Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #51, 12 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

Gröbner bases and the Lefschetz properties for powers of a general linear form in the squarefree algebra

Filip Jonsson Kling[1], Samuel Lundqvist[1], Fatemeh Mohammadi[2], Matthias Orth[3], and Eduardo Sáenz-de-Cabezón[4]

[1]*Department of Mathematics, Stockholm University, Sweden*
[2]*Departments of Mathematics and Computer Science, KU Leuven, Belgium*
[3]*Department of Mathematics, University of Kassel, Germany*
[4]*Departamento de Matemáticas y Computación, Universidad de La Rioja, Spain*

Abstract. For the almost complete intersection ideals $(x_1^2, \dots, x_n^2, (x_1 + \cdots + x_n)^k)$, we compute their reduced Gröbner basis for any term ordering, revealing a combinatorial structure linked to lattice paths, elementary symmetric polynomials, and Catalan numbers. Using this structure, we classify the weak Lefschetz property for these ideals. Additionally, we provide a new proof of the well-known result that the squarefree algebra satisfies the strong Lefschetz property.

Résumé. Pour les idéaux d'intersection presque complète $(x_1^2, \dots, x_n^2, (x_1 + \cdots + x_n)^k)$, nous calculons leur base de Gröbner réduite pour tout ordre monomial, mettant en évidence une structure combinatoire liée aux chemins de réseaux, aux polynômes symétriques élémentaires et aux nombres de Catalan. En utilisant cette structure, nous classifions la propriété de Lefschetz faible pour ces idéaux. Nous fournissons également une nouvelle démonstration du résultat bien connu selon lequel l'algèbre sans carrés satisfait la propriété de Lefschetz forte.

Keywords: Gröbner basis, Catalan numbers, lattice paths, almost complete intersections, general linear forms, elementary symmetric polynomials, Lefschetz properties

1 Introduction

Let $\mathbf{k}$ be a field of characteristic zero and let $R = \mathbf{k}[x_1, \dots, x_n]$ be the polynomial ring in n variables. A monomial order $\prec$ on R is a total order on the monomials of R where any subset of monomials has a smallest element, and such that $m_1 \prec m_2$ implies that $m \cdot m_1 \prec m \cdot m_2$ for all monomials m in R. A Gröbner basis for an ideal I with respect to $\prec$ is a subset $\{g_1, \dots, g_s\}$ of I such that the largest monomial of any f in I is divisible by the largest monomial of one of the g_i. Gröbner bases are a key tool in computer algebra systems for performing computations. However, they are also known for their lack of respect for symmetry, strong dependence on the monomial ordering, and often involved explicit description. In contrast, our main result gives, for any term ordering, a Gröbner basis for

$$I_{n,k} = (x_1^2, \dots, x_n^2, (x_1 + \cdots + x_n)^k)$$

which exhibits a different behavior. First, all Gröbner basis elements, except for the squares, are elementary symmetric polynomials in a subset of the variables, and in particular, they have $0, 1$-coefficients. Second, the sequence formed by the number of Gröbner basis elements of degree greater than 2 corresponds to the $(k-1)$-fold convolution of Catalan numbers.

Notice that we can present our ideals $I_{n,k}$ as $(x_1^2, \dots, x_n^2, e_k(x_1, \dots, x_n))$, where e_k denotes the k'th elementary symmetric polynomial. This gives a connection to earlier work by Haglund, Rhoades, and Shimozon [9] on the Delta conjecture [8] where they determine the Gröbner basis with respect to the lexicographical monomial order for ideals of the form $(x_1^k, \dots, x_n^k, e_n(x_1, \dots, x_n), \dots, e_{n-k+1}(x_1, \dots, x_n))$, and surprisingly show that its elements can be described in terms of Demazure characters, which provides an overlap with our class in the trivial cases $I_{n,n-1}$ and $I_{n,n}$.

The ideals $I_{n,k}$ are also well studied for their relations to the Lefschetz properties. An algebra $A = R/I$, where I is an ideal generated by homogeneous polynomials, so called forms, can be written as $A = A_0 \oplus A_1 \oplus \cdots$, where A_i is the vector space consisting of zero and all forms of degree i. The algebra A has the weak Lefschetz property (WLP) if there is an ℓ in A_1 such that, for every i, the map $A_i \to A_{i+1}$ given by $a \mapsto \ell \cdot a$ is either injective or surjective. If also all powers of ℓ give maps that are always injective or surjective, then A is said to have the strong Lefschetz property (SLP). It is well known that the Lefschetz properties often serve as a bridge between different areas of mathematics, with well-established connections to commutative algebra, algebraic geometry, combinatorics, representation theory, and algebraic topology. In this spirit, one can view the present paper as strengthening the link to combinatorics, and contributing with a link to computer algebra. Indeed, we use our combinatorially described Gröbner basis to show that

$$R/I_{n,k} \text{ has the WLP if and only if } \begin{cases} k \geq \frac{n-3}{2} & \text{for } n \text{ odd}, \\ k \geq \frac{n}{2} & \text{for } n \text{ even}, \end{cases}$$

and we give a new proof of the fact the squarefree algebra $R/(x_1^2, \dots, x_n^2)$ has the strong Lefschetz property, originally shown independently by Stanley [19] and Watanabe [22]. As a corollary, we also obtain a minor result related to the Fröberg conjecture [5].

The ideal $(x_1^2, \dots, x_n^2, (x_1 + \cdots + x_n)^k)$, which may initially appear quite specialized, is in fact a general object. After a linear change of coordinates, we obtain the isomorphism

$$R/(x_1^2, \dots, x_n^2, (x_1 + \cdots + x_n)^k) \cong R/(\ell_1^2, \dots, \ell_n^2, \ell_{n+1}^k),$$

where the ℓ_i are general linear forms. This reveals a deeper geometric connection, particularly in the context of the interpolation of general fat points via Macaulay's inverse system, as described by Emsalem and Iarrobino [4]. In light of this, it is natural to further explore the structure of algebras generated by $n+1$ powers of general linear forms.

We have included only the proofs of the main results and refer to [13] for the more detailed proofs.

Remark 1. While finalizing the paper, we were informed that Booth, Singh, and Vraciu in [2] independently have provided a description of the initial ideal of $(x_1^d, \ldots, x_n^d, (x_1 + \cdots + x_n)^d)$ for $d = 2, 3$, using different techniques. Their work overlaps with ours regarding the initial ideal for $I_{n,2}$. They also find relations for establishing failure of the weak Lefschetz property (WLP) due to injectivity of $I_{n,2}$ for certain n, which are similar to but distinct from special cases of the relations we derive for the failure of the WLP for $I_{n,k}$ for certain n and k.

2 The Gröbner basis

We fix notation as follows. Let $[n] = \{1, \ldots, n\}$. For any subset $S = \{i_1, \ldots, i_s\} \subseteq [n]$, denote by x_S the monomial $x_{i_1} \cdots x_{i_s}$ in $\mathbf{k}[x_1, \ldots, x_n]$. We fix integers n and $k \geq 2$.

2.1 Constructing the Gröbner basis elements

In this section, we define a collection of polynomials $g_{A,n,k}$ combinatorially and prove that they are in the ideal $I_{n,k}$. Later, we will show that they form a Gröbner basis for $I_{n,k}$.

Definition 1. *The* squarefree part *of a polynomial $f \in \mathbf{k}[x_1, \ldots, x_n]$ is its normal form with respect to the monomial ideal $(x_1^2, \ldots, x_n^2)$. We write* $\mathrm{SFP}(f)$ *for the squarefree part of f. For a subset of polynomials $P \subseteq \mathbf{k}[x_1, \ldots, x_n]$ we write* $\mathrm{SFP}(P) = \{\mathrm{SFP}(f) \mid f \in P\}$.

Note that in Definition 1, the word squarefree does not refer to irreducible factors of a polynomial f, but to the exponents of the variables $x_1, \ldots, x_n$ in the terms of its support. In other words, the squarefree part of a polynomial is the part obtained by removing all terms that contain a square. For example,

$$\mathrm{SFP}((x_1 + \cdots + x_5)^2) = 2(x_1x_2 + x_1x_3 + \cdots + x_4x_5) = 2e_2(x_1, \ldots, x_5),$$

where e_2 is the elementary symmetric polynomial of degree two. Since $(x_1^2, \ldots, x_n^2) \subset I_{n,k}$ for all n and k, the crucial part of the analysis of the homogeneous ideals $I_{n,k}$ is the degreewise description of their squarefree parts $\mathrm{SFP}((I_{n,k})_{(d)})$.

Lemma 1. *1. Let $S \subseteq [n]$ and $f_{S,n,k} = \frac{1}{k!}\mathrm{SFP}(x_S(x_1 + \cdots + x_n)^k)$. Then $f_{S,n,k}$ is squarefree of degree $k + |S|$ and can be written as*

$$f_{S,n,k} = \sum_{\substack{u \text{ squarefree} \\ \deg(u)=k+|S| \\ x_S \mid u}} u.$$

2. Let $k \leq d \leq n$. Then $\mathrm{SFP}((I_{n,k})_{(d)})$ is generated as a vector space by the polynomials $f_{S,n,k}$ with $|S| = d - k$.

Lemma 2. *Let $k \leq d \leq n$. For any $S \subseteq [n]$ with $|S| = d - k$ let $\lambda_S \in \mathbf{k}$ be a scalar. Then*

$$\sum_{|S|=d-k} \lambda_S f_{S,n,k} = \sum_{\substack{u \text{ squarefree} \\ \deg(u)=d}} \left(\sum_{\substack{x_S \mid u \\ |S|=d-k}} \lambda_S \right) u.$$

Proof. Write $f = \sum_{|S|=d-k} \lambda_S f_{S,n,k}$. As a $\mathbf{k}$-linear combination of squarefree polynomials of degree d, f is also squarefree of degree d. By Lemma 1, the coefficient of a squarefree term u of degree d in the polynomial $f_{S,n,k}$ is λ_S if $x_S \mid u$ and zero otherwise. Thus the coefficient of u in f is the sum of all λ_S where $x_S \mid u$, as claimed. □

Definition 2. *Let $k \leq d \leq n$, and $A = \{i_1, \ldots, i_d\} \subseteq [n]$ with $\max(A) \leq 2d - k$. We define*

$$g_{A,n,k} = \sum_{\substack{u \text{ squarefree} \\ \deg(u)=d \\ \mathrm{supp}(u)\cap(\{1,\ldots,2d-k\}\setminus A))=\emptyset}} u. \tag{2.1}$$

Example 1. *For the values $n = 5$, $k = 2$, and $d = 3$, consider the set $A = \{1, 3, 4\}$. Note that $\max(A) = 4 \leq 4 = 2 \cdot d - k$. In this case, $g_{A,n,k}$ is given by*

$$x_1x_3x_4 + x_1x_3x_5 + x_1x_4x_5 + x_3x_4x_5,$$

the elementary symmetric polynomial $e_3(x_1, x_3, x_4, x_5)$ of degree d supported on the variables indexed by $A \cup \{5\} = A \cup \big([n] \setminus \{1, \ldots, 2d - k\}\big)$.

We now show that $g_{A,n,k} \in I_{n,k}$, which is a major step needed for the proof of Theorem 2.

Theorem 1. *Let $f_{S,n,k} \in I_{n,k}$ for $S \subseteq [n]$ be the squarefree part of the polynomial $x_S(x_1 + \cdots + x_n)^k$. Then the elements $g_{A,n,k}$ can be written as*

$$g_{A,n,k} = \sum_{i=0}^{d-k} (-1)^i \frac{k}{(k+i)\binom{d}{k+i}} \sum_{S \in \mathcal{T}_i(A)} f_{S,n,k}$$

where $d = |A|$, $\max(A) \leq 2d - k$, and

$$\mathcal{T}_i(A) = \{S \subseteq [n] : |S| = d - k \text{ and } |S \cap \{1, \ldots, 2d - k\} \setminus A| = i\}.$$

Proof. Let $C = \{1, \dots, 2d-k\} \setminus A$. Write $f = \sum_{i=0}^{d-k} \lambda_i \left(\sum_{S \in \mathcal{T}_i} f_{S,n,k}\right)$. By construction, f is a sum of squarefree monomials of degree d. Let u be a squarefree term with $\deg(u) = d$. By Lemma 2, its coefficient in f is $\sum_{i=0}^{d-k} |\{S \in \mathcal{T}_i : x_S \mid u\}| \cdot \lambda_i$. On the other hand, the coefficient of u in $g_{A,n,k}$ is 1 if $|\mathrm{supp}(u) \cap C| = 0$, and zero otherwise. Thus, $f = g_{A,n,k}$ holds if and only if the coefficients λ_i solve the inhomogeneous linear system of $\binom{n}{d}$ equations (labeled by the terms u)

$$\sum_{i=0}^{d-k} |\{S \in \mathcal{T}_i : x_S \mid u\}| \cdot \lambda_i = \begin{cases} 1, & \text{if } |\mathrm{supp}(u) \cap C| = 0 \\ 0, & \text{if } |\mathrm{supp}(u) \cap C| > 0. \end{cases} \tag{2.2}$$

If $|\mathrm{supp}(u) \cap C| = j$ and $S \in \mathcal{T}_i$ with $i > j$, then $x_S \nmid u$. Thus, we can rewrite (2.2) as

$$\sum_{i=0}^{j} |\{S \in \mathcal{T}_i : x_S \mid u\}| \cdot \lambda_i = \begin{cases} 1, & \text{if } |\mathrm{supp}(u) \cap C| = 0 \\ 0, & \text{if } |\mathrm{supp}(u) \cap C| = j > 0. \end{cases} \tag{2.3}$$

We now claim that the system (2.3) contains only $d - k + 1$ distinct equations. First, it is easy to see that for $|\mathrm{supp}(u) \cap C| = 0$ we have $|\{S \in \mathcal{T}_0 : x_S \mid u\}| = \binom{d}{d-k}$ independently of u. Now, consider u with $|\mathrm{supp}(u) \cap C| = j > 0$. Note that $|\mathrm{supp}(u) \setminus C| = d - j$. Furthermore, consider $S \in \mathcal{T}_i$ with $0 \le i \le j$. Then $|S \cap C| = i$ and $|S \setminus C| = d - k - i$. The condition $x_S \mid u$ is then equivalent to satisfying both $S \cap C \subseteq \mathrm{supp}(u) \cap C$ and $S \setminus C \subseteq \mathrm{supp}(u) \setminus C$. The number of sets $S \in \mathcal{T}_i$ that fulfill this is $\binom{j}{i}\binom{d-j}{d-k-i}$ independently of u.

The system (2.3) now simplifies to

$$\begin{cases} \binom{d}{d-k}\lambda_0 &= 1 \\ \sum_{i=0}^{1} \binom{1}{i}\binom{d-1}{d-k-i}\lambda_i &= 0 \\ \vdots & \\ \sum_{i=0}^{d-k} \binom{d-k}{i}\binom{k}{d-k-i}\lambda_i &= 0 \end{cases} . \tag{2.4}$$

This is a lower triangular system with diagonal elements $\binom{d-j}{d-k-j} = \binom{d-j}{k} \neq 0$ (recall that $k \le d$ and $d - j \ge d - (d-k) = k$). In particular, it is straightforward to verify that its unique solution is $\lambda_i = (-1)^i \frac{k}{(k+i)\binom{d}{k+i}}$ for $0 \le i \le d - k$. $\square$

2.2 Hilbert series via Lattice paths

In what follows, we will establish a Gröbner basis with respect to any monomial order with $x_1 \succ \cdots \succ x_n$ of $I_{n,k}$ that is the union of $\{x_1^2, \dots, x_n^2\}$ with the set of $g_{A,n,k}$ whose leading term is minimal with respect to division. We will need a counting argument, and, to this end, we introduce a certain type of lattice paths.

Definition 3. *An* (N,E)-lattice path *is a path on the lattice* $\mathbb{Z}^2$ *that begins at* $(0,0)$ *and consists only of northward steps (in the direction* $(0,1)$*, denoted* N*) and eastward steps (in the direction* $(1,0)$*, denoted* E*).*

There exists a bijection τ *that maps an* (N,E)*-lattice path of length* n *to the monomial* $\prod_{j\in J} x_j$*, where the subset* $J \subseteq [n]$ *contains* j *if and only if the* j*-th step in the path is north.*

Lemma 3. *Let* $k \le d \le n$*. Consider polynomials* $g_{A,n,k}$ *for the sets* $A = \{i_1, \dots, i_d\} \subseteq [n]$ *with* $\max(A) \le 2d-k$ *as in* (2.1)*. Then, we have that*

1. $\operatorname{in}(g_{A,n,k}) = x_{i_1}\cdots x_{i_d}$,
2. *the* (N,E)*-lattice path* $\tau^{-1}(\operatorname{in}(g_{A,n,k}))$ *touches the line* $y = x+k$*,*
3. *if* P *is an* (N,E)*-lattice path of length* n *that intersects the line* $y = x+k$*, then there exists a polynomial* $g_{A,n,k} \in I_{n,k}$ *such that* $\operatorname{in}(g_{A,n,k})$ *divides* $\tau(P)$*.*

We now proceed to compute the number of elements $g_{A,n,k}$, where A is minimal with respect to inclusion. Their leading terms minimally generate $\operatorname{SFP}(\operatorname{in}(I_{n,k}))$.

Definition 4. *Let* $(a_i)_{i=0}^{\infty}$ *be a sequence of integers. The* k*-fold* self-convolution *of* $(a_i)_{i=0}^{\infty}$*, denoted* $(a_i^k)_{i=0}^{\infty}$*, is defined as the sequence of coefficients of the power series* $\left(\sum_{i=0}^{\infty} a_i t^i\right)^{k+1}$*.*

One sequence that is central to our discussion is the Catalan numbers.

Definition 5. *The* Catalan numbers $(C_n)_{n=0}^{\infty}$ *are given by* $C_n = \binom{2n}{n} - \binom{2n}{n-1}$*. The* r*-th number of the Catalan* $(k-1)$*-fold* convolution *is denoted by* C_r^{k-1}*.*

Corollary 1. *The number of polynomials* $g_{A,n,k}$ *such that* A *is minimal with respect to inclusion, and of degree* $k+r$*, is zero if* $n < 2r+k$*; otherwise, it is given by* C_r^{k-1}*. In particular, for* $k=1$ *and* $k=2$*, it is the* r*-th and* $(r+1)$*-th Catalan number, respectively.*

Proof. By Lemma 3, the polynomials $g_{A,n,k}$ of degree $k+r$ are in bijection with (N,E)-lattice paths that take at most n steps and touch the line $y = x+k$ exactly once, specifically at the last step after $k+r$ steps north. Since such a path comprises a total of $2r+k$ steps, there can be no such paths if $n < 2r+k$, resulting in no Gröbner basis elements of degree $r+k$. For $n \ge 2r+k$, we find that, disregarding the last step, these paths are also in bijection with paths that remain below the line $y = x+k-1$ and terminate on that same line after $k-1+r$ steps north.

Next, by [21, Corollary 16], C_i^j represents the number of (N,E)-lattice paths that start at $(i,0)$, do not cross the line $y=x$, and end at $(i+j, i+j)$. By shifting all such paths left by i steps, we can also interpret C_i^j as counting (N,E)-paths starting at the origin that remain below the line $y = x+i$ and terminate at $(j, i+j)$. By combining this count with the previously established enumeration for the number of polynomials $g_{A,n,k}$ of degree $k+r$, we arrive at the complete enumeration. Specifically, for $k=1$ we get $C_r^0 = C_r$ polynomials in degree r and for $k=2$, the claim follows from the well-known fact that $C_n^1 = C_{n+1}$. □

The following lemma presents our main counting argument and serves as the final step needed to complete the main part of the proof of Theorem 2.

Lemma 4. *Let d be an integer with $0 \leq d \leq n$, and consider the terms of degree d outside the degree reverse lexicographic initial ideal $\mathrm{in}(I_{n,k})$. If $2d - k \geq n$, there are no such terms. Otherwise, these terms are squarefree, and their number is at most $\binom{n}{d} - \binom{n}{d-k}$.*

Corollary 2. *The Hilbert series of $R/I_{n,k}$ is given by $\left[(1+t)^n(1-t^k)\right]$, where the brackets indicate truncation at the first non-positive coefficient.*

Before we state our main result, recall that a Gröbner basis is *reduced* if it is in bijection with the minimal generating set of $\mathrm{in}(I)$, all its elements have 1 as leading coefficient, and the monomials appearing after the leading monomial of every polynomial in it are not in the initial ideal.

Theorem 2. *Consider the family $\mathcal{A}$ of subsets $A \subseteq [n]$ satisfying $\max(A) = 2|A| - k$ for some $k \geq 2$, and minimal with respect to inclusion. Then, the reduced Gröbner basis of the ideal $I_{n,k} = (x_1^2, \dots, x_n^2, (x_1 + \cdots + x_n)^k)$ with respect to degree reverse lexicographic ordering is given by*

$$G_{n,k} = \{x_1^2, \dots, x_n^2\} \cup \bigcup_{d=k}^{k+\lfloor (n-k)/2 \rfloor} \{g_{A,n,k} \mid A \in \mathcal{A}, |A| = d\},$$

where for $|A| = d$,

$$g_{A,n,k} = e_d(x_{i_1}, \dots, x_{i_{n-d+k}})$$

is the elementary symmetric polynomial of degree d in the variables indexed by the set

$$\{i_1, \dots, i_{n-d+k}\} = A \cup \{2d - k + 1, \dots, n\}$$

with leading term $x_A = \prod_{a \in A} x_a$. In particular, $g_{A,n,k}$ is supported on $\binom{n-d+k}{d}$ terms. Moreover, for fixed k the sequence of cardinalities $|\{A \in \mathcal{A} : |A| = d\}|$ is a $(k-1)$-fold convolution of the sequence of Catalan numbers.

Proof. First, by Lemma 3 and Corollary 2, the squarefree leading terms of polynomials in $I_{n,k}$ are in bijection with n-step (N, E)-lattice paths that touch the line $y = x + k$, and each of these terms is divisible by some leading term $\mathrm{in}(g_{A,n,k})$. Among these leading terms, the minimal ones with respect to division are those that touch the line $y = x + k$ exactly once, occurring after their last step north. Moreover, the penultimate step cannot be east; if it were, there would be an earlier intersection with the line. This establishes the defining conditions of the set $\mathcal{A}$. Regarding non-squarefree leading terms, clearly they are divisible by some x_i^2. By Theorem 1, $G_{n,k} \subset I_{n,k}$. Thus, $G_{n,k}$ is a minimal Gröbner basis of $I_{n,k}$.

To show that $G_{n,k}$ is reduced, first observe that all its elements are monic. Now, consider $g_{A,n,k} \in G_{n,k}$. Let $(a,b) \in \mathbb{Z}^2$ be the lattice point where $\tau^{-1}(\mathrm{in}(g_{A,n,k}))$ touches the line $y = x + k$. Any term $u \in g_{A,n,k} - \mathrm{in}(g_{A,n,k})$ is of the form

$$u = x_{j_1} \cdots x_{j_m} \mathrm{in}(g_{A,n,k}) / (x_{\ell_1} \cdots x_{\ell_m}),$$

where $0 < m \leq |A|$, $j_1, \ldots, j_m > \max(A)$, and $\ell_1, \ldots, \ell_m \in A$. Consequently, the lattice path $\tau^{-1}(u)$ reaches the point $(a+m, b-m)$ after $\max(A)$ steps. The closest it could approach the line $y = x + k$ is at the point $(a+m, b)$ after $\max(A) + m$ steps. However, this point lies below the line since $m > 0$ and (a,b) is on the line. Therefore, $u \notin \mathrm{in}(I_{n,k})$.

The last claim follows from Corollary 1. □

Example 2. *We set $n = 5$ and $k = 2$. The reduced Gröbner basis of $I_{5,2}$ is*

$$G_{5,2} = \{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, g_{\{1,2\},5,2}, g_{\{1,3,4\},5,2}, g_{\{2,3,4\},5,2}\},$$

where

$$\begin{aligned} g_{\{1,2\},5,2} &= x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + x_3x_4 + x_1x_5 + x_2x_5 + x_3x_5 + x_4x_5, \\ g_{\{1,3,4\},5,2} &= x_1x_3x_4 + x_1x_3x_5 + x_1x_4x_5 + x_3x_4x_5, \\ g_{\{2,3,4\},5,2} &= x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5. \end{aligned}$$

Remark 2. The Gröbner basis of $I_{n,k}$ is independent of the monomial ordering once we fix the ordering $x_1 \succ \cdots \succ x_n$. It is also invariant under permutations of the first k variables and the action of transpositions of the form $(k+2i-1, k+2i)$ for $i \geq 1$. In particular, the number of distinct Gröbner bases of $I_{n,k}$ is given by the multinomial coefficient $\binom{n}{k,2,\ldots,2,1}$, where the entry 1 is included if and only if $n - k \equiv 1 \mod 2$. See [13, Proposition 2.29] for details.

3 The Lefschetz properties

We now apply the results from the previous section to study the Lefschetz properties. We refer to [11, 16] for more background and details on the Lefschetz properties.

3.1 Monomial complete intersections have the SLP

As a first application of our Gröbner basis results, specifically Corollary 2, to the Lefschetz properties, we establish the well-known result that the squarefree algebra possesses the SLP.

Corollary 3. *The squarefree algebra $A = R/(x_1^2, \ldots, x_n^2)$ has the SLP.*

Proof. Note that multiplication by a form f of degree d on an algebra B has full rank if and only if

$$\mathrm{HS}(B/(f);t) = [(1-t^d)\mathrm{HS}(B;t)],$$

where the square brackets indicate truncation of the polynomial at the first non-positive coefficient. Since $\mathrm{HS}(A;t) = (1+t)^n$, Corollary 2 says that $\mathrm{HS}(A/(\ell^d);t) = [(1-t^d)\mathrm{HS}(A;t)]$ for the linear form $\ell = x_1 + \cdots + x_n$ and any d. Thus any power of ℓ gives a map that has full rank on A, so A has the SLP. □

The SLP for artinian monomial complete intersections can be derived from the squarefree algebra, as shown in a short argument by Hara and Watanabe [10], so in this regard the squarefree case serves as the building block for the SLP for this class of algebras.

While Stanley and Watanabe established the SLP in the 1980s, the result for squarefree algebras dates back to the 1970s, studied in the context of zero-one matrices by Kantor [14], Graver and Jurkat [7], and Wilson [23], with references highlighted by Lindström. Besides these, we note Hara and Watanabe's proof for $d = 2$, Ikeda's [12] from the 1990s, and Phuong and Tran's [18] more recent proof. So, to our knowledge, our proof is the seventh for the base case $d = 2$.

3.2 The WLP for powers of linear forms in the squarefree algebra

As $R/(\ell_1^{a_1}, \ldots, \ell_n^{a_n})$ has the WLP for general ℓ_i, it is natural to ask if also $R/(\ell_1^d, \ldots, \ell_{n+1}^d)$ has the WLP. However, this is not the case, and was first observed by Fröberg and Hollman [6] for the case $(n,d) = (5,2)$, which via results by Cruz and Iarrobino [3] and Sturmfels and Xu [20] for $d = 2$, and Migliore, Miro-Róig and Nagel [15], Nagel and Trok [17] for general d, ended up in a classification by Boij and the second author [1], who showed that for $n \geq 4$, the WLP holds only in sporadic cases. The proof relies on Macaulay's inverse system and a reduction to the case $d = 2$, where it is shown that the WLP fails due to lack of surjectivity.

In the following theorem, we fully characterize the pairs (n,k) for which the algebra $R/(\ell_1^2, \ldots, \ell_n^2, \ell_{n+1}^k)$, defined by powers of general linear forms $\ell_1, \ldots, \ell_{n+1}$, has the WLP. In this case, the WLP can be derived from the structure of $R/(\ell_1^2, \ldots, \ell_n^2, \ell_{n+1}^2)$. Unlike the equigenerated case, however, the failure of the WLP here is due to a lack of injectivity. This requires identifying non-trivial identities within the squarefree algebra; see Proposition 1.

The classification result that we will prove is stated in the following theorem.

Theorem 3. *Let $\mathbf{k}$ be a field of characteristic zero. Then for general linear forms $\ell_1, \ldots, \ell_{n+1}$, the algebra $R/(\ell_1^2, \ldots, \ell_n^2, \ell_{n+1}^k) \cong R/(x_1^2, \ldots, x_n^2, (x_1 + \cdots + x_n)^k)$ has the weak Lefschetz property if and only if*

$$\begin{cases} k \geq \frac{n-3}{2} & \text{for } n \text{ odd,} \\ k \geq \frac{n}{2} & \text{for } n \text{ even.} \end{cases}$$

Remark 3. *Theorem 3 implies that the Fröberg conjecture is true for $a_1 = \cdots = a_{n+1} = 2$ and $a_{n+2} = k$, where*

$$\begin{cases} k \geq \frac{n-2}{2} & \text{for } n \text{ even,} \\ k \geq \frac{n+1}{2} & \text{for } n \text{ odd.} \end{cases}$$

The statement is trivial when n is odd, as in this case, k is the socle degree or higher. However, for even n, the case $k = \frac{n-2}{2}$ offers a new, but minor contribution for the conjecture.

In the following lemmas, it is useful to work with the ideal $I_{n-1,2}$ in place of $I_{n,k}$ to streamline the analysis and simplify computations; see [13] for proofs. Let us fix the algebras

$A = \mathbf{k}[x_1, \ldots, x_{n-1}]/(x_1^2, \ldots, x_{n-1}^2, (x_1 + \cdots + x_{n-1})^2)$ and $B = \mathbf{k}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$

Lemma 5. *The algebra $R/(x_1^2, \ldots, x_n^2, (x_1 + \cdots + x_n)^k)$ has the WLP if and only if, for a general linear form ℓ in $\mathbf{k}[x_1, \ldots, x_{n-1}]$, the multiplication by ℓ^k yields a full rank map on A.*

The following proposition will be used to establish the failure of the WLP in Theorem 3.

Proposition 1. *Let $\ell = a_1 x_1 + \cdots + a_n x_n$ be a general linear form. Then:*

(i) *If $n = 2p + 1$ for some $p \geq 1$, then there exists another linear form ℓ' and a degree $(p-1)$ form g such that in the algebra B, we have $\ell^p \ell' = (x_1 + \cdots + x_n)^2 g$.*

(ii) *If $n = 2p$ for some $p \geq 3$, then there exist a degree 2 form f and a degree $p-2$ form g such that $\ell^{p-2} f = (x_1 + \cdots + x_n)^2 g$ in B and f is not a multiple of $(x_1 + \cdots + x_n)^2$.*

While Proposition 1 will be used to establish failure of the WLP, the next lemma will be used to lift this failure to more degrees and address the nontrivial cases with the WLP.

Lemma 6. *Let ℓ be a general linear form in the algebra B. Then:*

(i) *If $n \geq 2p + 2$, then $\cdot \ell^p : B_1 \to B_{p+1}$ is injective.*

(ii) *If $n = 2p$ and $p \geq 3$, then for q, $2 < q < p$, the map $\cdot \ell^{q-2} : B_2 \to B_q$ is injective.*

With this preparation, we are now ready to prove Theorem 3.

Proof of Theorem 3. By Lemma 5, this is equivalent to showing that the k-th power of a general linear form has full rank on A. for the same values of n and k. We begin by examining the ideals for which we want to show the WLP.

Note that the socle degree of A is given by the smallest integer d for which $\binom{n-1}{d} - \binom{n-1}{d-2} > 0$, which is $(n-1)/2$ if n is odd, and $n/2$ if n is even. Since any multiplication by a power of a linear form from A_0, or into a zero-dimensional vector space, has full rank, it follows that any k greater than $(n-1)/2$ (for n odd) or greater than $n/2$ (for n even) works. Additionally, when $n = 2k+3$, the socle degree is $k+1$, so the only map potentially lacking full rank is from degree 1 to degree $k+1$. However, this map is injective by Lemma 6(i), establishing all the required full-rank maps.

We are now left to show that the remaining values of n, k give algebras that do not have the WLP. Assume first that $n = 2p+2$. Then we claim that for any general linear form ℓ in A, the map $\cdot\ell^k : A_{p+1-k} \to A_{p+1}$ does not have full rank. For $k = p$, Proposition 1(i) provides a linear form $\ell' \in A_1$ such that $\ell^p\ell' = 0$ in A. Additionally, by Lemma 6(i), we have $\ell^k\ell' \neq 0$ for any $k < p$. Therefore, $\ell^{p-k}\ell'$ is a nonzero element in the kernel of ℓ^k for all $k = 2, \dots, p-1$, demonstrating that $\cdot\ell^k : A_{p+1-k} \to A_{p+1}$ cannot be injective. Moreover, this map cannot be surjective either. Using the theory of inverse systems, Boij and the second author in [1, Theorem 5.2] have shown that $A/(\ell^k)$ is always nonzero in degree $p+1$ for any $k \geq 2$. This proves all required failures of maximal rank for the case where n is even.

The case where $n = 2p+1$ is odd follows a similar proof. First, the map $\cdot\ell^k : A_{p-k} \to A_p$ is not injective for $k = p-2$ by Proposition 1(ii), as there exists a nonzero form f of degree 2 with $\ell^{p-2}f = 0$ in A. Furthermore, by Lemma 6(ii), we have $\ell^{p-k-2}f \neq 0$ in A for $k < p-2$, which means that $\ell^{p-k-2}f$ is a nonzero element in the kernel of $\cdot\ell^k : A_{p-k} \to A_p$, demonstrating that this map is never injective. Additionally, by [1, Theorem 5.2], setting the last variable to zero shows that $A/(\ell^k)$ is nonzero in degree p for any $k \geq 2$. This completes the verification of all required failures of full rank, proving the theorem. □

References

[1] M. Boij and S. Lundqvist. "A classification of the weak Lefschetz property for almost complete intersections generated by uniform powers of general linear forms". *Algebra Number Theory* **17**.1 (2023), pp. 111–126. DOI.

[2] M. D. Booth, P. Singh, and A. Vraciu. "On the weak Lefschetz property for ideals generated by powers of general linear forms". 2024. arXiv:2410.22542.

[3] C. D'Cruz and A. Iarrobino. "High-order vanishing ideals at $n+3$ points of P^n". *J. Pure Appl. Algebra* **152**.1 (2000), pp. 75–82. DOI.

[4] J. Emsalem and A. Iarrobino. "Inverse system of a symbolic power. I". *J. Algebra* **174**.3 (1995), pp. 1080–1090. DOI.

[5] R. Fröberg. "An inequality for Hilbert series of graded algebras". *Math. Scand.* **56**.2 (1985), pp. 117–144. DOI.

[6] R. Fröberg and J. Hollman. "Hilbert series for ideals generated by generic forms". *J. Symbolic Comput.* **17**.2 (1994), pp. 149–157. DOI.

[7] J. Graver and W. Jurkat. "The module structure of integral designs". *J. Combin. Theory Ser. A* **15**.1 (1973), pp. 75–90. DOI.

[8] J. Haglund, J. Remmel, and A. Wilson. "The delta conjecture". *Trans. Amer. Math. Soc.* **370**.6 (2018), pp. 4029–4057. DOI.

[9] J. Haglund, B. Rhoades, and M. Shimozono. "Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture". *Adv. Math.* **329** (2018), pp. 851–915. DOI.

[10] M. Hara and J. Watanabe. "The determinants of certain matrices arising from the Boolean lattice". *Discret. Math.* **308**.23 (2008), pp. 5815–5822. DOI.

[11] T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi, and J. Watanabe. "Lefschetz Properties". *The Lefschetz Properties*. Vol. 2080. Springer, Berlin, Heidelberg, 2013, pp. 97–140. DOI.

[12] H. Ikeda. "Results on Dilworth and Rees numbers of Artinian local rings". *Japan. J. Math. (N.S.)* **22**.1 (1996), pp. 147–158. DOI.

[13] F. Jonsson Kling, S. Lundqvist, F. Mohammadi, M. Orth, and E. Sáenz-de Cabezón. "Gröbner bases, resolutions, and the Lefschetz properties for powers of a general linear form in the squarefree algebra". 2024. arXiv:2411.10209.

[14] W. M. Kantor. "On incidence matrices of finite projective and affine spaces". *Math. Z.* **124** (1972), pp. 315–318. DOI.

[15] J. C. Migliore, R. M. Miró-Roig, and U. Nagel. "On the weak Lefschetz property for powers of linear forms". *Algebra Number Theory* **6**.3 (2012), pp. 487 –526. DOI.

[16] J. C. Migliore and U. Nagel. "Survey Article: A tour of the weak and strong Lefschetz properties". *J. Commut. Algebra.* **5**.3 (2013), pp. 329 –358. DOI.

[17] U. Nagel and B. Trok. "Interpolation and the weak Lefschetz property". *Trans. Amer. Math. Soc.* **372**.12 (2019), pp. 8849–8870. DOI.

[18] H. V. N. Phuong and Q. H. Tran. "A new proof of Stanley's theorem on the strong Lefschetz property". *Colloq. Math.* **173**.1 (2023), pp. 1–8. DOI.

[19] R. P. Stanley. "Weyl Groups, the Hard Lefschetz Theorem, and the Sperner Property". *SIAM J. Algebraic Discrete Methods* **1**.2 (1980), pp. 168–184. DOI.

[20] B. Sturmfels and Z. Xu. "Sagbi bases of Cox-Nagata rings". *J. Eur. Math. Soc. (JEMS)* **12**.2 (2010), pp. 429–459. DOI.

[21] S. Tedford. "Combinatorial interpretations of convolutions of the Catalan numbers". *Integers* **11**.1 (2011), pp. 35–45. DOI.

[22] J. Watanabe. "The Dilworth number of Artin Gorenstein rings". *Adv. Math.* **76**.2 (1989), pp. 194–199. DOI.

[23] R. M. Wilson. "The necessary conditions for t-designs are sufficient for something". *Util. Math.* **4** (1973), pp. 207–215.