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Descents and flag major index on conjugacy classes of colored permutation groups without short cycles

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Abstract. We consider the descent and flag major index statistics on the colored permutation groups $\mathfrak{S}_{n,r} = \mathbb{Z}_r \wr \mathfrak{S}_n$ and their conjugacy classes. We show that the k-th moments of these statistics on $\mathfrak{S}_{n,r}$ will coincide with the corresponding moments on all conjugacy classes with no cycles of lengths $1, 2, \ldots, 2k$. Using this, we establish the asymptotic normality of the descent and flag major index statistics on conjugacy classes of $\mathfrak{S}_{n,r}$ with sufficiently long cycles. Our results generalize prior work of Fulman involving the descent and major index statistics on the symmetric group $\mathfrak{S}_n$. Our methods involve an intricate extension of Fulman's work on $\mathfrak{S}_n$ combined with the theory of degrees for colored permutation statistics, as introduced by Campion Loth, Levet, Liu, Sundaram, and Yin.

Keywords: colored permutation groups, permutation statistics, descents, flag major index, asymptotic normality, cycle type

1 Introduction

Statistics on the symmetric group $\mathfrak{S}_n$ and its generalizations are a major area of study in combinatorics. We consider statistics defined over the colored permutation groups, which are wreath products $\mathfrak{S}_{n,r} = \mathbb{Z}_r \wr \mathfrak{S}_n$. Colored permutation groups play an essential role in the classification of complex reflection groups [17], and they contain the symmetric groups $\mathfrak{S}_n \cong \mathfrak{S}_{n,1}$ and the signed symmetric groups $B_n \cong \mathfrak{S}_{n,2}$ as special cases.

Numerous statistics on $\mathfrak{S}_{n,r}$ have been studied, many of which generalize corresponding ones on $\mathfrak{S}_n$ and B_n. See [18] and [7] for examples. We will focus on the descent and flag major index statistics on $\mathfrak{S}_{n,r}$, which respectively generalize the descent

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and major index statistics on $\mathfrak{S}_n$. The descent statistic $\mathrm{des}_{n,r}$ on $\mathfrak{S}_{n,r}$ was introduced by Steingrímsson [18], who showed that $\mathrm{des}_{n,r}$ is equidistributed with the excedance statistic on $\mathfrak{S}_{n,r}$, and its generating function satisfies

$$\frac{1}{(1-q)^{n+1}} \sum_{(\omega,\tau)\in\mathfrak{S}_{n,r}} q^{\mathrm{des}_{n,r}(\omega,\tau)} = \sum_{i=0}^{\infty}(ir+1)^n q^i. \tag{1.1}$$

The flag major index statistic $\mathrm{fmaj}_{n,r}$ was introduced by Adin and Roichman [1] in the study of stable algebras. They showed that $\mathrm{fmaj}_{n,2}$ on the signed symmetric group $B_n \cong \mathfrak{S}_{n,2}$ is equidistributed with the length statistic on B_n. Subsequent work by Haglund, Loehr, and Remmel [12] established that the general distribution of $\mathrm{fmaj}_{n,r}$ is given by

$$\sum_{(\omega,\tau)\in\mathfrak{S}_{n,r}} q^{\mathrm{fmaj}_{n,r}(\omega,\tau)} = [r]_q[2r]_q\cdots[nr]_q, \tag{1.2}$$

where $[ir]_q = 1+q+q^2+\cdots+q^{ir-1}$ is the q-integer. This coincides with the Poincáre polynomial of $\mathfrak{S}_{n,r}$ as a complex reflection group [10, Theorem 1.4 and Table 1].

Main results

We study the statistics $\mathrm{des}_{n,r}$ and $\mathrm{fmaj}_{n,r}$ on conjugacy classes of $\mathfrak{S}_{n,r}$ with sufficiently long cycles. Recall that a conjugacy class in $\mathfrak{S}_n$ is uniquely determined by the common cycle type of the permutations in the class, and this cycle type is recorded using a partition λ of n. Elements in $\mathfrak{S}_{n,r}$ can also be expressed in cycle notation, and this leads to a generalized notion of cycle type that determines conjugacy classes of $\mathfrak{S}_{n,r}$. Similar to the usage of C_λ for conjugacy classes of $\mathfrak{S}_n$, we use $C_{\boldsymbol{\lambda}}$ to denote the conjugacy classes of $\mathfrak{S}_{n,r}$ indexed by $\boldsymbol{\lambda}$.

Though there is some prior work involving statistics on conjugacy classes of $\mathfrak{S}_n$ [5, 8, 11] and $B_n \cong \mathfrak{S}_{n,2}$ [9, 16], statistics on conjugacy classes of general colored permutation groups have not been explored heavily. The main theoretical advance appears in recent work by Campion Loth, Levet, Liu, Sundaram, and Yin [3]. Our main result strengthens [3, Theorem 1.1] for the statistics $\mathrm{des}_{n,r}$ and $\mathrm{fmaj}_{n,r}$.

Theorem 1.1. *Let $C_{\boldsymbol{\lambda}}$ be a conjugacy class of $\mathfrak{S}_{n,r}$. If $C_{\boldsymbol{\lambda}}$ has no cycles of lengths $1,2,\ldots,2k$, then the k-th moments of $\mathrm{des}_{n,r}$ and $\mathrm{fmaj}_{n,r}$ on $C_{\boldsymbol{\lambda}}$ match the respective k-th moments on $\mathfrak{S}_{n,r}$.*

The descent and flag major index statistics are known to be asymptotically normal on $\mathfrak{S}_{n,r}$ [4]. Combining this fact with the Method of Moments and Theorem 1.1, we obtain the following corollary, which shows asymptotic normality of $\mathrm{des}_{n,r}$ and $\mathrm{fmaj}_{n,r}$ on conjugacy classes with sufficiently long cycles.

Corollary 1.2. *For every $n \geq 1$, let $C_{\boldsymbol{\lambda}_n}$ be a conjugacy class of $\mathfrak{S}_{n,r}$ such that for all i, the number of cycles of length i in $\boldsymbol{\lambda}_n$ approaches 0 as $n \to \infty$. Let stat_n for $n \geq 1$ be either the descent or flag major index statistic on $C_{\boldsymbol{\lambda}_n}$ with mean μ_n and variance σ_n^2. Then as $n \to \infty$, the random variable $(\mathrm{stat}_n - \mu_n)/\sigma_n$ converges in distribution to the standard normal distribution.*

2 Preliminaries

We begin with preliminaries on the colored permutation groups $\mathfrak{S}_{n,r}$, their conjugacy classes, and the specific statistics considered in this paper. Our definitions are primarily based on what is given in [18] and [4]. For properties of the conjugacy classes of $\mathfrak{S}_{n,r}$, we use [14] as a reference, which contains a more general treatment of wreath products.

2.1 Colored permutation groups and statistics

Let $\mathbb{Z}_r$ be the group of integers modulo r and $\mathfrak{S}_n$ be the symmetric group on $[n] = \{1, 2, \ldots, n\}$. The *colored permutation group* $\mathfrak{S}_{n,r}$ is the wreath product $\mathbb{Z}_r \wr \mathfrak{S}_n$, which is the semidirect product $\mathbb{Z}_r^n \rtimes \mathfrak{S}_n$ formed from the permutation action of $\mathfrak{S}_n$ on $\mathbb{Z}_r^n$. An element in $\mathfrak{S}_{n,r}$ is called a *colored permutation*, and it will be denoted (ω, τ), where $\omega \in \mathfrak{S}_n$ and $\tau : [n] \to \mathbb{Z}_r$ is a function referred to as a *coloring*. For brevity, we will usually express τ in the form $(\tau(1), \ldots \tau(n))$. From its construction as a wreath product, the group operation on $\mathfrak{S}_{n,r}$ is defined as

$$(\omega_1, \tau_1)(\omega_2, \tau_2) = (\omega_1\omega_2, (\tau_1 \circ \omega_2) + \tau_2).$$

The colored permutation group $\mathfrak{S}_{n,r}$ can be embedded as a subgroup of the symmetric group $\mathfrak{S}_{rn}$, which we describe explicitly. Let $[n]^r$ denote the set of rn elements

$$\{i^c : i \in [n],\, c \in \mathbb{Z}_r\},$$

where the superscript indicates the *color* of an element in $[n]$. One can view the colored permutation (ω, τ) as a bijection on $[n]^r$. We abuse notation and also denote this bijection (ω, τ), and it is defined by $(\omega, \tau)(i^c) = \omega(i)^{\tau(i)+c}$ for all $i \in [n]$ and $c \in \mathbb{Z}_r$. Since the images of i^0 for $i \in [n]$ are sufficient for determining $(\omega, \tau) \in \mathfrak{S}_{n,r}$, one can use these to form the two-line and one-line notations of (ω, τ).

Example 2.1. Consider $\omega = [3, 8, 5, 6, 2, 1, 4, 7] \in \mathfrak{S}_8$ and $\tau = (1, 0, 0, 1, 2, 2, 0, 1) \in \mathbb{Z}_3^8$. This defines an element in $\mathfrak{S}_{8,3}$ whose two-line and one-line notations are

$$(\omega, \tau) = \begin{bmatrix} 1^0 & 2^0 & 3^0 & 4^0 & 5^0 & 6^0 & 7^0 & 8^0 \\ 3^1 & 8^0 & 5^0 & 6^1 & 2^2 & 1^2 & 4^0 & 7^1 \end{bmatrix} = \begin{bmatrix} 3^1 & 8^0 & 5^0 & 6^1 & 2^2 & 1^2 & 4^0 & 7^1 \end{bmatrix}. \tag{2.1}$$

We now define the statistics relevant for our work. For any $(\omega, \tau) \in \mathfrak{S}_{n,r}$, an index $i \in [n]$ is a *descent* of (ω, τ) if $\tau(i) > \tau(i+1)$, or $\tau(i) = \tau(i+1)$ and $\omega(i) > \omega(i+1)$, where we use the convention $\tau(n+1) = 0$ and $\omega(n+1) = n+1$. One can alternatively fix the total order on $[n]^r$

$$1^0 < 2^0 < 3^0 < \cdots < 1^1 < 2^1 < 3^1 < \cdots < 1^{r-1} < 2^{r-1} < 3^{r-1} < \cdots \tag{2.2}$$

and define a descent to be any $i \in [n]$ such that $(\omega, \tau)(i^0) > (\omega, \tau)((i+1)^0)$, with the convention that $(\omega, \tau)((n+1)^0) = (n+1)^0$.

Letting $\operatorname{Des}(\omega, \tau)$ denote the set of descents of $(\omega, \tau) \in \mathfrak{S}_{n,r}$, the *descent* and *major index* statistics on $\mathfrak{S}_{n,r}$ are respectively defined as

$$\operatorname{des}_{n,r}(\omega, \tau) = |\operatorname{Des}_{n,r}(\omega, \tau)| \quad \text{and} \quad \operatorname{maj}_{n,r}(\omega, \tau) = \sum_{i \in \operatorname{Des}_{n,r}(\omega,\tau) \cap [n-1]} i.$$

The *color* and *flag major index* statistics on $\mathfrak{S}_{n,r}$ are the nonnegative integers defined by

$$\operatorname{col}_{n,r}(\omega, \tau) = \sum_{i=1}^{n} \tau(i) \quad \text{and} \quad \operatorname{fmaj}_{n,r}(\omega, \tau) = r \cdot \operatorname{maj}_{n,r}(\omega, \tau) + \operatorname{col}_{n,r}(\omega, \tau).$$

Note that the $\operatorname{col}_{n,r}$ statistic uses $\{0, 1, \ldots, r-1\}$ as representative elements in $\mathbb{Z}_r$ and adds them in $\mathbb{Z}$. In the case when $r = 1$, the statistics $\operatorname{des}_{n,r}$ and $\operatorname{fmaj}_{n,r}$ align with the usual descent and major index statistics on $\mathfrak{S}_n$.

Example 2.2. Consider the permutation $(\omega, \tau) \in \mathfrak{S}_{8,3}$ from Example 2.1. The descent set of (ω, τ) is $\{1, 2, 5, 6, 8\}$, and the sum of the colors that appear is 7. Then

$$\operatorname{des}_{8,3}(\omega, \tau) = 5, \quad \operatorname{maj}_{8,3}(\omega, \tau) = 14, \quad \text{and} \quad \operatorname{fmaj}_{8,3}(\omega, \tau) = 3 \cdot 14 + 7 = 49.$$

For any statistic $X : \mathfrak{S}_{n,r} \to \mathbb{R}$, we can consider it as a random variable by equipping $\mathfrak{S}_{n,r}$ with the uniform distribution. The corresponding probability distribution is

$$\Pr_{\mathfrak{S}_{n,r}}[X = i] = |X^{-1}(i)| / |\mathfrak{S}_{n,r}|.$$

For each positive integer k, the k-th moment of X will be denoted $\mathbb{E}_{\mathfrak{S}_{n,r}}[X^k]$. For the descent and flag major index statistics, Chow and Mansour established the following results involving their asymptotic distributions.

Theorem 2.3. *[4] For any positive integers n and r, $\operatorname{des}_{n,r}$ has mean $\mu_{n,r} = \frac{rn+r-2}{2r}$ and variance $\sigma_{n,r}^2 = \frac{n+1}{12}$, and as $n \to \infty$, the standardized random variable $\frac{\operatorname{des}_{n,r} - \mu_{n,r}}{\sigma_{n,r}}$ converges to a standard normal distribution.*

Theorem 2.4. *[4] For any positive integers n and r, $\operatorname{fmaj}_{n,r}$ has mean $\mu_{n,r} = \frac{n(rn+r-2)}{4}$ and variance $\sigma_{n,r}^2 = \frac{2r^2n^3+3r^2n^2+(r^2-6)n}{72}$, and as $n \to \infty$, the standardized random variable $\frac{\operatorname{fmaj}_{n,r} - \mu_{n,r}}{\sigma_{n,r}}$ converges to a standard normal distribution.*

Our results will utilize a tool called the Method of Moments, as described in [2, Section 30]. In general, two different probability distributions can share the same moments. We will be primarily interested in normal distributions, which are uniquely determined by their moments. This allows us to apply the following theorem.

Theorem 2.5 (Method of Moments)**.** *Suppose $\{X_n\}_{n \geq 1}$ and Y are real-valued random variables with finite k-th moments for all k. If Y is uniquely determined by its moments and*

$$\lim_{n \to \infty} \mathbb{E}[X_n^k] = \mathbb{E}[Y^k],$$

for all k, then X_n converges in distribution to Y.

2.2 Conjugacy classes of colored permutation groups

Our work will focus on conjugacy classes of $\mathfrak{S}_{n,r}$, which we now describe. Similar to permutations in $\mathfrak{S}_n$, colored permutations also have a cycle notation. Starting with (ω, τ), one can express ω in the usual cycle notation with color 0 on all elements and then insert $\omega(i)^{\tau(i)}$ under i^0 for each $i \in [n]$. We will refer to this as the *two-line cycle notation*. Removing the first row in every cycle then results in the *cycle notation* for (ω, τ).

Example 2.6. Consider the permutation given in Example 2.1. The two-line and one-line cycle notations are given by

$$(\omega, \tau) = \begin{pmatrix} 1^0 & 3^0 & 5^0 & 2^0 & 8^0 & 7^0 & 4^0 & 6^0 \\ 3^1 & 5^0 & 2^2 & 8^0 & 7^1 & 4^0 & 6^1 & 1^2 \end{pmatrix} = (3^1 5^0 2^2 8^0 7^1 4^0 6^1 1^2).$$

The cycle notation leads to a notion of cycle type for colored permutations. An *r-partition* of $n \in \mathbb{Z}_+$ is an r-tuple of partitions $\boldsymbol{\lambda} = (\lambda^j)_{j=0}^{r-1}$ where each λ^j is a partition of some nonnegative integer n_j such that $\sum_{j=0}^{r-1} n_j = n$. For any cycle in the cycle notation of $(\omega, \tau) \in \mathfrak{S}_{n,r}$, its *length* is the number of elements in it, and its *color* is the sum of the colors that appear (as an element in $\mathbb{Z}_r$). The *cycle type* of $(\omega, \tau) \in \mathfrak{S}_{n,r}$ is the r-partition $\boldsymbol{\lambda}$ where λ^j records the cycle lengths for the cycles with color j.

Example 2.7. Consider the colored permutation in $\mathfrak{S}_{9,3}$ with cycle notation

$$(\omega, \tau) = (1^0 3^2 7^1 6^0)(2^1)(4^2 5^0)(8^0)(9^1).$$

Since $r = 3$, the cycle type of this colored permutation is

$$\boldsymbol{\lambda} = (\lambda^0, \lambda^1, \lambda^2) = ((1, 4), (1^2), (2)),$$

where each partition has been expressed in multiplicative notion $(1^{a_1}, 2^{a_2}, \ldots, n^{a_n})$.

As in $\mathfrak{S}_n$, the conjugacy classes of $\mathfrak{S}_{n,r}$ are determined by cycle type.

Proposition 2.8. *[14, Theorem 4.2.8 and Lemmas 4.2.9-4.2.10] Two elements $(\omega, \tau), (\omega', \tau') \in \mathfrak{S}_{n,r}$ are conjugate if and only if they share the same cycle type.*

Throughout, we use $C_{\boldsymbol{\lambda}}$ for the conjugacy class consisting of all colored permutations with cycle type $\boldsymbol{\lambda}$. For a statistic X on $\mathfrak{S}_{n,r}$, we can restrict X to $C_{\boldsymbol{\lambda}}$ and equip $C_{\boldsymbol{\lambda}}$ with the uniform distribution to consider X as a random variable. X then has a discrete probability distribution

$$\Pr_{C_{\boldsymbol{\lambda}}}[X = i] = |X^{-1}(i) \cap C_{\boldsymbol{\lambda}}| / |C_{\boldsymbol{\lambda}}|.$$

Note that this is equivalent to the conditional distribution $\Pr_{\mathfrak{S}_{n,r}}[X = i \mid C_{\boldsymbol{\lambda}}]$, and the above notation is introduced for brevity. For each positive integer k, the corresponding notation for the k-th moment of X on $C_{\boldsymbol{\lambda}}$ will be $\mathbb{E}_{C_{\boldsymbol{\lambda}}}[X^k]$.

2.3 Statistics on conjugacy classes with sufficiently long cycles

The paper [3] analyzes moments of statistics on conjugacy classes of $\mathfrak{S}_{n,r}$ with sufficiently long cycles. We will describe the parts of this work relevant to our results and refer the reader to [3] for a detailed account. See also [13] for additional results specific to the symmetric group.

A *partial colored permutation* on $\mathfrak{S}_{n,r}$ is a pair (K,κ) where $K = \{(i_h, j_h)\}_{h=1}^m$ consists of distinct ordered pairs of elements in $[n]$ and $\kappa : \{i_1, \ldots, i_m\} \to \mathbb{Z}_r$ is any function. We call m the *size* of (K,κ), and also denote this as $|(K,\kappa)|$. One can alternatively express (K,κ) using a single set of ordered pairs of elements in $[n]^r$ as

$$(K,\kappa) = \left\{ \left(i_h^0, j_h^{\kappa(i_h)} \right) \right\}_{h=1}^m .$$

Indeed, the correspondence between these notations is clear.

A permutation $\omega \in \mathfrak{S}_n$ *satisfies* K if $\omega(i_h) = j_h$ for all $h \in [m]$. A coloring $\tau : [n] \to \mathbb{Z}_r$ *satisfies* κ if $\tau(i_h) = \kappa(i_h)$ for all $h \in [m]$. A colored permutation $(\omega, \tau) \in \mathfrak{S}_{n,r}$ *satisfies* (K,κ) if ω satisfies K and τ satisfies κ. Viewing (ω,τ) as a bijection on $[n]^r$, this is equivalent to (ω,τ) mapping i_h^0 to $j_h^{\kappa(i_h)}$ for all $h \in [m]$. We use $I_{(K,\kappa)} : \mathfrak{S}_{n,r} \to \{0,1\}$ to denote the indicator function for a colored permutation satisfying (K,κ).

One can view each $I_{(K,\kappa)}$ as locally checking certain values in a colored permutation, and one key insight of [3] is that these indicator functions $I_{(K,\kappa)}$ can be viewed as building blocks for colored permutation statistics. Formally, a colored permutation statistic $X : \mathfrak{S}_{n,r} \to \mathbb{R}$ has *degree m* if it is in the $\mathbb{R}$-vector space spanned by $\{I_{(K,\kappa)} : |(K,\kappa)| \leq m\}$ and not in the vector-space spanned by $\{I_{(K,\kappa)} : |(K,\kappa)| \leq m-1\}$. We give examples below using the statistics relevant to this paper.

Example 2.9. The statistics $\mathrm{des}_{n,r}$ and $\mathrm{fmaj}_{n,r}$ have degree at most 2, as

$$\mathrm{des}_{n,r} = \sum_{i=1}^{n-1} \sum_{j_1^{c_1} < j_2^{c_2}} I_{\{(i^0, j_2^{c_2}), ((i+1)^0, j_1^{c_1})\}} + \sum_{j=1}^{n} \sum_{c=1}^{r-1} I_{\{(n^0, j^c)\}},$$

$$\mathrm{fmaj}_{n,r} = r \cdot \sum_{i=1}^{n-1} \sum_{j_1^{c_1} < j_2^{c_2}} i \cdot I_{\{(i^0, j_2^{c_2}), ((i+1)^0, j_1^{c_1})\}} + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{c=0}^{r-1} c \cdot I_{\{(i^0, j^c)\}}.$$

The condition $j_1^{c_1} < j_2^{c_2}$ is with respect to the total order given in (2.2). One can show that in general, $\mathrm{des}_{n,r}$ and $\mathrm{fmaj}_{n,r}$ have degree exactly 2, but this is not needed for our work.

We will utilize the following result involving degree. Notice that our Theorem 1.1 strengthens the following result for $\mathrm{des}_{n,r}$ and $\mathrm{fmaj}_{n,r}$, as Example 2.9 shows these statistics have degree at most 2.

Theorem 2.10. *[3, Theorem 1.1] Suppose $X : \mathfrak{S}_{n,r} \to \mathbb{R}$ has degree at most m. For any $k \geq 1$, the k-th moment $\mathbb{E}_{C_\lambda}[X^k]$ coincides on all conjugacy classes C_λ of $\mathfrak{S}_{n,r}$ with no cycles of lengths $1, 2, \ldots, mk$.*

3 Descents

In this section, we consider Theorem 1.1 and Corollary 1.2 for $\mathrm{des}_{n,r}$. Throughout, define X_i to be the indicator function for a descent at position i,

$$X_i(\omega, \tau) = \begin{cases} 1 & \text{if } i \in \mathrm{Des}_{n,r}(\omega, \tau) \\ 0 & \text{otherwise.} \end{cases}$$

The descent statistic can be expressed as $\mathrm{des}_{n,r} = \sum_{i=1}^{n} X_i$, so

$$\mathrm{des}_{n,r}^k = \sum_{a_1,\ldots,a_k \in [n]} X_{a_1} \cdots X_{a_k}. \tag{3.1}$$

Since expectation is linear, an understanding of the mean of $X_{a_1} \cdots X_{a_k}$ on $\mathfrak{S}_{n,r}$ or C_λ informs us of the k-th moments of $\mathrm{des}_{n,r}$ on these sets. We begin by considering the mean of $X_{a_1} \ldots X_{a_k}$ on $\mathfrak{S}_{n,r}$, starting with the following definitions based on [8]. Our modifications account for the possibility of a descent at position n in $\mathfrak{S}_{n,r}$, which cannot occur in $\mathfrak{S}_n$.

Definition 3.1. The *Young subgroup* generated by $a_1, \ldots, a_k \in [n]$ is the subgroup J of $\mathfrak{S}_n$ generated by the adjacent transpositions

$$\{(a_1, a_1+1), \ldots, (a_k, a_k+1)\} \setminus \{(n, n+1)\}.$$

The *blocks* induced by $a_1, \ldots, a_k \in [n]$ are the equivalence classes $\mathscr{B}_1, \ldots, \mathscr{B}_t \subseteq [n]$ generated by the following property: $i, j \in [n]$ are in the same equivalence class if some $\omega \in J$ maps i to j. Observe that one can alternatively express $J = \mathfrak{S}_{\mathscr{B}_1} \times \cdots \times \mathfrak{S}_{\mathscr{B}_t}$, where $\mathfrak{S}_{\mathscr{B}_i}$ is the group of permutations on the elements in $\mathscr{B}_i$.

Fulman [8, Proof of Theorem 3] gives an explicit formula for $\mathbb{E}_{\mathfrak{S}_n}[X_{a_1} X_{a_2} \cdots X_{a_k}]$ when $a_1, \ldots, a_k \in [n-1]$. In $\mathfrak{S}_{n,r}$, we will derive the corresponding formulas, and there will be two cases depending on whether or not $a_1, \ldots, a_k$ contains n.

Lemma 3.2. *Let $a_1, \ldots, a_k \in [n]$ with induced blocks $\mathscr{B}_1, \ldots, \mathscr{B}_t$, where $\mathscr{B}_t$ contains n. If $n \notin \{a_1, \ldots, a_k\}$, then*

$$\mathbb{E}_{\mathfrak{S}_{n,r}}[X_{a_1} X_{a_2} \cdots X_{a_k}] = \prod_{i=1}^{t} \frac{1}{|\mathscr{B}_i|!}. \tag{3.2}$$

If $n \in \{a_1, \ldots, a_k\}$, then

$$\mathbb{E}_{\mathfrak{S}_{n,r}}[X_{a_1} \cdots X_{a_k}] = \left(\frac{r-1}{r}\right)^{|\mathscr{B}_t|} \cdot \prod_{i=1}^{t} \frac{1}{|\mathscr{B}_i|!}.$$

We now consider $X_{a_1}\cdots X_{a_k}$ on any C_λ without cycles of lengths $1,2,\dots,2k$. Fix $a_1,\dots,a_k \in [n]$, let $\mathscr{B}_1,\dots,\mathscr{B}_t \subseteq [n]$ be the blocks induced by $a_1,\dots,a_k$, and let $J = \mathfrak{S}_{\mathscr{B}_1}\times\cdots\times\mathfrak{S}_{\mathscr{B}_t}$ be the Young subgroup of $\mathfrak{S}_n$ generated by $a_1,\dots,a_k$. Define an action of J on $\mathfrak{S}_{n,r}$ as follows: for all $\pi \in J$ and $(\omega,\tau) \in \mathfrak{S}_{n,r}$,

$$\pi\cdot(\omega,\tau) = (\pi,\mathbf{0})(\omega,\tau)(\pi,\mathbf{0})^{-1}, \tag{3.3}$$

where $\mathbf{0}$ is the zero coloring. Alternatively, this is the conjugation action of J on $\mathfrak{S}_{n,r}$ after identifying J with the subgroup $J\times\mathbf{0}$. The following result describes orbits under the action given in (3.3).

Lemma 3.3. *Let $(\omega,\tau)\in\mathfrak{S}_{n,r}$. Let $\pi\in\mathfrak{S}_n$ and $\mathbf{0}$ be the zero coloring. If $(i_1^{c_1}, i_2^{c_2},\dots,i_\ell^{c_\ell})$ is a cycle in (ω,τ), then $(\pi(i_1)^{c_1},\pi(i_2)^{c_2},\dots,\pi(i_\ell)^{c_\ell})$ is a cycle in $(\pi,\mathbf{0})(\omega,\tau)(\pi,\mathbf{0})^{-1}$.*

Lemma 3.3 implies that the orbit of any $(\omega,\tau)\in\mathfrak{S}_{n,r}$ under the action in (3.3) consists of colored permutations that can be obtained by fixing a cycle notation of (ω,τ) and permuting elements within each block $\mathscr{B}_1,\dots,\mathscr{B}_t$ without changing the location of colors. On conjugacy classes C_λ without cycles of lengths $1,2,\dots,2k$, we will show that these orbits are particularly well-behaved.

Lemma 3.4. *Let $a_1,\dots,a_k\in[n-1]$ with induced blocks $\mathscr{B}_1,\dots,\mathscr{B}_t$, and let $J=\mathfrak{S}_{\mathscr{B}_1}\times\cdots\times\mathfrak{S}_{\mathscr{B}_t}$ act on a conjugacy class C_λ of $\mathfrak{S}_{n,r}$ by (3.3). If C_λ contains no cycles of lengths $1,2,\dots,2k$, then each orbit under this action has size $|J| = \prod_{i=1}^t |\mathscr{B}_i|!$. Furthermore, there is a unique element in each orbit that has descents at $a_1,\dots,a_k$.*

Our method of proving Lemma 3.4 involves an algorithm that identifies the unique element in the J-orbit that has descents at $a_1,\dots,a_k$. This algorithm will extend the one used by Fulman in [8, Proof of Theorem 3] to general colored permutation groups. Since our algorithm is very technical, we provide an extended example.

Example 3.5. Consider indices $1,2,4,5\in[9]$ and the 9-cycle with color 2

$$(\omega,\tau) = (1^0 3^1 8^2 5^2 2^0 7^0 4^1 9^0 6^2)\in\mathfrak{S}_{9,3}.$$

The blocks induced by $1,2,4,5$ are $\mathscr{B}_1=\{1,2,3\}$, $\mathscr{B}_2=\{4,5,6\}$, $\mathscr{B}_3=\{7\}$, $\mathscr{B}_4=\{8\}$ and $\mathscr{B}_5=\{9\}$. We wish to find an element in the orbit of (ω,τ) under the action in (3.3) that has descents at positions 1, 2, 4, and 5. We start by replacing all elements with the smallest number in its corresponding block, resulting in $(1^0 1^1 8^2 4^2 1^0 7^0 4^1 9^0 4^2)$.

We must now find an appropriate way to replace the instances of 1 and 4 with elements in the same block. Ignoring colors for the moment, we observe that the elements 7, 8, and 9 appear exactly once, and they are respectively preceded by 1, 1, and 4. For simplicity, we focus on the largest element 9, which is preceded by a 4 that appears multiple times. The elements directly after appearances of 4's are 1^0, 9^0, and 1^0. Regardless of how these two appearances of 1 are replaced with other elements in $\mathscr{B}_1 = \{1,2,3\}$, the element 9^0 will still be the largest. Then for descents at positions 4 and 5 to occur, the element 4^0 must map to 9^0. Using this, we next consider $(1^0 1^1 8^2 5^2 1^0 7^0 4^1 9^0 5^2)$, as we have determined the image of 4^0, but we have not determined the images of 5^0 or 6^0.

Continuing, the next iterations of this algorithm result in $(2^0 1^1 8^2 5^2 3^0 7^0 4^1 9^0 5^2)$ and then $(2^0 1^1 8^2 5^2 3^0 7^0 4^1 9^0 6^2)$. Observe that this is in the orbit of (ω, τ) under the action in (3.3), and it has descents at positions $1, 2, 4$, and 5.

Algorithm 1: `ColoredDescents`

Input: $(\omega, \tau) \in \mathfrak{S}_{n,r}$ with no cycles of lengths $1, 2, \dots, 2k$; indices $a_1, \dots, a_k \in [n]$
Output: a colored permutation $(\omega', \tau') \in \mathfrak{S}_{n,r}$ in the orbit of (ω, τ) under (3.3)

- $\mathscr{B}_1, \dots, \mathscr{B}_t :=$ blocks induced by $a_1, \dots, a_k$
- $\sigma_1, \dots, \sigma_m :=$ cycles of (ω, τ)
- $\sigma'_1, \dots, \sigma'_m :=$ cycles obtained by starting with $\sigma_1, \dots, \sigma_m$ and replacing each $i \in [n]$ with the smallest number from the block that contains it
- **while** *$\sigma'_1, \dots, \sigma'_m$ contains repeated integers from $[n]$* **do**
  - $S :=$ subset of $[n]$ consisting of elements that appear exactly once in $\sigma'_1, \dots, \sigma'_m$
  - $j :=$ largest element in S whose preceding element i in $\sigma'_1, \dots, \sigma'_m$ appears multiple times
  - $\mathscr{B} :=$ block containing i
  - $i_1, \dots, i_\ell :=$ elements in $\sigma'_1, \dots, \sigma'_m$ that are in the block $\mathscr{B}$
  - $j_1^{c_1}, \dots, j_\ell^{c_\ell} :=$ elements respectively following $i_1, \dots, i_\ell$ in $\sigma'_1, \dots, \sigma'_m$
  - $\leq :=$ partial order on $j_1^{c_1}, \dots, j_\ell^{c_\ell}$ given by (2.2) with repeated elements treated as distinct, incomparable elements
  - $\preceq :=$ partial order on $i_1, \dots, i_\ell$ formed by starting with $\leq$, replacing each $j_h^{c_h}$ with i_h, and reversing the relation in $\leq$
  - $\sigma'_1, \dots, \sigma'_m := \sigma'_1, \dots, \sigma'_m$ after replacing instances of $i_1, \dots, i_\ell$ with minimal elements in $\mathscr{B}$ in a manner that respects $\preceq$
- **return** $\sigma'_1, \dots, \sigma'_m$

Using the `ColoredDescents` algorithm, we establish Lemma 3.4. This in turn allows us to establish an analog of Lemma 3.2 on conjugacy classes without short cycles.

Lemma 3.6. *Let $a_1, \dots, a_k \in [n]$ with induced blocks $\mathscr{B}_1, \dots, \mathscr{B}_t$, where $\mathscr{B}_t$ contains n. Let C_λ be a conjugacy class of $\mathfrak{S}_{n,r}$ with no cycles of lengths $1, 2, \dots, 2k$. If $a_1, \dots, a_k \in [n-1]$, then*

$$\mathbb{E}_{C_\lambda}[X_{a_1} X_{a_2} \cdots X_{a_k}] = \prod_{i=1}^{t} \frac{1}{|\mathscr{B}_i|!}. \tag{3.4}$$

If $n \in \{a_1, \dots, a_k\}$, then

$$\mathbb{E}_{C_\lambda}[X_{a_1} X_{a_2} \cdots X_{a_k}] = \left(\frac{r-1}{r}\right)^{|\mathscr{B}_t|} \cdot \prod_{i=1}^{t} \frac{1}{|\mathscr{B}_i|!}. \tag{3.5}$$

Theorem 1.1 and Corollary 1.2 for $\mathrm{des}_{n,r}$ can now be established using (3.1), the results of this section, Theorem 2.5, and Theorem 2.3.

4 Flag major index

In this section, we consider Theorem 1.1 and Corollary 1.2 for the flag major index statistic $\mathrm{fmaj}_{n,r}$. Our general approach combines our work for $\mathrm{des}_{n,r}$ and the theory of degrees for colored permutation statistics, as described in Section 2.3.

Throughout this section, we define $Y_{i,c}$ to be the indicator function for the color of $i \in [n]$ being $c \in \mathbb{Z}_r$,

$$Y_{i,c}(\omega, \tau) = \begin{cases} 1 & \text{if } \tau(i) = c \\ 0 & \text{otherwise.} \end{cases}$$

Using the same X_i indicator functions for descents, this allows us to express $\mathrm{fmaj}_{n,r}$ as

$$\mathrm{fmaj}_{n,r} = r \cdot \sum_{i=1}^{n-1} iX_i + \sum_{i=1}^{n} \sum_{c=0}^{r-1} cY_{i,c}. \tag{4.1}$$

In particular, $\mathrm{fmaj}_{n,r}^k$ can be expressed as linear combinations of the random variables

$$X_{a_1} \cdots X_{a_j} Y_{a_{j+1},c_{j+1}} \cdots Y_{a_k,c_k} \tag{4.2}$$

where $a_1, \ldots, a_j \in [n-1]$, $a_{j+1}, \ldots, a_k \in [n]$, and $c_{j+1}, \ldots, c_k \in \mathbb{Z}_r$. We will show the expectation of (4.2) aligns on $\mathfrak{S}_{n,r}$ and all C_λ with no cycles of lengths $1, 2, \ldots, 2k$. We start with a definition.

Definition 4.1. Let $a_1, \ldots, a_j \in [n-1]$, $a_{j+1}, \ldots, a_k \in [n]$, and $c_{j+1}, \ldots, c_k \in \mathbb{Z}_r$. The *essential set* of the statistic $X_{a_1} \cdots X_{a_j} Y_{a_{j+1},c_{j+1}} \cdots Y_{a_k,c_k}$ is

$$\mathrm{Ess}(X_{a_1} \cdots X_{a_j} Y_{a_{j+1},c_{j+1}} \cdots Y_{a_k,c_k}) = \left(\bigcup_{i=1}^{j} \{a_i, a_i + 1\} \right) \bigcup \left(\bigcup_{i=j+1}^{k} \{a_i\} \right).$$

In (4.2), there can be numerous scenarios involving descents at positions $a_1, \ldots, a_j$ and the colors at $a_{j+1}, \ldots, a_k$. Consequently, analyzing the expectation of (4.2) on $\mathfrak{S}_{n,r}$ and C_λ can be difficult. The following two results allow us to reduce to the case where $\mathrm{Ess}(X_{a_1} \cdots X_{a_j}) = \mathrm{Ess}(Y_{a_{j+1},c_{j+1}} \cdots Y_{a_k,c_k})$.

Lemma 4.2. *Let $a_1, \ldots, a_j \in [n-1]$, $a_{j+1}, \ldots, a_k \in [n]$, and $c_{j+1}, \ldots, c_k \in \mathbb{Z}_r$. Then $Z = X_{a_1} \cdots X_{a_j} Y_{a_{j+1},c_{j+1}} \cdots Y_{a_k,c_k}$ has degree at most $j + k$. Consequently, its mean coincides on all conjugacy classes C_λ of $\mathfrak{S}_{n,r}$ without cycles of lengths $1, 2, \ldots, j + k$. The same holds for $ZY_{i,c}$ when $i \in \mathrm{Ess}(Z)$ and $c \in \mathbb{Z}_r$ is arbitrary.*

Lemma 4.3. *Let $a_1, \ldots, a_j \in [n-1]$, $a_{j+1}, \ldots, a_k \in [n]$, and $c_{j+1}, \ldots, c_k \in \mathbb{Z}_r$. If $a_k \notin \mathrm{Ess}(X_{a_1} \cdots X_{a_j} Y_{a_{j+1},c_{j+1}} \cdots Y_{a_{k-1},c_{k-1}})$, then*

$$\mathbb{E}_{\mathfrak{S}_{n,r}}[X_{a_1} \cdots X_{a_j} Y_{a_{j+1},c_{j+1}} \cdots Y_{a_{k-1},c_{k-1}} Y_{a_k,c_k}] = \frac{1}{r} \cdot \mathbb{E}_{\mathfrak{S}_{n,r}}[X_{a_1} \cdots X_{a_j} Y_{a_{j+1},c_{j+1}} \cdots Y_{a_{k-1},c_{k-1}}].$$

The same holds on any C_λ with no cycles of lengths $1, 2, \ldots, j+k$.

Repeated application of the preceding lemmas allows us to consider the expectation of (4.2) on $\mathfrak{S}_{n,r}$ and C_λ only when $\mathrm{Ess}(X_{a_1} \cdots X_{a_j}) = \mathrm{Ess}(Y_{a_{j+1},c_{j+1}} \cdots Y_{a_k,c_k})$. Our `ColoredDescents` algorithm can be used in this case to establish the following result.

Lemma 4.4. *Let $a_1, \ldots, a_j \in [n-1]$, $a_{j+1}, \ldots, a_k \in [n]$, and $c_{j+1}, \ldots, c_k \in \mathbb{Z}_r$. Then on any conjugacy class C_λ of $\mathfrak{S}_{n,r}$ with no cycles of lengths $1, 2, \ldots, j+k$, we have*

$$\mathbb{E}_{\mathfrak{S}_{n,r}}[X_{a_1} \cdots X_{a_j} Y_{a_{j+1},c_{j+1}} \cdots Y_{a_k,c_k}] = \mathbb{E}_{C_\lambda}[X_{a_1} \cdots X_{a_j} Y_{a_{j+1},c_{j+1}} \cdots Y_{a_k,c_k}].$$

Theorem 1.1 and Corollary 1.2 for $\mathrm{fmaj}_{n,r}$ can now be established through technical arguments that apply the results of this section, Theorem 2.5, and Theorem 2.3.

5 Conclusion

In this paper, we analyzed the moments and asymptotic distributions of $\mathrm{des}_{n,r}$ and $\mathrm{fmaj}_{n,r}$ on conjugacy classes C_λ of $\mathfrak{S}_{n,r}$ with sufficiently long cycles. Our methods showed that the moments and asymptotic distributions of these statistics on C_λ coincide with those on $\mathfrak{S}_{n,r}$. Two natural problems arise from our work. Several prior results on $\mathfrak{S}_n$ and B_n may be relevant for these open problems, e.g., see [3, 6, 15, 16].

Problem 5.1. Study the distributions of $\mathrm{des}_{n,r}$ and $\mathrm{fmaj}_{n,r}$ on conjugacy classes of $\mathfrak{S}_{n,r}$.

Problem 5.2. Determine the asymptotic distribution for $\mathrm{des}_{n,r}$ on arbitrary conjugacy classes of $\mathfrak{S}_{n,r}$.

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