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Proof of the Newell–Littlewood saturation conjecture

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Abstract. By inventing the notion of *honeycombs*, A. Knutson and T. Tao proved the saturation conjecture for Littlewood–Richardson coefficients. The Newell–Littlewood numbers are a generalization of the Littlewood–Richardson coefficients. By introducing honeycombs on a Möbius strip, we prove the saturation conjecture for Newell–Littlewood numbers posed by S. Gao, G. Orelowitz and A. Yong.

Keywords: Littlewood–Richardson coefficients, representation theory, Lie groups, saturation, honeycombs, Möbius strips

1 Introduction

1.1 Background

The irreducible polynomial representations V_λ of $\mathrm{GL}_n\mathbb{C}$ are indexed by the set of partitions

$$\mathrm{Par}_n := \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}; \tag{1.1}$$

see, *e.g.*, [3]. For each $\mu, \nu \in \mathrm{Par}_n$,

$$V_\mu \otimes V_\nu \cong \bigoplus_{\lambda \in \mathrm{Par}_n} V_\lambda^{\oplus c^\lambda_{\mu,\nu}}. \tag{1.2}$$

The tensor product multiplicities $c^\lambda_{\mu,\nu}$ are the **Littlewood–Richardson coefficients**.

For each $k \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and $\lambda \in \mathrm{Par}_n$, let $k\lambda := (k\lambda_1, \cdots, k\lambda_n)$.

Theorem 1 (Saturation of Littlewood–Richardson coefficients [14])**.** *Let $\lambda, \mu, \nu \in \mathrm{Par}_n$. If there exists $k \in \mathbb{N}$ such that $c^{k\lambda}_{k\mu,k\nu} > 0$, then $c^\lambda_{\mu,\nu} > 0$.*

A. Knutson and T. Tao proved Theorem 1 using *honeycombs* [14]. Honeycombs are combinatorial objects used to count Littlewood–Richardson coefficients. This paper concerns a generalization of Theorem 1 and its proof.

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The significance of the saturation theorem stems from *Horn's conjecture* [8] which gives a recursive description of linear inequalities, called *Horn's inequalities*, on the eigenvalues of $n \times n$ Hermitian matrices A, B and $A+B$. Theorem 1 combined with earlier work of A. A. Klyachko [13] proved Horn's conjecture; see W. Fulton's survey [4].

1.2 Main result

We generalize Theorem 1 and its proof to the **Newell–Littlewood numbers**, which are defined, using the Littlewood–Richardson coefficients, as follows:

$$N_{\lambda,\mu,\nu} := \sum_{\alpha,\beta,\gamma \in \mathrm{Par}_n} c^{\lambda}_{\beta,\gamma} c^{\mu}_{\gamma,\alpha} c^{\nu}_{\alpha,\beta} \quad (\lambda,\mu,\nu \in \mathrm{Par}_n). \tag{1.3}$$

For each $\lambda \in \mathrm{Par}_n$, let $|\lambda| := \lambda_1 + \cdots + \lambda_n$. If $c^{\lambda}_{\mu,\nu} \neq 0$, then $|\mu| + |\nu| = |\lambda|$. According to [6, Lemma 2.2],

$$|\mu| + |\nu| = |\lambda| \quad \Rightarrow \quad N_{\lambda,\mu,\nu} = c^{\lambda}_{\mu,\nu}. \tag{1.4}$$

Thus, Newell–Littlewood numbers generalize Littlewood–Richardson coefficients.

In 2021, S. Gao, G. Orelowitz and A. Yong [6, Conjecture 5.5, 5.6] conjectured a generalization of Theorem 1. In *ibid.*, this conjecture was proved for the special cases that $\lambda = \mu = \nu$ [6, Theorem 4.1] and for $n = 2$ [6, Theorem 4.1]. In [5, Corollary 6.1], S. Gao, G. Orelowitz, N. Ressayre, and A. Yong gave a computational proof of the cases when $n \leq 5$. Our main result is a complete proof of said conjecture from [6, Conjecture 1.1], by modifying the proof of Theorem 1 in [14].

Theorem 2 (Newell–Littlewood saturation [17, Theorem 1.2])**.** *Let $\lambda, \mu, \nu \in \mathrm{Par}_n$ satisfying $|\lambda| + |\mu| + |\nu| \equiv 0 \pmod 2$. If there exists $k \in \mathbb{N}$ such that $N_{k\lambda,k\mu,k\nu} > 0$, then $N_{\lambda,\mu,\nu} > 0$.*

This follows from the technical center of this paper, Theorem 7, and is introduced at the end. In view of (1.4), Theorem 2 immediately implies the saturation of Littlewood–Richardson coefficients.

We now discuss consequences of proving Theorem 2. Analogous to the Horn's inequalities, S. Gao, G. Orelowitz and A. Yong [7, Theorem 1.3] defined *extended Horn inequalities* (which we will not restate here) and proved that they are necessary conditions for $N_{\lambda,\mu,\nu} > 0$. Additionally, they conjectured the converse; our paper also confirms this conjecture, due to [5, Corollary 8.5].

Corollary 1. [7, Conjecture 1.4] *If $(\lambda,\mu,\nu) \in (\mathrm{Par}_n)^3$ satisfies the extended Horn inequalities and $|\lambda| + |\mu| + |\nu| \equiv 0 \pmod 2$, then $N_{\lambda,\mu,\nu} > 0$.*

Therefore, the extended Horn inequalities and $|\lambda| + |\mu| + |\nu| \equiv 0 \pmod 2$ completely determine the set

$$\mathrm{NL} := \{(\lambda,\mu,\nu) \in (\mathrm{Par}_n)^3 \mid N_{\lambda,\mu,\nu} > 0\}. \tag{1.5}$$

Another application is to the eigenvalues of a family of complex matrices. Let

$$\mathrm{Par}_n^{\mathbb{Q}} := \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Q}^n \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}, \tag{1.6}$$

$$\text{NL-sat}(n) := \{(\lambda, \mu, \nu) \in (\mathrm{Par}_n^{\mathbb{Q}})^3 \mid \exists k > 0, \quad N_{k\lambda,k\mu,k\nu} > 0\}. \tag{1.7}$$

In [5, Proposition 3.1], S. Gao, G. Orelowitz, N. Ressayre and A. Yong proved that NL-sat(n) describes an analogue of the Horn problem for matrices in $\mathfrak{sp}_{2n}\mathbb{C} \cap \mathfrak{u}_{2n}\mathbb{C}$. Theorem 2 shows that NL-sat(n) can be simplified to NL.

Lastly, Theorem 2 is related to the conjecture suggested in [14, Section 7]. Given a split reductive group G over $\mathbb{C}$, it has a root system and its irreducible representation is indexed by a dominant integral weight λ. Write the dual weight as λ^* and the tensor product multiplicities by $c^{\lambda}_{\mu,\nu}(G)$.

Theorem 3. [10, Theorem 1.1] *Let G be a split reductive group over $\mathbb{C}$. Then there exists $k_G \in \mathbb{N}$ with following property: if λ, μ, ν are dominant integral weights such that $\lambda^* + \mu + \nu$ is in the root lattice,*

$$\exists k \in \mathbb{N} \text{ such that } c^{k\lambda}_{k\mu,k\nu}(G) > 0 \quad \Rightarrow \quad c^{k_G\lambda}_{k_G\mu,k_G\nu}(G) > 0. \tag{1.8}$$

Conjecture 1. [11, Conjecture 1.4] *If the root system of G is simply laced, then k_G can be chosen as* 1*.*

In particular, we are interested in the cases when $G = \mathrm{SO}_{2n+1}\mathbb{C}, \mathrm{Sp}_{2n}\mathbb{C}, \mathrm{SO}_{2n}\mathbb{C}$. In [10, Theorem 1.1], M. Kapovich and J. J. Millson proved that $k_G = 4$. Additionally, P. Belkale and S. Kumar [1, Theorem 6, 7] proved that $k_G = 2$ if G is $\mathrm{SO}_{2n+1}\mathbb{C}$ or $\mathrm{Sp}_{2n}\mathbb{C}$. S. V. Sam [18, Theorem 1.1] proved that $k_G = 2$ when $G = \mathrm{SO}_{2n+1}\mathbb{C}, \mathrm{Sp}_{2n}\mathbb{C}, \mathrm{SO}_{2n}\mathbb{C}$, by using quiver representations, extending the proof of Theorem 1 given by H. Derksen and J. Weyman [2].

The possibility that $k_G = 1$ when $G = \mathrm{SO}_{2n}\mathbb{C}$ remains open. For recent work concerning $\mathrm{SO}_{2n}\mathbb{C}$ and $\mathrm{Spin}_{2n}\mathbb{C}$, see, *e.g.*, [9, 12].

Let $G = \mathrm{SO}_{2n+1}\mathbb{C}, \mathrm{Sp}_{2n}\mathbb{C}, \mathrm{SO}_{2n}\mathbb{C}$. For the classical Lie groups, irreducible representations are indexed by the set of partitions Par_n; see, *e.g.*, [3, 16]. $l(\lambda)$ denotes the number of non-zero components of $\lambda = (\lambda_1, \cdots, \lambda_n)$. According to [15, Theorem 3.1],

$$l(\mu) + l(\nu) \leq n \quad \Rightarrow \quad N_{\lambda,\mu,\nu} = c^{\lambda}_{\mu,\nu}(G). \tag{1.9}$$

The condition (1.9) imposed on $\mu, \nu \in \mathrm{Par}_n$ is called the *stable range*. The next result is an immediate consequence of Theorem 2:

Corollary 2. *Let $G = \mathrm{SO}_{2n+1}\mathbb{C}, \mathrm{Sp}_{2n}\mathbb{C}, \mathrm{SO}_{2n}\mathbb{C}$. Suppose $\lambda, \mu, \nu \in \mathrm{Par}_n$ and $l(\mu) + l(\nu) \leq n$. If there exists $k \in \mathbb{N}$ such that $c^{k\lambda}_{k\mu,k\nu}(G) > 0$, then $c^{\lambda}_{\mu,\nu}(G) > 0$.*

Thus, k_G from Conjecture 1 may be taken as 1 for $G = \mathrm{SO}_{2n+1}\mathbb{C}, \mathrm{Sp}_{2n}\mathbb{C}, \mathrm{SO}_{2n}\mathbb{C}$ if (μ, ν) is in the stable range.

2 Saturation of Littlewood–Richardson coeffcients

In this paper, B is fixed to be the two-dimensional real vector space

$$B := \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}. \tag{2.1}$$

Let the **lattice points** of B be

$$B_{\mathbb{Z}} := \{(x, y, z) \in \mathbb{Z}^3 \mid x + y + z = 0\}. \tag{2.2}$$

A. Knutson and T. Tao constructed a directed graph Δ_n; see [14, Figure 3]. Write V_{Δ_n} as the set of vertices of Δ_n. If $h : V_{\Delta_n} \to B$ satisfies conditions to be a *configuration*, h is called a **honeycomb**; see [14, Section 2.1] for details. Also, see [14, Section 2.2] to check how they read off three partitions $\lambda, \mu, \nu \in \mathrm{Par}_n$ from a honeycomb h, which is *boundary condition* of h denoted by $\partial(h)$.

Theorem 4 ([14, Theorem 4])**.** *Let $\lambda, \mu, \nu \in \mathrm{Par}_n$. Then $c^{\lambda}_{\mu,\nu}$ counts the number of honeycombs h satisfying:*

- $\partial(h) = (\lambda, \mu, \nu)$, *and*
- $\forall v \in V_{\Delta_n}$, $h(v) \in B_{\mathbb{Z}}$.

Theorem 5 ([14, Theorem 2])**.** *For any honeycomb h with $\partial(h) \in \mathbb{Z}^{3n}$, there exists a honeycomb g such that:*

- $\partial(g) = \partial(h)$, *and*
- $\forall v \in V_{\Delta_n}$, $g(v) \in B_{\mathbb{Z}}$.

See [14, Section 5] to check the construction of honeycomb g in Theorem 5, named as *largest lift*. As a corollary, we have Theorem 1.

Proof of Theorem 1. Suppose $\lambda, \mu, \nu \in \mathrm{Par}_n$ and $k \in \mathbb{N}$ such that $c^{k\lambda}_{k\mu,k\nu} > 0$. By Theorem 4, there exists a honeycomb h such that

$$\partial(h) = (k\lambda, k\mu, k\nu). \tag{2.3}$$

Since $k > 0$, $\frac{1}{k}h$ is also a honeycomb; for instance, see [17, Lemma 2.1]. Then

$$\partial\left(\frac{1}{k}h\right) = (\lambda, \mu, \nu). \tag{2.4}$$

Apply Theorem 5 to find a honeycomb g such that $\partial(g) = \partial(\frac{1}{k}h)$ and $g(v) \in B_{\mathbb{Z}}$ for all $v \in V_{\Delta_n}$. By Theorem 4 once more, $c^{\lambda}_{\mu,\nu} > 0$. □

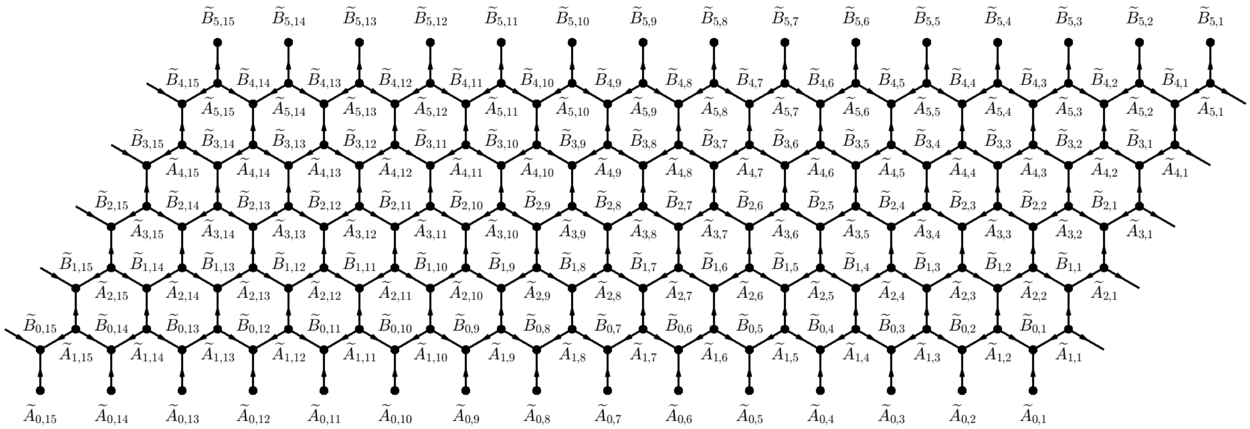

Figure 1: The directed graph $\widetilde{\Gamma}_5$.

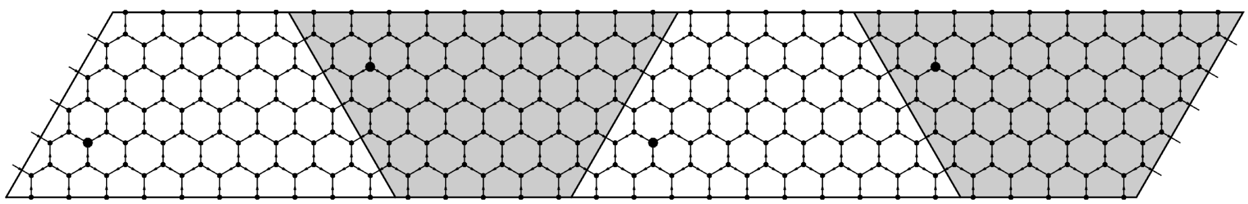

Figure 2: Identify $\widetilde{\Gamma}_5$ to have Γ_5.

3 Graphs embedded in Möbius strips

Define a directed graph $\widetilde{\Gamma}_n$ of which vertices and edges are

$$V_{\widetilde{\Gamma}_n} = \{\widetilde{A}_{i,j} \mid i,j \in \mathbb{Z}, 0 \le i \le n\} \cup \{\widetilde{B}_{i,j} \mid i,j \in \mathbb{Z}, 0 \le i \le n\}, \tag{3.1}$$

$$E_{\widetilde{\Gamma}_n} = \{(\widetilde{A}_{i,j}, \widetilde{B}_{i,j}) \mid i,j \in \mathbb{Z}, 0 \le i \le n\} \tag{3.2}$$

$$\cup \{(\widetilde{A}_{i,j}, \widetilde{B}_{i-1,j}) \mid i,j \in \mathbb{Z}, 1 \le i \le n\} \tag{3.3}$$

$$\cup \{(\widetilde{A}_{i,j}, \widetilde{B}_{i-1,j-1}) \mid i,j \in \mathbb{Z}, 1 \le i \le n\}. \tag{3.4}$$

Here, we denote a directed edge from U to W as (U, W). As in Figure 1, $\widetilde{\Gamma}_n$ is an infinite strip composed of $(n-1)$-number of layers of hexagons. There are vertices connected to exactly one edge in Figure 1, namely $\widetilde{A}_{0,j}, \widetilde{B}_{n,j}$ for $j \in \mathbb{Z}$. Such vertices of $\widetilde{\Gamma}_n$ are the **boundary vertices** in $\widetilde{\Gamma}_n$.

We now define a graph Γ_n, which will be a "quotient graph" of $\widetilde{\Gamma}_n$. Intuitively, "slice" $\widetilde{\Gamma}_n$ into pieces by using trapezoids as in Figure 2. We want to identify all trapezoids as one, which corresponds to the quotient graph Γ_n. For instance, four bold vertices of $\widetilde{\Gamma}_5$ in Figure 2 are identified as a vertex of Γ_5.

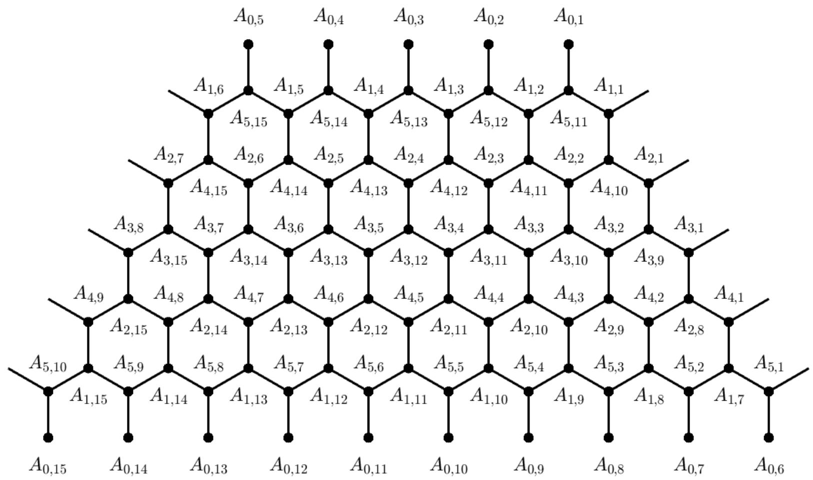

Figure 3: The graph Γ_5.

To be precise, identify the vertices of $\widetilde{\Gamma}_n$ using the equivalence relation $\sim$ defined by

$$\widetilde{A}_{i,j} \sim \widetilde{B}_{-i+n,-i+j+2n}, \quad \widetilde{B}_{i,j} \sim \widetilde{A}_{-i+n,-i+j+2n}. \quad (i,j \in \mathbb{Z}, 0 \le i \le n). \tag{3.5}$$

The vertices of Γ_n are representatives of the equivalence classes $[\widetilde{P}]$ for each $\widetilde{P} \in V_{\widetilde{\Gamma}_n}$; we have the quotient map induced by the equivalence relation:

$$p_v : V_{\widetilde{\Gamma}_n} \to V_{\Gamma_n}, \quad \widetilde{P} \mapsto [\widetilde{P}]. \tag{3.6}$$

Next, we define an equivalence relation $\equiv$ on the edges in $\widetilde{\Gamma}_n$. Write a directed edge $\widetilde{e} = (\mathsf{tail}(\widetilde{e}), \mathsf{head}(\widetilde{e}))$. For each $\widetilde{e} = (\widetilde{A}, \widetilde{B})$ and $\tilde{e}' = (\widetilde{A}', \widetilde{B}')$, set

$$\widetilde{e} \equiv \tilde{e}' \iff \widetilde{A} \sim \widetilde{A}', \widetilde{B} \sim \widetilde{B}' \text{ or } \widetilde{A} \sim \widetilde{B}', \widetilde{B} \sim \widetilde{A}'. \tag{3.7}$$

The edges of Γ_n are representatives of equivalence classes $[\widetilde{e}]$ for each $\widetilde{e} \in E_{\widetilde{\Gamma}_n}$. Here, $[\widetilde{e}]$ is a *non-directed* edge connecting $p_v(\mathsf{tail}(\widetilde{e}))$ and $p_v(\mathsf{head}(\widetilde{e}))$. We denote a non-directed edge $e = \{A, B\}$ if e connects vertices A and B. The quotient map is defined by

$$p_e : E_{\widetilde{\Gamma}_n} \to E_{\Gamma_n}, \quad \widetilde{e} \mapsto [\widetilde{e}]. \tag{3.8}$$

From (3.5), $\widetilde{A}_{i,j} \sim \widetilde{A}_{i,j+3n}$ and $\widetilde{B}_{i,j} \sim \widetilde{B}_{i,j+3n}$ for all indices. Therefore, there are $3n(n+1)$-many equivalence classes in $V_{\widetilde{\Gamma}_n}$, each represented by $\widetilde{A}_{i,j}$ for $0 \le i \le n, 1 \le j \le 3n$. Set $A_{i,j} := p_v(\widetilde{A}_{i,j})$ for $0 \le i \le n, 1 \le j \le 3n$.

In summary, Γ_n is a finite graph embedded in a Möbius strip. For instance, consider Γ_5 in Figure 3. Each of the vertices $A_{1,1}, A_{2,1}, A_{3,1}, A_{4,1}, A_{5,1}$ are connected to $A_{5,10}, A_{4,9}, A_{3,8}, A_{2,7}, A_{1,6}$, respectively.

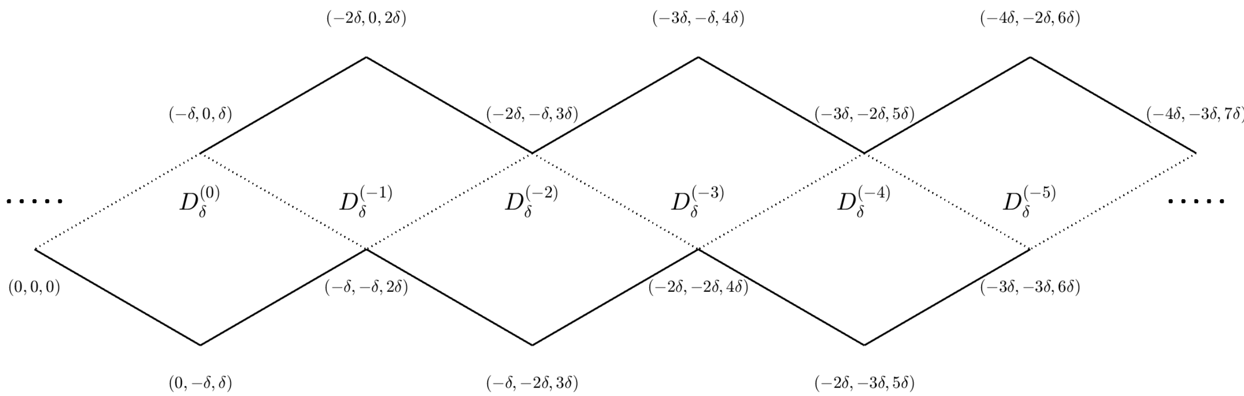

Figure 4: The infinite strip $\widetilde{B}_\delta$ contained in B.

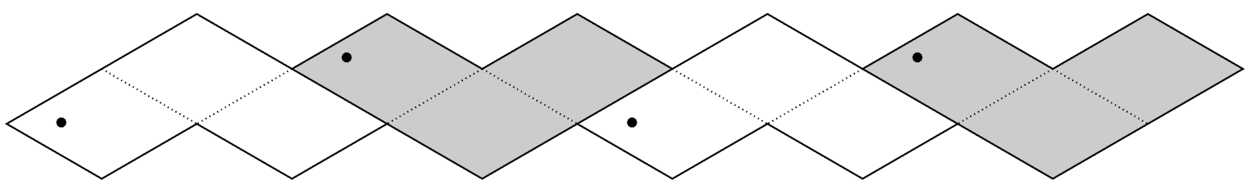

Figure 5: The Möbius strip B_δ and its covering space $\widetilde{B}_\delta$.

4 Covering space of Möbius strips

Fix $\delta \in \mathbb{N}$. For each $k \in \mathbb{Z}$, define subsets of B

$$D_\delta^{(2k)} := \{(x, y, z) \in B \mid (k-1)\delta \leq x \leq k\delta, \quad (k-1)\delta \leq y \leq k\delta\}, \tag{4.1}$$

$$D_\delta^{(2k+1)} := \{(x, y, z) \in B \mid (k-1)\delta \leq x \leq k\delta, \quad k\delta \leq y \leq (k+1)\delta\}, \tag{4.2}$$

$$\widetilde{B}_\delta := \bigcup_{k \in \mathbb{Z}} D_\delta^{(k)}. \tag{4.3}$$

$\widetilde{B}_\delta$ is depicted in Figure 4, as an infinite zigzag strip. Here, $D_\delta^{(k)}$ is a rhombus. In Figure 4, there are six rhombi, which are $D_\delta^{(0)}, D_\delta^{(-1)}, \cdots, \cdots D_\delta^{(-5)}$, from left to right.

We want to define a quotient space B_δ of $\widetilde{B}_\delta$. Intuitively, we "slice" $\widetilde{B}_\delta$ into pieces and identify them into one to construct B_δ. See Figure 5, where the four bold points are identified as one element in B_δ.

To write a formal definition, define an equivalence relation on B, namely

$$(x, y, z) \sim (y - 2\delta, x - \delta, z + 3\delta). \tag{4.4}$$

Denote the quotient map by $q : B \to B/\sim$. Define $B_\delta := q(\widetilde{B}_\delta)$. $\widetilde{B}_\delta$ is an infinite strip whereas B_δ is a Möbius strip; see Figure 5.

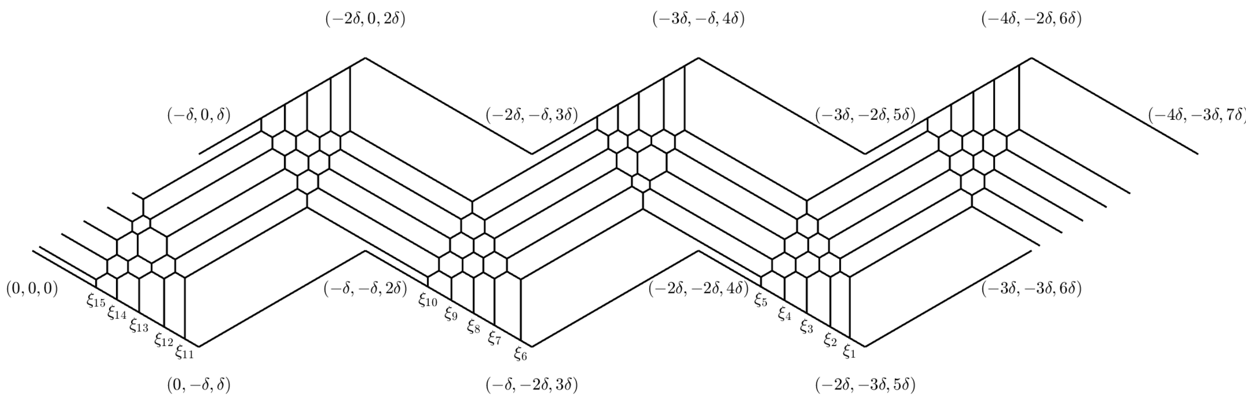

Figure 6: Image of $\widetilde{h}$ contained in $\widetilde{B}_\delta$ when $n = 5$.

By the equivalence relation on B, $D_\delta^{(k)}$ is identified to $D_\delta^{(k-3)}$ for all $k \in \mathbb{Z}$. For instance, $D_\delta^{(0)}$ and $D_\delta^{(-3)}$, $D_\delta^{(-1)}$ and $D_\delta^{(-4)}$, $D_\delta^{(-2)}$ and $D_\delta^{(-5)}$ are identified by the map q in Figure 4.

5 Möbius honeycombs

Define a direction map $d : E_{\widetilde{\Gamma}_n} \to B$ by mapping

$$(\widetilde{A}_{i,j}, \widetilde{B}_{i-1,j-1}) \mapsto (0,-1,1), \quad (\widetilde{A}_{i,j}, \widetilde{B}_{i-1,j}) \mapsto (1,0,-1), \quad (\widetilde{A}_{i,j}, \widetilde{B}_{i,j}) \mapsto (-1,1,0). \quad (5.1)$$

As in Figure 1, d maps each southeast edges to $(0,-1,1)$, southwest edges to $(1,0,-1)$, and north edges to $(-1,1,0)$. Define a function $\widetilde{h} : V_{\widetilde{\Gamma}_n} \to B$ satisfying

$$\widetilde{h}(\mathsf{head}(\widetilde{e})) - \widetilde{h}(\mathsf{tail}(\widetilde{e})) \in \{a \cdot v \in B \mid a \geq 0, v = d(\widetilde{e})\}, \quad \widetilde{e} \in E_{\widetilde{\Gamma}_n}. \quad (5.2)$$

Consider $\widetilde{A}_{0,1}, \widetilde{A}_{0,2}, \cdots, \widetilde{A}_{0,3n}$, which are representatives of equivalence classes of boundary vertices. For instance, in Figure 1, these vertices are on the lowest level, from right to left. Add conditions on $\widetilde{h}$ so that it satisfies

$$\widetilde{h}(\widetilde{A}_{0,j}) \in \{(-2\delta, 2\delta - \xi, \xi) \mid 4\delta \leq \xi \leq 5\delta\}, \quad (1 \leq j \leq n) \quad (5.3a)$$

$$\widetilde{h}(\widetilde{A}_{0,j}) \in \{(-\delta, \delta - \xi, \xi) \mid 2\delta \leq \xi \leq 3\delta\}, \quad (n+1 \leq j \leq 2n) \quad (5.3b)$$

$$\widetilde{h}(\widetilde{A}_{0,j}) \in \{(0, -\xi, \xi) \mid 0 \leq \xi \leq \delta\}, \quad (2n+1 \leq j \leq 3n). \quad (5.3c)$$

When $n = 5$, for each $1 \leq j \leq 5$, $\widetilde{A}_{0,j}$ should be mapped to the line segment connecting $(-2\delta, -3\delta, 5\delta)$ and $(-2\delta, -2\delta, 4\delta)$, which is in the boundary of $D_\delta^{(-4)}$; see Figure 4. The cases of $6 \leq j \leq 10$ and $11 \leq j \leq 15$ can be interpreted in similar fashion.

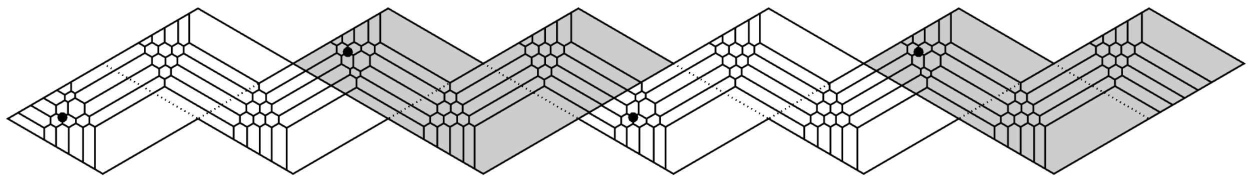

Figure 7: Images of $\widetilde{h}$ repeated due to identification.

The last condition on $\widetilde{h}$ is

$$\widetilde{P}_1 \sim \widetilde{P}_2 \ \in V_{\widetilde{\Gamma}_n} \quad \Rightarrow \quad \widetilde{h}(\widetilde{P}_1) \sim \widetilde{h}(\widetilde{P}_2) \ \in B. \tag{5.4}$$

For fixed $\delta \in \mathbb{N}$, $\widetilde{h} : V_{\widetilde{\Gamma}_n} \to B$ is a **Möbius honeycomb** of size δ if $\widetilde{h}$ satisfies (5.2), (5.3) and (5.4). In (5.3), write ξ_i as the z-coordinate of $\widetilde{h}(\widetilde{A}_{0,i})$. Define the **boundary condition** of $\widetilde{h}$ as $(\xi_1, \cdots, \xi_{3n})$ and denote it as $\partial(\widetilde{h})$. See Figure 6 and 7 for illustrations of Möbius honeycombs.

6 Saturation of Newell–Littlewood numbers

Our goal is to generalize Theorem 4 and Theorem 5 from Littlewood–Richardson coefficients to Newell–Littlewood numbers. As a result, we have Theorem 6 and Theorem 7, leading to Theorem 2.

Theorem 6 ([17, Theorem 3.1])**.** *Let $\lambda, \mu, \nu \in \mathrm{Par}_n$ and $\delta \in \mathbb{N}$ such that $\delta \geq \lambda_1, \mu_1, \nu_1$. Then $N_{\lambda,\mu,\nu}$ counts the number of Möbius honeycombs $\widetilde{h}$ of size δ satisfying:*

- $\partial(\widetilde{h}) = (\lambda_1 + 4\delta, \cdots, \lambda_n + 4\delta, \mu_1 + 2\delta, \cdots, \mu_n + 2\delta, \nu_1, \cdots, \nu_n)$*, and*
- $\forall \widetilde{W} \in V_{\widetilde{\Gamma}_n}$, $\widetilde{h}(\widetilde{W}) \in B_{\mathbb{Z}}$.

Proof. By Theorem 4, $c^{\lambda}_{\beta,\gamma} c^{\mu}_{\gamma,\alpha} c^{\nu}_{\alpha,\beta}$ is the number of ordered triples $(h_\lambda, h_\mu, h_\nu)$ of honeycombs satisfying:

- $\partial(h_\lambda) = (\lambda, \beta, \gamma)$, $\partial(h_\mu) = (\mu, \gamma, \alpha)$, $\partial(h_\nu) = (\nu, \alpha, \beta)$, and
- $\forall v \in V_{\Delta_n}$, $h_\lambda(v), h_\mu(v), h_\nu(v) \in B_{\mathbb{Z}}$.

If $c^{\lambda}_{\beta,\gamma} c^{\mu}_{\gamma,\alpha} c^{\nu}_{\alpha,\beta} \neq 0$, then $\delta \geq \alpha_1, \beta_1, \gamma_1$ follows from $\delta \geq \lambda_1, \mu_1, \nu_1$. As a result,

$$\forall v \in V_{\Delta_n}, \quad h_\lambda(v), h_\mu(v), h_\nu(v) \in D^{(0)}_\delta. \tag{6.1}$$

We have infinite copies of three different types of rhombi depicted in Figure 8. Each type of rhombi is arranged in B as follows.

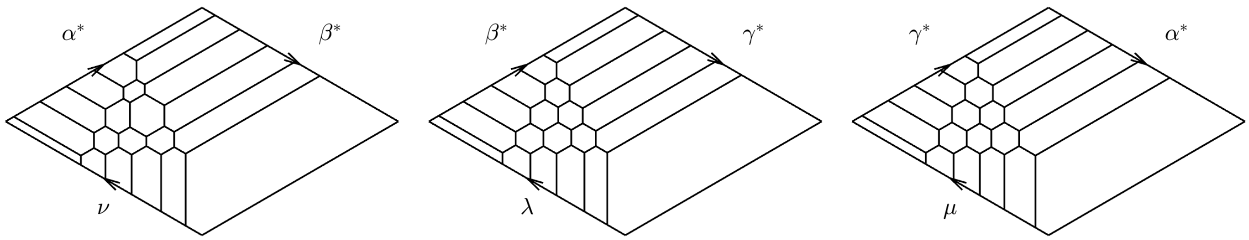

Figure 8: Image of h_λ, h_μ, h_ν contained in $D_\delta^{(0)}$ when $n = 5$.

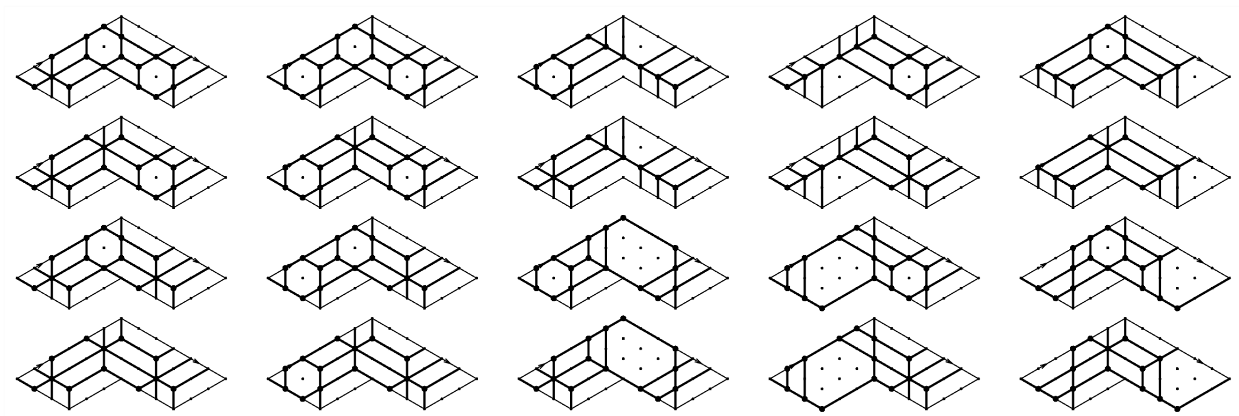

Figure 9: $n = 3$, $\delta = 3$, $\lambda = \mu = \nu = (3, 2, 1)$. Then $N_{\lambda,\mu,\nu} = 20$.

- h_λ rhombus: $\cdots, D_\delta^{(-4)}, D_\delta^{(-1)}, D_\delta^{(2)}, D_\delta^{(5)}, \cdots$
- h_μ rhombus: $\cdots, D_\delta^{(-5)}, D_\delta^{(-2)}, D_\delta^{(1)}, D_\delta^{(4)}, \cdots$
- h_ν rhombus: $\cdots, D_\delta^{(-6)}, D_\delta^{(-3)}, D_\delta^{(0)}, D_\delta^{(3)}, \cdots$

Gluing pieces along the line segments α^*, β^* and γ^*, we have $\widetilde{h}$ satisfying the given conditions. Therefore, the number of $\widetilde{h}$ satisfying the given conditions is greater than or equal to $N_{\lambda,\mu,\nu}$.

Conversely, by slicing $\widetilde{B}_\delta$ into $D_\delta^{(k)}$, we can reverse the process above, proving the other side of the inequality. □

In short, $c^\lambda_{\beta,\gamma} c^\nu_{\gamma,\alpha} c^\nu_{\alpha,\beta}$ leads to gluing three honeycombs, which is a Möbius strip. Here, the boundary of Möbius strip is chosen as in Figure 7 so that gluing process can be reversed.

For instance, let $n = 3$ and $\lambda = \mu = \nu = (3, 2, 1)$. Since $\lambda_1 = \mu_1 = \nu_1 = 3$, take $\delta = 3$. In Figure 9, the number of Möbius honeycombs satisfying the conditions is 20. Therefore, $N_{\lambda,\mu,\nu} = 20$.

Theorem 7 ([17, Theorem 3.2]). *Let $\delta \in \mathbb{N}$ and $\widetilde{h}$ be a Möbius honeycomb of size δ such that $\partial(\widetilde{h}) = (\xi_1, \cdots, \xi_{3n})$ in $\mathbb{Z}^{3n}$ and $\sum_{1\leq j\leq 3n} \xi_j \equiv 0 \pmod 2$. Then there exists a Möbius honeycomb $\widetilde{g}$ of size δ with:*

- $\partial(\widetilde{g}) = \partial(\widetilde{h})$, *and*
- $\forall \widetilde{W} \in V_{\widetilde{\Gamma}_n}, \widetilde{g}(\widetilde{W}) \in B_{\mathbb{Z}}$.

For the existence of $\widetilde{g}$ in Theorem 7, see [17, Section 4] for the construction of *largest lifts* of Möbius honeycombs and [17, Section 5] for readjustment. As a corollary, we have Theorem 2.

Proof of Theorem 2. Suppose that $N_{k\lambda,k\mu,k\nu} > 0$. Choose $\delta \in \mathbb{N}$ such that $\delta \geq \lambda_1, \mu_1, \nu_1$. By Theorem 6, there exists a Möbius honeycomb $\widetilde{h}$ of size $k\delta$ satisfying

$$\partial(\widetilde{h}) = (k\lambda_1 + 4k\delta, \cdots, k\lambda_n + 4k\delta, k\mu_1 + 2k\delta, \cdots, k\mu_n + 2k\delta, k\nu_1, \cdots, k\nu_n). \tag{6.2}$$

Due to [17, Lemma A.7], $\frac{1}{k}\widetilde{h}$ is a Möbius honeycomb of size δ and

$$\partial\left(\frac{1}{k}\widetilde{h}\right) = (\lambda_1 + 4\delta, \cdots, \lambda_n + 4\delta, \mu_1 + 2\delta, \cdots, \mu_n + 2\delta, \nu_1, \cdots, \nu_n). \tag{6.3}$$

In particular, $\partial\left(\frac{1}{k}\widetilde{h}\right) \in \mathbb{Z}^{3n}$ and the sum of its components is $|\lambda| + |\mu| + |\nu| + 6n\delta$, which is an even integer due to the condition $|\lambda| + |\mu| + |\nu| \equiv 0 \pmod 2$. Apply Theorem 7 to find a Möbius honeycomb $\widetilde{g}$ of size δ such that

$$\partial\left(\frac{1}{k}\widetilde{h}\right) = \partial(\widetilde{g}) \text{ and } \forall \widetilde{W} \in V_{\widetilde{\Gamma}_n},\ \widetilde{g}(\widetilde{W}) \in B_{\mathbb{Z}}. \tag{6.4}$$

Due to the existence of $\widetilde{g}$, $N_{\lambda,\mu,\nu} > 0$ follows from Theorem 6. □

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References

[1] P. Belkale and S. Kumar. "Eigencone, saturation and Horn problems for symplectic and odd orthogonal groups". *J. Algebraic Geom.* **19**.2 (2010), pp. 199–242. DOI.

[2] H. Derksen and J. Weyman. "Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients". *J. Amer. Math. Soc.* **13**.3 (2000), pp. 467–479. DOI.

[3] W. Fulton and J. Harris. *Representation Theory: A First Course*. Graduate Texts in Mathematics. Springer New York, 1991. DOI.

[4] W. Fulton. "Eigenvalues, invariant factors, highest weights, and Schubert calculus". *Bull. Amer. Math. Soc. (N.S.)* **37**.3 (2000), pp. 209–249. DOI.

[5] S. Gao, G. Orelowitz, N. Ressayre, and A. yong. "Newell-Littlewood numbers III: eigencones and GIT-semigroups". 2022. arXiv:2107.03152.

[6] S. Gao, G. Orelowitz, and A. Yong. "Newell-Littlewood numbers". *Trans. Amer. Math. Soc.* **374**.9 (2021), pp. 6331–6366. DOI.

[7] S. Gao, G. Orelowitz, and A. Yong. "Newell-Littlewood numbers II: extended Horn inequalities". *Algebr. Comb.* **5**.6 (2022), pp. 1287–1297. DOI.

[8] A. Horn. "Eigenvalues of sums of Hermitian matrices". *Pacific J. Math.* **12** (1962), pp. 225–241. Link.

[9] M. Kapovich, S. Kumar, and J. J. Millson. "The eigencone and saturation for Spin(8)". *Pure Appl. Math. Q.* **5**.2, Special Issue: In honor of Friedrich Hirzebruch. Part 1 (2009), pp. 755–780. DOI.

[10] M. Kapovich and J. J. Millson. "A path model for geodesics in Euclidean buildings and its applications to representation theory". *Groups Geom. Dyn.* **2**.3 (2008), pp. 405–480. DOI.

[11] M. Kapovich and J. J. Millson. "Structure of the tensor product semigroup". *Asian J. Math.* **10**.3 (2006), pp. 493–539. DOI.

[12] J. Kiers. "On the saturation conjecture for $\mathrm{Spin}(2n)$". *Exp. Math.* **30**.2 (2021), pp. 258–267. DOI.

[13] A. A. Klyachko. "Stable bundles, representation theory and Hermitian operators". *Selecta Math. (N.S.)* **4**.3 (1998), pp. 419–445. DOI.

[14] A. Knutson and T. Tao. "The honeycomb model of $\mathrm{GL}_n(\mathbf{C})$ tensor products. I. Proof of the saturation conjecture". *J. Amer. Math. Soc.* **12**.4 (1999), pp. 1055–1090. DOI.

[15] K. Koike. "On the decomposition of tensor products of the representations of the classical groups: by means of the universal characters". *Adv. Math.* **74**.1 (1989), pp. 57–86. DOI.

[16] K. Koike and I. Terada. "Young-diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, D_n". *J. Algebra* **107**.2 (1987), pp. 466–511. DOI.

[17] J. Min. "Proof of the Newell-Littlewood saturation conjecture". 2024. arXiv:2409.00233.

[18] S. V. Sam. "Symmetric quivers, invariant theory, and saturation theorems for the classical groups". *Adv. Math.* **229**.2 (2012), pp. 1104–1135. DOI.