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Equivariant γ-nonnegativity of order polytopes of graded posets

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Abstract. The equivariant Ehrhart theory of lattice polytopes has been introduced by Stapledon, developed by many researchers, and the study of the equivariant h^*-polynomials is getting one of the trends in the theory of lattice polytopes. On the other hand, the h^*-polynomials of order polytopes of sign-graded posets are known to be γ-nonnegative by Brändén. In this manuscript, we prove that order polytopes of sign-graded posets are always equivariant γ-nonnegative. Namely, the γ-polynomials of the equivariant h^*-polynomials of order polytopes of graded posets have its coefficients with actual characters.

Keywords: equivariant Ehrhart theory, equivariant h^*-polynomials, γ-polynomials, γ-nonnegative, order polytopes, sign-graded posets

1 Introduction

1.1 Introduction to equivariant Ehrhart theory

First, let us briefly recall the classical Ehrhart theory before explaining the equivariant version. See [1] for the introduction to Ehrhart theory. Let $P \subset \mathbb{R}^d$ be a lattice (resp. rational) polytope, i.e., a convex polytope all of whose vertices belong to $\mathbb{Z}^d$ (resp. $\mathbb{Q}^d$), of dimension d. Consider the *Ehrhart series* of P, which is the generating function $1 + \sum_{m \geq 1} |mP \cap \mathbb{Z}^d| t^m$, where $mP = \{m\alpha : \alpha \in P\}$. Then it is known that if P is a lattice polytope, then this Ehrhart series becomes of the form:

$$1 + \sum_{m \geq 1} |mP \cap \mathbb{Z}^d| t^m = \frac{h_0^* + h_1^* t + \cdots + h_d^* t^d}{(1-t)^{d+1}}.$$

In this case, we call the polynomial $h_0^* + h_1^* t + \cdots + h_d^* t^d$ of degree at most d appearing in the numerator the *h^*-polynomial* of P, denoted by $h^*(P;t)$.

In [10], the equivariant Ehrhart series is introduced. Next, we recall what the equivariant Ehrhart theory is. See, e.g., [4] and [11] for more recent developments on the

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equivariant Ehrhart theory. Let G be a finite group linearly acting on $\mathbb{Z}^d$. Then the action of G can be extended to the lattice $\mathbb{Z}^{d+1} = \mathbb{Z}^d \oplus \mathbb{Z}$ by fixing the second summand. Let $\rho : G \to \mathrm{GL}_d(\mathbb{C})$ be the G-representation defined by this action. Let $P \subset \mathbb{R}^d$ be a lattice polytope of dimension d and assume that P is G-invariant. Then the dilated polytope mP is also G-invariant for any $m \in \mathbb{Z}_{>0}$. In particular, G acts on $mP \cap \mathbb{Z}^d$ as a permutation. Let $\chi_{mP} \in R(G)$ be the character of this permutation representation, where $R(G)$ denotes the representation ring of G and we identify each representation with its character.

Definition 1.1. With the notation above, the *equivariant Ehrhart series* of a G-invariant lattice polytope P is given by

$$\mathrm{EE}(P, \rho; t) := 1 + \sum_{m \geq 1} \chi_{mP} t^m \in R(G)[[t]].$$

Then we write $\mathrm{EE}(P, \rho; t)$ as follows:

$$\mathrm{EE}(P, \rho; t) = \frac{h^*(P, \rho; t)}{(1-t)\det(\mathrm{Id} - \rho t)},$$

where $h^*(P, \rho; t)$ is a power series in $R(G)[[t]]$ and Id is the identity matrix. Note that $((1-t)\det(\mathrm{Id} - \rho t))^{-1}$ is an element in $R(G)[[t]]$ (cf. [10, Lemma 3.1]), and so is $h^*(P, \rho; t)$ which is not necessarily a polynomial. We call $h^*(P, \rho; t)$ the *equivariant h^*-series* (or *equivariant h^*-polynomial* if it is a polynomial) of P with respect to ρ.

By evaluating $\mathrm{EE}(P, \rho; t)$ with $g \in G$, we obtain the Ehrhart series of $P^g := \{x \in P : g \cdot x = x\}$ by definition. In particular, we recover the Ehrhart series of P by evaluating $\mathrm{EE}(P, \rho; t)$ with the unit e of G.

Note that each coefficient of the equivariant h^*-series is a virtual character of G. We say that $h^*(P, \rho; t)$ is *effective* if each coefficient is an actual character, i.e., each coefficient has a unique expression $\sum a_\chi \chi$ by irreducible characters χ and nonnegative integer coefficients $a_\chi \in \mathbb{Z}_{\geq 0}$.

The following conjecture is one of the main topics in the equivariant Ehrhart theory:

Conjecture 1.2 ([10, Conjecture 12.1])**.** With the above setting, if $h^*(P, \rho; t)$ is a polynomial, then $h^*(P, \rho; t)$ is effective.

This conjecture is known as the *effectiveness conjecture*. Note that the effectiveness of $h^*(P, \rho; t)$ always implies its polynomiality. This conjecture has been verified in several cases. See, e.g., [3], [5] and [11]. In the non-equivariant setting, the nonnegativity of the h^*-polynomial is well known ([1, Theorem 3.12]). The effectiveness of the equivariant h^*-polynomial corresponds to the nonnegativity in the equivariant setting.

1.2 Equivariant γ-nonnegativity

Let $f(t) = \sum_{i=0}^{s} a_i t^i$ be a polynomial of degree s with $a_i \in \mathbb{Z}_{\geq 0}$.

- We say that $f(t)$ is *real-rooted* if all of the roots of $f(t)$ are real numbers.
- We say that $f(t)$ is *unimodal* if there is $0 \leq k \leq s$ with $a_0 \leq \cdots \leq a_k \geq \cdots \geq a_s$.
- We say that $f(t)$ is *palidromic* if $a_i = a_{s-i}$ for $i = 0, \ldots, s$.
- Assume that $f(t)$ is palindromic. Then there exist coefficients $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor s/2 \rfloor}$ such that $f(t) = \sum_{i=0}^{\lfloor s/2 \rfloor} \gamma_i t^i (t+1)^{s-2i}$. We call $\gamma(t) = \sum_{i=0}^{\lfloor s/2 \rfloor} \gamma_i t^i$ the *γ-polynomial* of $f(t)$. We say that $f(t)$ is *γ-nonnegative* (sometimes, *γ-positive*) if all γ_i's are nonnegative.

For a palindromic polynomial, the following implications hold:

$$f(t) \text{ is real-rooted} \Longrightarrow f(t) \text{ is } \gamma\text{-nonnegative} \Longrightarrow f(t) \text{ is unimodal.}$$

The real-rootedness and γ-nonnegativity of the h^*-polynomials of lattice polytopes are one of the main topics in Ehrhart theory and many results are known. Some of them are collected in [6] as a survey paper.

Now, it is quite natural to think of the equivariant version of γ-nonnegativity. For a given palindromic polynomial $f(t) = \sum_{i=0}^{s} \chi_i t^i \in R(G)[t]$ of degree s whose coefficients are virtual characters of a finite group G, we say that $f(t)$ is *equivariant γ-nonnegative* (or *γ-effective*) if the γ-polynomial of $f(t)$ is effective. Namely, we call $f(t)$ equivariant γ-nonnegative if it can be written as

$$f(t) = \sum_{i=0}^{\lfloor s/2 \rfloor} \gamma_i t^i (1+t)^{s-2i} \in R(G)[t]$$

and each $\gamma_i \in R(G)$ is an actual character. Then the following question naturally arises:

Question 1.3. Let G be a finite group and let P be a G-invariant lattice polytope. Assume that $h^*(P;t)$ is palindromic and γ-nonnegative. Then, is $h^*(P,\rho;t)$ equivariant γ-nonnegative?

1.3 Order polytopes of labeled posets

Order polytopes were introduced in [9] by Stanley. Let us recall what order polytopes of labeled posets are. Let (P, ω) be a labeled poset, i.e., $P = \{p_1, \ldots, p_d\}$ is a poset equipped with a partial order $\prec$ and a bijection $\omega : P \to \{1, \ldots, d\}$. We define the "half-open" order polytope of (P, ω) as follows:

$$\begin{aligned} \mathcal{O}(P,\omega) = \{(x_1, \ldots, x_d) \in [0,1]^d : \; & x_i \geq x_j \text{ if } p_i \preceq p_j, \\ & x_i > x_j \text{ if } p_i \prec p_j \text{ and } \omega(p_i) > \omega(p_j)\}. \end{aligned}$$

Note that $\mathcal{O}(P,\omega)$ is a usual polytope if ω is order-preserving (a.k.a. a natural labeling), i.e., $\omega(p) < \omega(p')$ whenever $p \prec p'$. We naturally define the Ehrhart series of the half-open polytope $\mathcal{O}(P,\omega)$ by setting

$$\sum_{n \geq 0} |n\mathcal{O}(P,\omega) \cap \mathbb{Z}^d| t^n = \frac{h^*(\mathcal{O}(P,\omega);t)}{(1-t)^{d+1}}.$$

Here, we regard $|n\mathcal{O}(P,\omega) \cap \mathbb{Z}^d| = 1$ for $n = 0$ if ω is a natural labeling, while $|n\mathcal{O}(P,\omega) \cap \mathbb{Z}^d| = 0$, otherwise. The h^*-polynomial $h^*(\mathcal{O}(P,\omega);t)$ of (P,ω) is also known as the *W-polynomial* of (P,ω).

We say that a poset P is *graded* if each maximal chain in P has the same length. Then we can associate the rank function $\rho : P \to \mathbb{Z}_{\geq 0}$ by letting $\rho(p)$ be the maximal length of the chain starting from a minimal element to p. The rank of a graded poset is the maximum of the lengths among all maximal chains.

Let (P,ω) be a labeled poset and let $E(P) = \{(p,p') \in P \times P : p' \text{ covers } p\}$, where we say that p' *covers* p if $p \prec p'$ and there is no p'' with $p \prec p'' \prec p'$. Then we can also associate the labeling ω to $E(P)$ defined by

$$\omega(p,p') = \begin{cases} 1 & \text{if } \omega(p) < \omega(p'), \\ -1 & \text{if } \omega(p) > \omega(p'). \end{cases}$$

A labeled poset (P,ω) is called *sign-graded* (or a poset P is *ω-graded*) if for every maximal chain $p_1 \prec p_2 \prec \cdots \prec p_{s+1}$ in P, the sum $\sum_{i=1}^{s} \omega(p_i, p_{i+1})$ is constant. The common value of this sum is called the *rank* of a sign-graded poset (P,ω), denoted by $r(\omega)$. Note that a poset P is graded if and only if P is ω-graded for some natural labeling ω.

We say that a poset P is *ω-consistent* if for each $p \in P$, $\Lambda_p := \{q \in P : q \preceq p\}$ is ω_p-graded, where ω_p is the restriction of ω to $E(\Lambda_p)$. Notice that if P is ω-graded, then P is ω-consistent. The *rank function* $\rho : P \to \mathbb{Z}$ of an ω-consistent poset P is defined by $\rho(p) = r(\omega_p)$. This agrees with the (usual) rank function on P as above if P is graded, i.e., ω-graded for some natural labeling ω.

It is known that $h^*(\mathcal{O}(P,\omega);t)$ is palindromic if and only if (P,ω) is sign-graded. Moreover, the following is proved by Brändén:

Theorem 1.4 ([2, Theorem 4.2])**.** For a sign-graded poset (P,ω), the h^*-polynomial of $\mathcal{O}(P,\omega)$ is γ-nonnegative.

1.4 Main Result

An *automorphism* of a labeled poset (P,ω) is a bijection φ on P satisfying that $p \preceq p' \iff \varphi(p) \preceq \varphi(p')$ and $\omega(p) > \omega(p') \iff \omega(\varphi(p)) > \omega(\varphi(p'))$. Let $\mathrm{Aut}(P,\omega)$ denote the automorphism group of (P,ω). Then $\mathrm{Aut}(P,\omega)$ naturally acts on $\mathcal{O}(P,\omega)$

as a linear action by permuting the corresponding coordinates. Hence, we can discuss the equivariant h^*-series of $\mathcal{O}(P,\omega)$. In view of Question 1.3 and Theorem 1.4, we are tempted to discuss the equivariant γ-nonnegativity of order polytopes of sign-graded posets. The main result of this manuscript is the following:

Theorem 1.5 (Main Result)**.** Let (P,ω) be a sign-graded poset admitting an action by a finite group G. Then the equivariant h^*-polynomial of $\mathcal{O}(P,\omega)$ is equivariant γ-nonnegative.

2 The γ-polynomials of the equivariant h^*-polynomials of order polytopes

The purpose of this section is to explain how to prove Theorem 1.5. To this end, we first recall an idea of the proof of Theorem 1.4, and we perform our proof by an example.

2.1 Fundamental facts on labeled posets

For the explanation of the proof of Theorem 1.4, we have to recall some notions and lemmas on labeled posets from [2].

Lemma 2.1 ([2, Corollary 2.4])**.** Let P be sign-graded with respect to different labelings ω and ω'. Then $h^*(\mathcal{O}(P,\omega');t) = t^{(r(\omega)-r(\omega'))/2}h^*(\mathcal{O}(P,\omega);t)$.

Remark 1. We say that ω is *canonical* if (P,ω) has a rank function ρ with $\rho(P) \subset \{0,1\}$, and $\rho(p) < \rho(p')$ implies $\omega(p) < \omega(p')$. It is known by [2, Theorem 2.5] that P admits a canonical labeling if P is ω-graded for some labeling ω. Therefore, for a sign-graded poset (P,ω), we may assume that ω is canonical for the computation of the h^*-polynomial of $\mathcal{O}(P,\omega)$ thanks to Lemma 2.1.

Let (P,ω) and (P',ω') be labeled posets equipped with partial orders $\prec_P$ and $\prec_{P'}$, respectively. The *ordinal sum* $P \oplus P'$ is the poset $P \sqcup P'$ equipped with the partial order defined by $x \prec y$ if $x \prec_P y$ in P or $x \prec_{P'} y$ in P' or $x \in P, y \in P'$. We define two kinds of labelings $\omega \oplus_{\pm 1} \omega'$ of $E(P \oplus P')$ as follows:

$$(\omega \oplus_1 \omega')(x,y) = \begin{cases} \omega(x,y) & \text{if } (x,y) \in E(P), \\ \omega'(x,y) & \text{if } (x,y) \in E(P'), \\ 1 & \text{if } x \in P, y \in P', \end{cases} \quad \text{and}$$

$$(\omega \oplus_{-1} \omega')(x,y) = \begin{cases} \omega(x,y) & \text{if } (x,y) \in E(P), \\ \omega'(x,y) & \text{if } (x,y) \in E(P'), \\ -1 & \text{if } x \in P, y \in P'. \end{cases}$$

By abuse the notation, we write $P \oplus_{\pm 1} P'$ when the labeling is given by $\omega \oplus_{\pm 1} \omega'$.

Regarding the h^*-polynomials in the case of the ordinal sums, we have the following:

Lemma 2.2 ([2, Proposition 3.3]). Let (P, ω) and (P', ω') be labeled posets. Then

$$h^*(\mathcal{O}(P \oplus P', \omega \oplus_1 \omega'); t) = h^*(\mathcal{O}(P, \omega); t) h^*(\mathcal{O}(P', \omega'); t) \text{ and}$$
$$h^*(\mathcal{O}(P \oplus P', \omega \oplus_{-1} \omega'); t) = t \cdot h^*(\mathcal{O}(P, \omega); t) h^*(\mathcal{O}(P', \omega'); t).$$

Let (P, ω) be a sign-graded poset. We say that (P, ω) is *saturated* if p and p' are comparable for all $p, p' \in P$ whenever $\rho(p)$ and $\rho(p')$ differ by 1. For two posets P and $\widetilde{P}$ with $P = \widetilde{P}$ as sets, we say that $\widetilde{P}$ *extends* P if $p \prec p'$ in $\widetilde{P}$ whenever $p \prec p'$ in P. We call $\widetilde{P}$ a *saturation* of P if a saturated poset $\widetilde{P}$ has the same rank function as (P, ω) and $\widetilde{P}$ extends P.

Lemma 2.3 (cf. [2, Theorem 3.2]). Let (P, ω) be a sign-graded poset. Then we have

$$h^*(\mathcal{O}(P, \omega); t) = \sum_{\widetilde{P}} h^*(\mathcal{O}(\widetilde{P}, \omega); t), \tag{2.1}$$

where the union runs over all saturations of (P, ω).

2.2 An idea of the proof of Theorem 1.4

Let (P, ω) be a sign-graded poset and assume that ω is canonical (see Remark 1).

The proof of Theorem 1.4 roughly consists of the following three steps.

The first step: By Lemma 2.3, it suffices to prove the γ-nonnegativity of $h^*(\mathcal{O}(\widetilde{P}, \omega); t)$ for each saturation $\widetilde{P}$ of P.

The second step: For each saturation $(\widetilde{P}, \omega)$, we know by [2, Proposition 3.4] that

$$(\widetilde{P}, \omega) = A_0 \oplus_1 A_1 \oplus_{-1} A_2 \oplus_1 \cdots \oplus_{\pm 1} A_k$$

holds, where each A_i is an anti-chain. Hence, we can directly compute $h^*(\mathcal{O}(\widetilde{P}, \omega); t)$ from $h^*(\mathcal{O}(A_i, \omega); t)$ by applying Lemma 2.2 and the γ-nonnegativity follows from that of each $h^*(\mathcal{O}(A_i, \omega); t)$.

The third step: The h^*-polynomial of the order polytope of an anti-chain corresponds to the h^*-polynomial of the unit cube $[0, 1]^a$. It is known that the h^*-polynomial of $[0, 1]^a$ is γ-nonnegative in general. See Subsection 2.3 for more details.

2.3 Equivariant Ehrhart series of cubes

For the extension of the proof of Theorem 1.4 to Theorem 1.5, the crucial part is the third step of Subsection 2.2. (We can see that the discussions of the first and second steps straightforwardly work also in the equivariant setting, but we omit the detail due to space limitations.) So, we discuss how to see the equivariant γ-nonnegativity of $[0,1]^d$ under the $\mathfrak{S}_d$-action. Although the group action in Theorem 1.5 is not necessarily the $\mathfrak{S}_d$-action, this action will play the essential role in our setting.

Let $P = [0,1]^d$ be the d-dimensional unit cube and consider the equivariant Ehrhart series of P with respect to the action of $\mathfrak{S}_d$. Then it is known that the equivariant h^*-polynomial is equivariant γ-nonnegative. See, e.g., [8]. In fact, we know that

$$h^*(P,\rho;t) = \sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \chi_{d,j} t^j (1+t)^{d-1-2j} \tag{2.2}$$

by using actual characters $\chi_{d,j}$ of $\mathfrak{S}_d$. In [7], an explicit way how to describe each of $\chi_{d,j}$ is given. For example, we see the following:

$$\begin{aligned} &\chi_{3,0} = 1,\ \ \chi_{3,1} = \chi^{\ydiagram{2,1}}; \quad \chi_{4,0} = 1,\ \ \chi_{4,1} = \chi^{\ydiagram{2,2}} + 2\chi^{\ydiagram{3,1}}; \\ &\chi_{5,0} = 1,\ \ \chi_{5,1} = 2\chi^{\ydiagram{3,2}} + 3\chi^{\ydiagram{4,1}},\ \ \chi_{5,2} = \chi^{\ydiagram{3,2}} + \chi^{\ydiagram{3,1,1}} + \chi^{\ydiagram{2,2,1}}, \end{aligned} \tag{2.3}$$

where "1" is the trivial character and χ^λ stands for the irreducible character of the Specht module S^λ corresponding to a partition λ (a Young diagram) of d (i.e., a certain conjugacy class of $\mathfrak{S}_d$). See [7, Theorem 1.1] for more details.

For the calculations later, we collect the character tables of $\mathfrak{S}_3$ and $\mathfrak{S}_4$:

$\mathfrak{S}_3$	1	$\chi^{\ydiagram{1,1,1}}$	$\chi^{\ydiagram{2,1}}$
e	1	1	2
$(1\,2)$	1	-1	0
$(1\,2\,3)$	1	1	-1

$\mathfrak{S}_4$	1	$\chi^{\ydiagram{1,1,1,1}}$	$\chi^{\ydiagram{2,2}}$	$\chi^{\ydiagram{3,1}}$	$\chi^{\ydiagram{2,1,1}}$
e	1	1	2	3	3
$(1\,2)$	1	-1	0	1	-1
$(1\,2\,3)$	1	1	-1	0	0
$(1\,2\,3\,4)$	1	-1	0	-1	1
$(1\,2)(3\,4)$	1	1	2	-1	-1

2.4 Example: how to prove Theorem 1.5

Let us describe how to see the equivariant γ-nonnegativity of $h^*(\mathcal{O}(P,\omega),\rho;t)$ with an example. Throughout this subsection, we consider the labeled poset $P = \{p_1,\ldots,p_8\}$ as in Figure 1 whose canonical and natural labeling ω is given by $\omega(p_i) = i$ for each i.

Let $G = D_4 = \langle \sigma, \tau : \sigma^4 = \tau^2 = e, \sigma\tau\sigma = \tau \rangle$ be the dihedral group of order 8. Then G

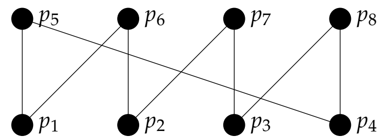


Figure 1: An example of a poset

acts on P as follows:

$$\sigma \cdot p_i = \begin{cases} p_{i+1} & \text{for } i = 1,2,3,5,6,7; \\ p_1 & \text{for } i = 4; \\ p_5 & \text{for } i = 8; \end{cases} \quad \text{and} \quad \tau \cdot p_i = \begin{cases} p_i & \text{for } i = 1,3; \\ p_4 & \text{for } i = 2; \\ p_2 & \text{for } i = 4; \\ p_{i+1} & \text{for } i = 5,7; \\ p_{i-1} & \text{for } i = 6,8. \end{cases}$$

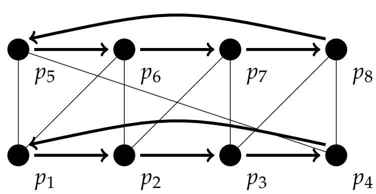


Figure 2: The action by σ

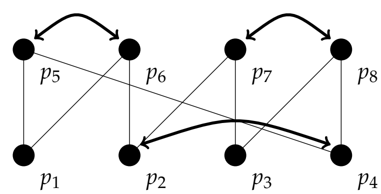


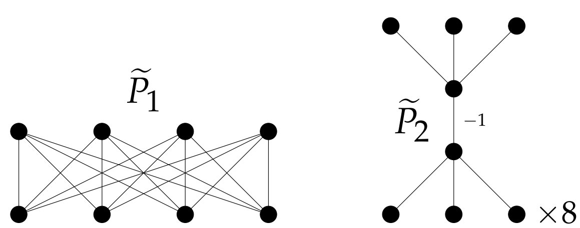
Figure 3: The action by τ

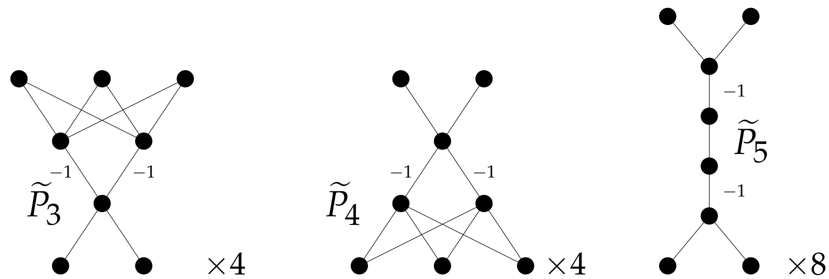
Note that this preserves the naturality of the labeling ω. Hence, G is isomorphic to a subgroup of $\mathrm{Aut}(P, \omega)$.

We describe the character table of D_4:

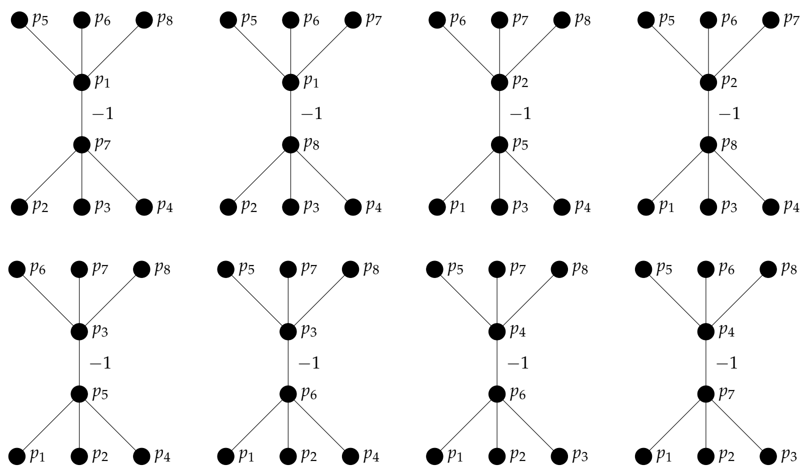
D_4	1	χ_1	χ_2	χ_3	χ_4
e	1	1	1	1	2
$\{\sigma, \sigma^3\}$	1	1	-1	-1	0
σ^2	1	1	1	1	-2
$\{\tau, \tau\sigma^2\}$	1	-1	1	-1	0
$\{\tau\sigma, \tau\sigma^3\}$	1	-1	-1	1	0

We also describe all saturations of P as follows:





Here, "-1" indicates the labeling of $E(P)$ and the labeling "1" is omitted. Note that each of Hasse diagrams gives a G-orbit. For example, regarding $\widetilde{P}_2$, "$\times 8$" means that there are 8 copies of the same poset with different labelings, which form a G-orbit. (The others are similar.) In fact, the labelings of the poset $\widetilde{P}_2$ are the following:



Below, we compute the γ-polynomial of the equivariant h^*-polynomial for each orbit. For later use, let $\mathfrak{S}_{a_1,\ldots,a_s} := \mathfrak{S}_{a_1} \times \cdots \times \mathfrak{S}_{a_s}$.

$\boxed{\widetilde{P}_1}$ Note that this class forms a single G-orbit. We see that this saturation is of the form $A_1 \oplus_1 A_2$, where $|A_1| = |A_2| = 4$. As described in (2.2), we know that the equivariant h^*-polynomial of $[0,1]^a$ with respect to the $\mathfrak{S}_a$-action is equivariant γ-nonnegative. Since $A_1 \oplus_1 A_2$ naturally admits the $\mathfrak{S}_{4,4}$-action and $h^*(\mathcal{O}(A_1 \oplus_1 A_2, \omega); t)$ is just the product of $h^*(\mathcal{O}(A_1, \omega); t)$ and $h^*(\mathcal{O}(A_2, \omega); t)$, we obtain the γ-polynomial of the equivariant h^*-polynomial of $\mathcal{O}(A_1 \oplus_1 A_2, \omega)$ as follows:

$$R(\mathfrak{S}_{4,4})[t] \ni (1 + \chi_{1,A_1} t)(1 + \chi_{1,A_2} t) =: 1 + \gamma_1^{(1)} t + \gamma_2^{(1)} t^2,$$

where χ_{1,A_1} and χ_{1,A_2} are certain characters of $\mathfrak{S}_4$ arising from its action to the cubes $\mathcal{O}(A_1, \omega)$ and $\mathcal{O}(A_2, \omega)$, respectively. (We know $\chi_{1,A_1} = \chi_{1,A_2} = \chi^{\yng(2,2)} + 2\chi^{\yng(3,1)}$ by (2.3).) Note that $\gamma_1^{(1)}$ is a direct sum of two characters of $\mathfrak{S}_4$ and $\gamma_2^{(1)}$ corresponds to the tensor product, so both of $\gamma_1^{(1)}$ and $\gamma_2^{(1)}$ correspond to certain characters of $\mathfrak{S}_{4,4}$. Here, in this case, we see that G is a subgroup of $\mathfrak{S}_{4,4}$. Hence, we can regard each of the characters $\gamma_1^{(1)}$ and $\gamma_2^{(1)}$ as a character of G by restriction.

Therefore, we conclude that each coefficient of the γ-polynomial of $h^*(\mathcal{O}(\widetilde{P}_1), \rho; t)$ is an actual character of G. The explicit descriptions of $\gamma_1^{(1)}, \gamma_2^{(1)} \in R(G)$ are as follows:

$$\gamma_1^{(1)} = 2 + 3\chi_2 + 3\chi_3 + 4\chi_4; \quad \gamma_2^{(1)} = 9 + 9\chi_1 + 7\chi_2 + 7\chi_3 + 16\chi_4.$$

We briefly explain how to get them. For $\gamma_1^{(1)}$ ($\gamma_2^{(1)}$ is similar), since each conjugacy class of G can be interpreted as that of $\mathfrak{S}_{4,4}$ by

$$\begin{aligned} e \in G &\longleftrightarrow (e, e) \in \mathfrak{S}_{4,4}; \\ \sigma, \sigma^3 \in G &\longleftrightarrow ((1\,2\,3\,4), (1\,2\,3\,4)) \in \mathfrak{S}_{4,4}; \\ \sigma^2 \in G &\longleftrightarrow ((1\,2)(3\,4), (1\,2)(3\,4)) \in \mathfrak{S}_{4,4}; \\ \tau, \tau\sigma^2 \in G &\longleftrightarrow ((1\,2), (1\,2)(3\,4)) \in \mathfrak{S}_{4,4}; \\ \tau\sigma, \tau\sigma^3 \in G &\longleftrightarrow ((1\,2)(3\,4), (1\,2)) \in \mathfrak{S}_{4,4}, \end{aligned}$$

we see that $(\chi_{1,A_1} + \chi_{1,A_2})(g)$ is equal to

$$\left((\chi^{\ydiagram{2,2}} + 2\chi^{\ydiagram{3,1}})|_{A_1} + (\chi^{\ydiagram{2,2}} + 2\chi^{\ydiagram{3,1}})|_{A_2} \right)(g) = \begin{cases} 16 & \text{if } g = e, \\ -4 & \text{if } g = \sigma, \sigma^3, \\ 0 & \text{if } g = \sigma^2, \\ 2 & \text{if } g = \tau, \tau\sigma^2, \\ 2 & \text{if } g = \tau\sigma, \tau\sigma^3. \end{cases}$$

$\boxed{\widetilde{P}_2}$ There are 8 copies of the same poset $\widetilde{P}_2$ which is of the form $A_1 \oplus_1 A_2 \oplus_{-1} A_3 \oplus_1 A_4$, where $|A_1| = |A_4| = 3$ and $|A_2| = |A_3| = 1$. Let $\widetilde{\mathcal{P}}_2$ be this G-orbit. Regarding the γ-polynomial of the equivariant h^*-polynomial of $\mathcal{O}(\widetilde{P}_2, \omega)$ with respect to the $\mathfrak{S}_{3,1,1,3}$-action, we have

$$R(\mathfrak{S}_{3,1,1,3})[t] \ni t \cdot (1 + \chi_{1,A_1} t)(1 + \chi_{1,A_4} t) =: t(1 + \gamma_1^{(2)} t + \gamma_2^{(2)} t^2),$$

where χ_{1,A_1} and χ_{1,A_4} are certain characters of $\mathfrak{S}_3$ arising from its action to $\mathcal{O}(A_1, \omega)$ and $\mathcal{O}(A_4, \omega)$, respectively. Actually, we know $\chi_{1,A_1} = \chi_{1,A_4} = \chi^{\ydiagram{2,1}}$ by (2.3).

Unlike $\widetilde{P}_1$, we notice that G is not a subgroup of $\mathfrak{S}_{3,1,1,3}$. Let $H = G \cap \mathfrak{S}_{3,1,1,3}$. In this case, we have $H = \{e\}$. By considering the induced character from H to G of the restrictions of the characters $\gamma_1^{(2)}, \gamma_2^{(2)} \in R(\mathfrak{S}_{3,1,1,3})$ to H, we obtain the description of the γ-polynomial of the sum of the equivariant h^*-polynomials of this G-orbit. Therefore, the equivariant γ-nonnegativity follows. More precisely, we see the following:

$$\left(\text{the } \gamma\text{-polynomial of } \sum_{\widetilde{P} \in \widetilde{\mathcal{P}}_2} h^*(\mathcal{O}(\widetilde{P}, \omega), \rho; t) \right) = \chi_{\mathrm{perm}} t(1 + 4t + 4t^2),$$

where $\chi_{\mathrm{perm}} = 1 + \chi_1 + \chi_2 + \chi_3 + 2\chi_4$. In fact, the restriction of $\gamma_1^{(2)}, \gamma_2^{(2)}$ to H is just the trivial character up to scalar, so the character χ_{perm} of the regular representation of G appears as the induced character.

$\boxed{\widetilde{P}_3}$ There are 4 copies of the poset $\widetilde{P}_3$ which is of the form $A_1 \oplus_1 A_2 \oplus_{-1} A_3 \oplus_1 A_4$, where $|A_1| = |A_3| = 2$, $|A_2| = 1$ and $|A_4| = 3$. Let $\widetilde{\mathcal{P}}_3$ be this G-orbit.

The γ-polynomial of the equivariant h^*-polynomial of $\mathcal{O}(\widetilde{P}_3, \omega)$ with respect to the $\mathfrak{S}_{2,1,2,3}$-action is $t \cdot (1 + \chi_{1,A_4} t) \in R(\mathfrak{S}_{2,1,2,3})[t]$, where $\chi_{1,A_4} = \chi^{\ydiagram{2,1}} \in R(\mathfrak{S}_3)$.

Here, we notice that G is not a subgroup of $\mathfrak{S}_{2,1,2,3}$, while the intersection $H = G \cap \mathfrak{S}_{2,1,2,3}$ becomes non-trivial. In this case, we have $H = \{e, \tau\sigma\}$ or $\{e, \tau\sigma^3\}$. Thus, we can regard the orbit $\widetilde{\mathcal{P}}_3$ as a coset representative of H, so we think of the permutation action on $\widetilde{\mathcal{P}}_3$ by G. By considering the induced character from H to G of the restrictions of the character $\chi_{1,A_4} \in R(\mathfrak{S}_{2,1,2,3})$ to H, we obtain the following explicit description of the γ-polynomial:

$$\left(\text{the } \gamma\text{-polynomial of } \sum_{\widetilde{P} \in \widetilde{\mathcal{P}}_3} h^*(\mathcal{O}(\widetilde{P}, \omega), \rho; t) \right) = t(1 + \chi_3 + \chi_4 + \chi_{\mathrm{perm}} t).$$

Note that $1 + \chi_3 + \chi_4$ corresponds to the character of the permutation representation of the action of G on G/H.

$\boxed{\widetilde{P}_4}$ Similarly to the case of $\widetilde{P}_3$, we obtain that

$$\left(\text{the } \gamma\text{-polynomial of } \sum_{\widetilde{P} \in \widetilde{\mathcal{P}}_4} h^*(\mathcal{O}(\widetilde{P}, \omega), \rho; t) \right) = t(1 + \chi_2 + \chi_4 + \chi_{\mathrm{perm}} t),$$

where $1 + \chi_2 + \chi_4$ corresponds to the character of the permutation representation of the action of G on G/H with $H = G \cap \mathfrak{S}_{3,2,1,2} = \{e, \tau\}$ or $\{e, \tau\sigma^2\}$.

$\boxed{\widetilde{P}_5}$ There are 8 copies of the same poset $\widetilde{P}_5$ which is of the form $A_1 \oplus_1 A_2 \oplus_{-1} A_3 \oplus_1 A_4 \oplus_{-1} A_5 \oplus_1 A_6$, where $|A_1| = |A_6| = 2$, and $|A_2| = |A_3| = |A_4| = |A_5| = 1$. Let $\widetilde{\mathcal{P}}_5$ be this G-orbit. Since the γ-polynomial of $\mathcal{O}(\widetilde{P}_5, \omega)$ with respect to the $\mathfrak{S}_{2,1,1,1,1,2}$-action is just t^2, we get

$$\left(\text{the } \gamma\text{-polynomial of } \sum_{\widetilde{P} \in \widetilde{\mathcal{P}}_5} h^*(\mathcal{O}(\widetilde{P}, \omega), \rho; t) \right) = \chi_{\mathrm{perm}} t^2.$$

By summarizing those computations, we conclude that

$$\begin{aligned}\gamma(\mathcal{O}(P,\omega),\rho;t) &= 1 + (2+3\chi_2+3\chi_3+4\chi_4)t + (9+9\chi_1+7\chi_2+7\chi_3+16\chi_4)t^2 \\ &\quad + t(\chi_{\mathrm{perm}} + 4\chi_{\mathrm{perm}}t + 4\chi_{\mathrm{perm}}t^2) \\ &\quad + t(1+\chi_3+\chi_4+\chi_{\mathrm{perm}}t) + t(1+\chi_2+\chi_4+\chi_{\mathrm{perm}}t) + \chi_{\mathrm{perm}}t^2 \\ &= 1 + (5+\chi_1+5\chi_2+5\chi_3+8\chi_4)t \\ &\quad + (16+16\chi_1+14\chi_2+14\chi_3+30\chi_4)t^2 + 4(1+\chi_1+\chi_2+\chi_3+2\chi_4)t^3.\end{aligned}$$

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References

[1] M. Beck and S. Robins. *Computing the continuous discretely*. Second edition. Undergraduate Texts in Mathematics. Springer, NY, 2015, xx+285 pp. DOI.

[2] P. Brändén. "Sign-graded posets, unimodality of W-polynomials and the Charney-Davis conjecture". *Electron. J. Combin.* **11**.2 (2004/06), Research Paper 9, 15 pp. DOI.

[3] O. Clarke, A. Higashitani, and M. Kölbl. "The equivariant Ehrhart theory of polytopes with order-two symmetries". *Proc. Am. Math. Soc.* **151**.9 (2023), pp. 4027–4041. DOI.

[4] A. D'Alì and E. Delucchi. "Equivariant Hilbert and Ehrhart series under translative group actions". 2024. arXiv:2312.14088.

[5] S. Elia, D. Kim, and M. Supina. "Techniques in equivariant Ehrhart theory". *Ann. Comb.* **28**.3 (2024), pp. 819–870. DOI.

[6] L. Ferroni and A. Higashitani. "Examples and counterexamples in Ehrhart theory". 2023. arXiv:2307.10852.

[7] T. Horiguchi, M. Masuda, T. Sato, J. Shareshian, and J. Song. "The representation of $\mathfrak{S}_n$ on the cohomology of the permutohedral variety and gamma vectors of partitioned permutohedra". 2024. arXiv:2405.09242.

[8] J. Shareshian and M. L. Wachs. "Gamma-positivity of variations of Eulerian polynomials". *J. Comb.* **11**.1 (2020), pp. 1–33. DOI.

[9] R. P. Stanley. "Two poset polytopes". *Discrete Comput. Geom.* **1**.1 (1986), pp. 9–23. DOI.

[10] A. Stapledon. "Equivariant Ehrhart theory". *Adv. Math.* **226**.4 (2011), pp. 3622–3654. DOI.

[11] A. Stapledon. "Equivariant Ehrhart theory, commutative algebra and invariant triangulations of polytopes". 2024. arXiv:2311.17273.