k) \tag{4.7}$$

gives rise (see [4, Lemma 4.3]) to an isomorphism of $H_T^*(\mathrm{Gr}(d, \mathbb{C}^k))$-modules

$$H_T^*(X_{n,k,d}) \cong H_T^*(\mathrm{Gr}(d, \mathbb{C}^k)) \otimes_{\mathbb{C}} H^*(X_{n,d}). \tag{4.8}$$

The standard Schubert cell decomposition of $\mathrm{Gr}(d, \mathbb{C}^k)$ gives rise to a T-invariant affine paving of $\mathrm{Gr}(d, \mathbb{C}^k)$ with $\binom{k}{d}$ cells. On the other hand, Pawlowski–Rhoades [13] proved that $H^*(X_{n,d})$ has vector space dimension $d! \cdot \mathrm{Stir}(n, d)$. It follows that $H_T^*(X_{n,k,d})$ is a free $\mathbb{C}[\mathbf{t}_k]$-module of rank

$$\binom{k}{d} \times d! \cdot \mathrm{Stir}(n, d) = \frac{k!}{(k-d)!} \cdot \mathrm{Stir}(n, d) = \# \mathcal{W}_{n,k,d}. \tag{4.9}$$

By Lemma 4.1, we conclude that φ is an epimorphism between free $\mathbb{C}[\mathbf{t}_k]$-modules of the same rank $\#\mathcal{W}_{n,k,d}$. It follows that φ is an isomorphism. □

Acknowledgements

The authors thank Leonardo Mihalcea, Ed Richmond, and Travis Scrimshaw for helpful conversations.

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