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A Toric Analogue for Greene's Rational Function

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Abstract. Given a finite poset, Greene introduced a rational function obtained by summing certain rational functions over the linear extensions of the poset. This function has interesting interpretations and for certain families of posets, it simplifies surprisingly. Greene evaluated this rational function for strongly planar posets in work on the Murnaghan–Nakayama formula.

Develin, Macauley, and Reiner introduced toric posets, which combinatorially are equivalence classes of posets (or rather acyclic quivers) under the operation of flipping maximum elements into minimum elements and vice versa. In this work, we introduce a toric analogue of Greene's rational function for toric posets, and study its properties. In addition, we use toric posets to show that the Kleiss–Kuijf relations, which appear in scattering amplitudes, are equivalent to a specific instance of Greene's evaluation of his rational function for strongly planar posets. We also give an algorithm for finding the set of toric total extensions of a toric poset.

Keywords: Toric, poset, linear extension, arrangement, rational function, Greene

1 Introduction

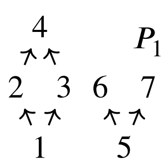
In 1992, C. Greene associated to every poset P on $[n] = \{1, 2, \ldots, n\}$ a rational function

$$\Psi^P(\mathbf{x}) = \sum_{w \in \mathcal{L}(P)} \frac{1}{(x_{w_1} - x_{w_2})(x_{w_2} - x_{w_3}) \cdots (x_{w_{n-1}} - x_{w_n})},$$

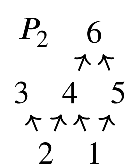
motivated by a combinatorial proof of the Murnaghan–Nakayama formula [13]. Here $\mathcal{L}(P)$ denotes the set of *linear extensions* $w = (w_1 < \cdots < w_n)$ of P from the partial order P to a total order. Part of the mathematical beauty in $\Psi^P(\mathbf{x})$ is that for certain poset families, $\Psi^P(\mathbf{x})$ simplifies surprisingly.

Example 1.1. We evaluate Greene's rational function for two posets.

$\Psi^{P_1}(\mathbf{x}) = 0$



$$\Psi^{P_2}(\mathbf{x}) = \frac{x_1 - x_6}{(x_2-x_3)(x_2-x_4)(x_1-x_4)(x_1-x_5)(x_4-x_6)(x_5-x_6)}$$



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A poset P is *strongly planar* if after adding a new bottom element $\hat{0}$ and top element $\hat{1}$, the *Hasse diagram* $H(P \sqcup \{\hat{0}, \hat{1}\})$ may be order-embedded in $\mathbb{R}^2$ without crossings. In this embedding, each bounded region δ of $\mathbb{R}^2 \setminus H(P)$ has a unique minimum element $min(\delta)$ and a unique maximum element $max(\delta)$. Greene proved that $\Psi^P(\mathbf{x})$ vanishes if $H(P)$ is disconnected, and otherwise

$$\Psi^P(\mathbf{x}) = \frac{\prod_{\delta \in \Delta}(x_{\min(\delta)} - x_{\max(\delta)})}{\prod_{i \lessdot_P j}(x_i - x_j)}. \tag{1.1}$$

In Example 1.1, the poset P_2 is a connected, strongly planar poset and in $H(P_2)$, there is exactly one bounded region δ, with $\max(\delta) = 6$ and $\min(\delta) = 1$.

Boussicault, Féray, Lascoux, and Reiner [4] interpreted $\Psi^P(\mathbf{x})$ geometrically and algebraically, extending Greene's results. E.g., they showed $H(P)$ is disconnected if and only if $\Psi^P(\mathbf{x}) = 0$.

Develin, Macauley, and Reiner introduced *toric posets* [9] (also seen in [16]). Geometrically, a toric poset corresponds to a toric chamber in the complement of a graphic toric hyperplane arrangement. This is similar to how a poset corresponds to a chamber in the complement of a graphic hyperplane arrangement; see [9, 14, 21, 25] and Section 2. Recently, toric hyperplane arrangements have received increased attention; see, e.g., [1, 3]. Combinatorially, toric posets can be thought of as an equivalence class $[Q]$ of acyclic (no directed cycles) quivers that are equivalent under the relation of flipping a sink vertex to a source vertex and vice versa. This flip operation has been well-studied and appears widely in different contexts [2, 7, 10, 17, 18, 22, 23, 24]. In fact, these equivalence classes are subsets of the mutation class of a quiver used in cluster algebras [11].

Just as a permutation $(w_1, w_2, \ldots, w_n)$ of $[n]$ may be thought of as a total order $w_1 < w_2 < \cdots < w_n$ or an acyclic orientation of the complete graph on $[n]$, a *toric total order* is the cyclic equivalence class $[(w_1, w_2, \ldots, w_n)]$ under rotation $(w_1, w_2, \ldots, w_n) \mapsto (w_2, w_3, \ldots, w_n, w_1)$, or the special case of a toric poset for an acyclic quiver whose underlying undirected graph is complete.

In this work, which is an abstract of the recent preprint [6], we define a toric analogue of Greene's rational function for toric posets, a sum of rational functions indexed by the set of *toric total extensions*, denoted $\mathcal{L}_{\text{tor}}([Q])$ (see Section 3).

Definition 1.2. Let $[Q]$ be a toric poset. Then, we define $\Psi_{\text{tor}}^{[Q]}(\mathbf{x})$ as

$$\Psi_{\text{tor}}^{[Q]}(\mathbf{x}) := \sum_{[w] \in \mathcal{L}_{\text{tor}}([Q])} \Psi_{\text{tor}}^{[w]}(\mathbf{x}),$$

where

$$\Psi_{\text{tor}}^{[w]}(\mathbf{x}) = \frac{1}{(x_{w_1} - x_{w_2})(x_{w_2} - x_{w_3}) \cdots (x_{w_{n-1}} - x_{w_n})(x_{w_n} - x_{w_1})}.$$

Motivated by the results in [4], we use Greene's results to prove similar results for $\Psi_{\text{tor}}^{[Q]}(\mathbf{x})$ – see our first two main results, Theorem 4.5 on its vanishing, and Theorem 4.8 on its denominator. To compute $\Psi_{\text{tor}}^{[Q]}(\mathbf{x})$ for a toric poset $[Q]$, it is necessary to compute the set $\mathcal{L}_{\text{tor}}([Q])$ of toric total extensions. We show in Theorem 3.14 that counting $\mathcal{L}_{\text{tor}}([Q])$ is a $\#P$-complete problem, so one should not expect efficient algorithms for finding this set. For theoretical purposes, we often use a decomposition (see Proposition 3.12 part (ii)) that expresses $\mathcal{L}_{\text{tor}}([Q])$ as a disjoint union indexed by the subset $[Q]_v$ of quivers in $[Q]$ having vertex v as a source. Although there is no efficient algorithm for computing $[Q]_v$, one of our main theorems derives a more efficient recursive algorithm to compute $\mathcal{L}_{\text{tor}}([Q])$ (see Theorem 5.3). In Corollary 4.3, another one of our main results, we exhibit a novel connection between toric posets and the *Kleiss–Kuijf relations*, which appear in scattering amplitudes. Full proofs can be found in [6].

Remark 1.3. The rational function $\Psi_{\text{tor}}^{[w]}(\mathbf{x})$ appears in scattering amplitude computations as Parke–Taylor factors [20]. Most recently, in [19], Parisi, Sherman-Bennett, Tessler, and Williams utilize $\Psi_{\text{tor}}^{[w]}(\mathbf{x})$ in order to prove a tiling conjecture for the $m = 2$ amplituhedron.

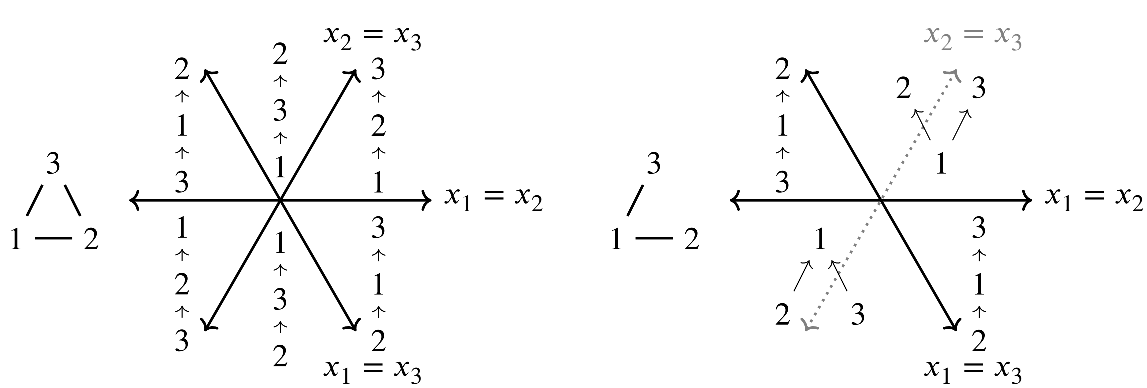
2 Posets and Graphic Hyperplane Arrangements

The definition of a toric poset relies on the well-studied association between posets and chambers in graphic hyperplane arrangements [9, 14, 21, 25], so we discuss this correspondence. A poset P on $[n]$ gives rise to an open polyhedral cone $c(P)$ in $\mathbb{R}^n$ where $c(P) := \{x \in \mathbb{R}^n : x_i < x_j \text{ if } i <_P j\}$.

Connected components in the complement of a graphic hyperplane arrangement are open polyhedral cones called *chambers*, and each cone $c(P)$ appears as a chamber in the complement of at least one graphic hyperplane arrangement. Let G be a simple, undirected graph on the vertex set $[n]$, so $G \subseteq \binom{[n]}{2}$. Then, the *graphic hyperplane arrangement* $\mathcal{A}(G)$ is defined to be $\mathcal{A}(G) := \bigcup_{\{i,j\} \in G} \mathcal{H}_{ij}$ where $\mathcal{H}_{ij}$ is the hyperplane $x_i = x_j$.

An acyclic quiver is a directed graph that contains no directed cycles. There is a one-to-one correspondence between chambers in $\mathbb{R}^n - \mathcal{A}(G)$ and acyclic quivers that have the same underlying graph G. Given such a chamber, for every pair of vertices i, j such that $\{i, j\} \in G$, we orient this edge $i \to j$ if $x_i < x_j$ and orient the edge $j \to i$ otherwise. Moreover, any acyclic quiver on n vertices induces a poset structure on n elements. In particular, we set $i < j$ in the poset whenever there is a directed path from i to j in the quiver.

Example 2.1. We show two graphic hyperplane arrangements. For each arrangement, we label the chambers by the posets induced by acyclic orientations of the corresponding graph.



A poset P is also determined by its set of linear extensions. Each extension $(w_1, w_2, \dots, w_n)$ corresponds to a chamber $c_w := \{\mathbf{x} \in \mathbb{R}^n : x_{w_1} < x_{w_2} < \cdots < x_{w_n}\}$ in the complement of the complete graphic hyperplane arrangement $\mathcal{A}(K_n)$, also known as the braid arrangement. From this observation, we have $\overline{c(P)} = \bigcup_{w \in \mathcal{L}(P)} \overline{c}_w$ where $\overline{(\cdot)}$ denotes topological closure. This equation demonstrates that when one fixes the graph G, posets (chambers) are determined by their sets of linear extensions. Posets may arise as chambers in several graphic hyperplane arrangements as the graph G varies. Although there is ambiguity when identifying P with an *acyclic quiver* Q, there are two natural choices: the transitive closure $\overline{P}$ and the Hasse diagram $H(P)$.

3 Toric Posets

In [9], Develin, Macauley, and Reiner introduce toric posets. We may distinguish toric posets from posets, by calling posets “ordinary" posets. Toric graphic hyperplane arrangements are the source of the name “toric” poset. Given an undirected graph G on n vertices, recall that there is an associated graphic hyperplane arrangement $\mathcal{A}(G)$ inside $\mathbb{R}^n$. We define a quotient map $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$. The *toric graphic hyperplane arrangement* associated to G is $\mathcal{A}_{\text{tor}}(G) = \pi(\mathcal{A}(G))$. A connected component of $\mathbb{R}^n/\mathbb{Z}^n - \mathcal{A}_{\text{tor}}(G)$ is a *toric chamber*. A *toric poset* is a set that arises as a toric chamber in a toric graphic hyperplane arrangement for at least one graph G.

Naturally, given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, these points lie in the same equivalence class in $\mathbb{R}^n/\mathbb{Z}^n$ exactly when for each coordinate $1 \le i \le n$, we have $x_i \mod 1 = y_i \mod 1$ where $x_i \mod 1$ and $y_i \mod 1$ are elements of $[0, 1)$. Therefore, we can still recover an acyclic quiver with underlying graph G for each point $[\mathbf{x}] \in \mathbb{R}^n/\mathbb{Z}^n$ by orienting $\{i, j\} \in G$ as $i \to j$ if $x_i \mod 1 < x_j \mod 1$ and orienting $\{i, j\}$ as $j \to i$ otherwise.

> *Key point:* By this construction, two points in the same toric chamber might not map to the same acyclic quiver. To account for this, the following *flip operation* is defined.

Definition 3.1 ([9]). Consider acyclic quivers Q_1, Q_2 that differ by converting one source vertex (all edges directed outward) to one sink vertex (all edges directed inward). Then, we say that Q_1, Q_2 are related by a *flip*. This flip operation induces an equivalence relation on the set of acyclic quivers with the same underlying graph G, and we denote this equivalence relation as $\equiv$.

Remark 3.2. This flip operation was studied by Mosesian and Pretzel in [18] and [22], respectively and has appeared in other works including Chen [7], Defant and Kravitz [8], Eriksson and Eriksson [10], Macauley and Mortveit [17], Speyer [24], and Propp [23]. This flip operation also appears in the context of reflection functors in quiver representations [2] and is an instance of quiver mutation at a sink or source vertex [11].

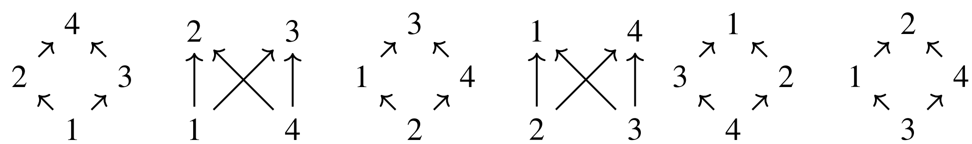
With ordinary posets, we saw that there is a bijection between chambers of $\mathcal{A}(G)$ and the set of acyclic quivers with underlying graph G. For toric posets, we have the following theorem.

Theorem 3.3 ([9, Theorem 1.4])**.** *There is a bijection between the chambers of $\mathcal{A}_{\text{tor}}(G)$ and equivalence classes under $\equiv$ of acyclic quivers with underlying graph G.*

With $\equiv$ defined, we can define toric posets in a combinatorial way.

Definition 3.4. A *toric poset* $[Q]$ is an equivalence class of acyclic quivers that are equivalent under the relation of flipping a sink vertex to a source vertex and vice versa.

Example 3.5. Let us consider the following toric poset $[Q]$:



Any two quivers $Q_1, Q_2 \in [Q]$ are related by a sequence of source to sink (or sink to source) flips.

3.1 Properties of Toric Posets

All quivers will be acyclic (no directed cycles) and simple, no parallel directed arcs; self-loops and anti-parallel directed arcs are already prevented due to the acyclic assumption.

For ordinary posets, the Hasse diagram and transitive closure depend on chains in the poset. A similar story is true for toric posets and *toric chains*. We first define a *toric directed path*.

Definition 3.6 ([9])**.** For $Q' \in [Q]$, elements $x_1, x_2, \dots, x_{k-1}, x_k$ form a *toric directed path* if Q' contains the chain $x_1 \to x_2 \to \cdots \to x_{k-1} \to x_k$ and the directed edge $x_1 \to x_k$.

Let C be the set of vertices in a toric directed path. The *length* of the toric directed path is $|C| - 1$. We note that an edge and a vertex is a toric directed path of length 1 and length 0, respectively.

Definition 3.7 ([9])**.** For a toric poset $[Q]$ that has underlying graph G with vertex set V, a *toric chain* is a subset $V' \subseteq V$ that is totally ordered for every poset induced by an acyclic quiver in $[Q]$.

Just as chains are closed under subsets in a poset, toric chains are closed under subsets in a toric poset. In [9, Proposition 6.3], Develin, Macauley, and Reiner show that a subset $V' \subseteq V$ is a toric chain if and only if the elements of V' lie along a toric directed path.

Definition 3.8 ([9])**.** Two elements a, b of a toric poset $[Q]$ are *torically comparable* if there exists a toric chain in $[Q]$ that a, b lie on together. Otherwise, elements a, b are *torically incomparable*.

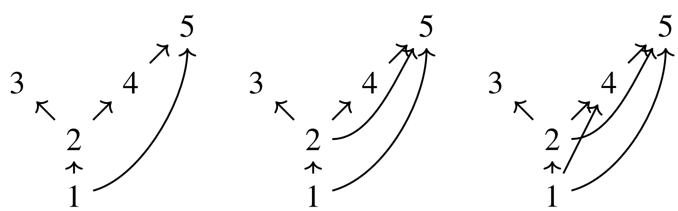
As with ordinary posets, a toric poset may arise as a chamber in the complement of toric graphic hyperplane arrangements for several graphs. However, there are two natural choices of such graphs:

i. $\overline{[Q]}$, the *toric transitive closure* of $[Q]$, and

ii. $[Q]_{\text{Hasse}}$, the *toric Hasse diagram* corresponding to $[Q]$.

Definition 3.9 ([9])**.** Let $[Q]$ be a toric poset and $Q' \in [Q]$. The *toric transitive closure* of Q', denoted $\overline{Q'}$ is the quiver where one adds to the underlying graph of Q' all edges $\{i, j\}$ if i and j live on a toric chain and directs $i \to j$ if there exists a toric directed path from i to j in Q'. Then, the *toric transitive closure* of $[Q]$, denoted $\overline{[Q]}$ is defined as $\overline{[Q]} := [\overline{Q'}]$. In [9, Corollary 7.3], the authors show that $\overline{[Q]}$ does not depend on the choice of representative $Q' \in [Q]$.

In contrast to the toric transitive closure, we can define the toric analogue of a Hasse diagram as follows. Let $[Q]$ be a toric poset and choose a representative $Q' \in [Q]$. Let Q'_{Hasse} be the quiver constructed from Q' by removing each edge $i \to j$ for which Q' contains a toric directed path from i to j that is both non-maximal and has length strictly greater than 1. The *toric Hasse diagram* of $[Q]$, denoted $[Q]_{\text{Hasse}}$, is defined as $[Q]_{\text{Hasse}} := [Q'_{\text{Hasse}}]$. In [9, Corollary 9.2], the authors show that the toric Hasse diagram does not depend on the choice of representative $Q' \in [Q]$. Below we show one representative of $[Q]_{\text{Hasse}}$, $[Q]$, and $\overline{[Q]}$ and we label these quivers Q_1, Q_2, Q_3, respectively.



Q_1 Q_2 Q_3

A *toric total order* is a cyclic equivalence class and corresponds to a chamber in the complement of the toric complete graphic arrangement $\mathcal{A}_{tor}(K_V)$. A toric total order is of the form

$$[w] := [(w_1, \ldots, w_n)] = \{(w_1, \ldots, w_{n-1}, w_n), (w_2, \ldots, w_n, w_1), \ldots (w_n, w_1, \ldots, w_{n-1})\}.$$

Definition 3.10 ([9])**.** Let $[Q]$ be a toric poset and let c be the toric chamber in the associated toric graphic hyperplane arrangement that corresponds to $[Q]$. A toric total order $[w]$ is a *toric total extension* of $[Q]$ if $c_{[w]} \subseteq c$, where $c_{[w]}$ is the toric chamber associated to $[w]$. We denote the set of toric total extensions of $[Q]$ as $\mathcal{L}_{\text{tor}}([Q])$.

In the following lemma, we show that the set of toric total extensions of $[Q]_{\text{Hasse}}$ is the same as the set of toric total extensions of $\overline{[Q]}$. Sometimes it is more convenient to work in the toric transitive closure rather than the toric Hasse diagram and vice versa. For instance, part (iii) of Theorem 5.3 is phrased in terms of the toric transitive closure.

Lemma 3.11. *Let $[Q]$ be a toric poset. Then,* $\mathcal{L}_{\text{tor}}([Q]_{\text{Hasse}}) = \mathcal{L}_{\text{tor}}([Q]) = \mathcal{L}_{\text{tor}}(\overline{[Q]})$.

Let $[Q]_v$ denote the set of quivers in $[Q]$ where v is a source. We prove the following proposition.

Proposition 3.12. For a toric poset $[Q]$, the set of toric total extensions can be written in terms of ordinary linear extensions in the following ways:

i. $\mathcal{L}_{\text{tor}}([Q]) = \{[w] : w \in \mathcal{L}(Q') \text{ for some } Q' \in [Q]\}$

ii. $\mathcal{L}_{\text{tor}}([Q]) = \bigsqcup_{Q' \in [Q]_v} \left\{[v\hat{w}] : \hat{w} \in \mathcal{L}(Q' - \{v\})\right\}$

where $\bigsqcup$ denotes disjoint union.

A *bounded* poset P is one that has a unique minimum $\hat{0}$ and a unique maximum $\hat{1}$. We prove the following proposition, which is used in the proof of Theorem 3.14.

Proposition 3.13. *Let P be a bounded poset, and let Q be the quiver resulting from adding the directed edge $\hat{0} \to \hat{1}$ to the Hasse diagram $H(P)$. Then one has a bijection*

$$\begin{aligned} \theta : \mathcal{L}(P) &\longrightarrow \mathcal{L}_{\text{tor}}([Q]) \\ (\hat{0}, w_2, \dots, w_{n-1}, \hat{1}) &\longmapsto [(\hat{0}, w_2, \dots, w_{n-1}, \hat{1})]. \end{aligned}$$

In 1991, Brightwell and Winkler showed that counting the number of linear extensions of an ordinary poset is a $\#P$-complete problem. We prove an analogous result for toric posets in Theorem 3.14. Further discussion regarding $\#P$-completeness can be found in [5, 12].

Theorem 3.14. *Counting the toric total extensions for a toric poset $[Q]$ is $\#P$-complete.*

Although part (ii) of Proposition 3.12 provides a more efficient process for finding the set of toric total extensions relative to Proposition 3.12 part (i), we look for more efficient ways to compute this set. In Section 5, we provide a recursive algorithm to more efficiently compute $\mathcal{L}_{\text{tor}}([Q])$.

4 Properties of $\Psi_{\text{tor}}^{[Q]}(\mathbf{x})$

As mentioned in Remark 1.3, the denominator of $\Psi_{\text{tor}}^{[w]}(\mathbf{x})$ appears as Parke–Taylor factors in scattering amplitude computations. We now show how to recover the *Kleiss–Kuijf relations*, an identity that appears in scattering amplitudes [15], by showing that these relations are, surprisingly, a specific instance of Greene's theorem for strongly planar posets (recall Equation (1.1)).

Proposition 4.1. *Suppose P is a bounded poset with minimum element $\hat{0}$ and maximum element $\hat{1}$. Let Q be the quiver resulting from adding the directed edge $\hat{0} \to \hat{1}$ to the Hasse diagram $H(P)$. Then, for the toric poset $[Q]$, we have*

$$\Psi_{\text{tor}}^{[Q]}(\mathbf{x}) = \frac{1}{x_{\hat{1}} - x_{\hat{0}}} \Psi^P(\mathbf{x}).$$

Corollary 4.2. *Let P be a bounded, strongly planar poset with minimum element $\hat{0}$ and maximum element $\hat{1}$. Let Δ be the set of bounded regions of P, and let Q be the quiver resulting from adding the directed edge $\hat{0} \to \hat{1}$ to $H(P)$. Then, by Proposition 4.1 and Equation (1.1), we have*

$$\Psi_{\text{tor}}^{[Q]}(\mathbf{x}) = \frac{1}{x_{\hat{1}} - x_{\hat{0}}} \frac{\prod_{\delta \in \Delta}(x_{\min(\delta)} - x_{\max(\delta)})}{\prod_{i \lessdot_P j}(x_i - x_j)}.$$

The following corollary is a special case of Corollary 4.2 applied to the poset P on the right of Figure 1. Let $\mathbf{b} = (b_1, b_2, \dots, b_k)$, $\mathbf{c} = (c_1, c_2, \dots, c_j)$, and let $\mathrm{rev}(\mathbf{b}) = (b_k, \dots, b_2, b_1)$. As convention, let $b_{k+1} = c_{j+1} = \hat{1}$ and $b_0 = c_0 = \hat{0}$.

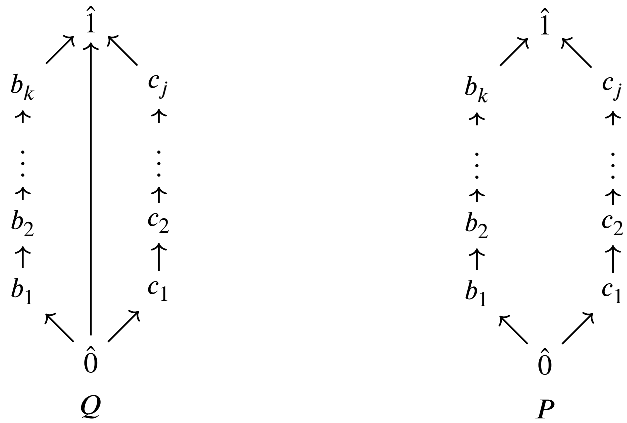


Figure 1

Corollary 4.3. *(Kleiss–Kuijf Shuffle Relations) For $\Psi^{[Q]}_{\mathrm{tor}}(\mathbf{x})$ where $[Q]$ is the toric poset represented by quiver Q shown on the left of Figure 1, we have*

$$\Psi^{[Q]}_{\mathrm{tor}}(\mathbf{x}) = \frac{(-1)^k}{\prod_{r=0}^{k}(x_{b_{r+1}} - x_{b_r}) \cdot \prod_{s=0}^{j}(x_{c_s} - x_{c_{s+1}})}, \tag{4.1}$$

or equivalently,

$$\sum_{\mathbf{a} \in \mathbf{b} \sqcup\!\sqcup \mathbf{c}} \Psi^{[(\hat{1},\hat{0},\mathbf{a})]}_{\mathrm{tor}}(\mathbf{x}) = (-1)^k \Psi^{[(\hat{1},\mathrm{rev}(\mathbf{b}),\hat{0},\mathbf{c})]}_{\mathrm{tor}}(\mathbf{x}). \tag{4.2}$$

Properties of $\Psi^P(\mathbf{x})$ shown by Boussicault, Féray, Lascoux, and Reiner in [4] as well as properties by Greene in [13] serve as motivation for the next few analogous properties of $\Psi^{[Q]}_{\mathrm{tor}}(\mathbf{x})$. Recall that in [4], the authors show that a poset P is disconnected if and only if $\Psi^P(\mathbf{x}) = 0$. We present a sufficient condition for when $\Psi^{[Q]}_{\mathrm{tor}}(\mathbf{x}) = 0$, but first present a computational lemma that will help in the proof of this result. It also appeared recently as [19, Proposition 7.17] with a different proof.

Lemma 4.4. *Let $\mathbf{a} = (a_1, a_2 \dots, a_m)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$. Then, $\sum_{\mathbf{c} \in \mathbf{a} \sqcup\!\sqcup \mathbf{b}} \Psi^{[(1,\mathbf{c})]}_{\mathrm{tor}}(\mathbf{x}) = 0$.*

We now present our next main result. Recall that a *cut vertex* is a vertex such that if it is removed, the number of connected components of the graph increases.

Theorem 4.5. *Let $[Q]$ be a toric poset, and let G be the underlying undirected graph of $[Q]$. If G is either disconnected with at least three vertices or has a cut vertex, then $\Psi^{[Q]}_{\mathrm{tor}}(\mathbf{x}) = 0$.*

In Theorem 4.5, we need to assume the toric poset $[Q]$ has at least three vertices since if $[Q]$ has exactly two vertices $1, 2$ and no arcs, then $\Psi_{\mathrm{tor}}^{[Q]}(\mathbf{x}) = \frac{1}{(x_1-x_2)(x_2-x_1)} = \frac{-1}{(x_1-x_2)^2} \neq 0$.

Remark 4.6. Theorem 4.5 gives a sufficient condition for the vanishing of $\Psi_{\mathrm{tor}}^{[Q]}(\mathbf{x})$. We depict a quiver Q whose toric poset $[Q]$ has $\Psi_{\mathrm{tor}}^{[Q]}(\mathbf{x}) = 0$, but the vanishing is not implied by Theorem 4.5.

$$\begin{array}{ccccc} & & 5 & & \\ & \nearrow & \uparrow & \nwarrow & \\ 2 & & 3 & & 4 \\ & \nwarrow & \uparrow & \nearrow & \\ & & 1 & & \end{array}$$

In [19, Theorem 7.11], the authors show that a certain sum over *cyclic extensions* of *partial cyclic orders* vanishes. Since cyclic extensions can be seen to be the same as toric total extensions, it is natural to wonder how their sum relates to our Theorem 4.5. In [6], we show that neither theorem implies the other, but acknowledge some overlap.

For ordinary posets, Boussicault, Féray, Lascoux, and Reiner show that linear terms in the denominator of $\Psi^P(\mathbf{x})$ correspond to cover relations of P.

Theorem 4.7 ([4, Corollary 5.2])**.** *For a connected poset P, the minimal denominator of $\Psi^P(\mathbf{x})$ is $\prod_{i \lessdot_P j}(x_i - x_j)$.*

For toric posets, we prove the following result.

Theorem 4.8. *For $[Q]$ a toric poset, $\Psi_{\mathrm{tor}}^{[Q]}(\mathbf{x})$ can always be expressed over the denominator of $\prod_{\{i,j\}\in[Q]_{\mathrm{Hasse}}}(x_i - x_j)$ where we take the product over all edges $\{i, j\}$ in $[Q]_{\mathrm{Hasse}}$.*

Remark 4.9. For $[Q]$ in Figure 1, we emphasize that the minimal denominator of $\Psi_{\mathrm{tor}}^{[Q]}(\mathbf{x})$ does not contain the linear factor $(x_{\hat{0}} - x_{\hat{1}})$, even though the edge $\{\hat{0}, \hat{1}\}$ does appear in $[Q]_{\mathrm{Hasse}}$.

5 An Algorithm for Finding Toric Total Extensions

Computing the set of toric total extensions $\mathcal{L}_{\mathrm{tor}}([Q])$ via Proposition 3.12 (ii) requires enumerating all quivers $Q' \in [Q]$ with a vertex v as a source; there is currently no good algorithm for this. Therefore, we are motivated to find methods that are more computationally efficient to compute $\mathcal{L}_{\mathrm{tor}}([Q])$. We provide a recurrence for finding $\mathcal{L}_{\mathrm{tor}}([Q])$ (see Theorem 5.3) that is similar to the following recurrence for finding the set of ordinary linear extensions of posets.

Lemma 5.1. *Let P be a poset, and let a, b be two incomparable elements of P. Then,*

$$\mathcal{L}(P) = \mathcal{L}(P_{a\to b}) \sqcup \mathcal{L}(P_{b\to a})$$

where $P_{a\to b}$ is obtained from P by adding the relation $a < b$ and $P_{b\to a}$ is defined similarly.

Theorem 5.2 is a key component in the proof of Theorem 5.3 and may be of independent interest. One can view Theorem 5.2 simply as a statement regarding sink-source mutation of acyclic quivers.

Theorem 5.2. *Let v be any vertex in an acyclic quiver Q, and let Q_1, Q_2 be any two acyclic quivers in the subset $[Q]_v$ of the source-sink flip-equivalence class $[Q]$, so v is a source in both Q_1 and Q_2. Then there exists a source-sink flip sequence from Q_1 to Q_2 such that every intermediate quiver in the sequence also has v as a source.*

The proof of Theorem 5.2 requires some work, and can be found in our preprint [6, Theorem 1.9]. Moreover, we note that Theorem 5.2 is not true if we drop the acyclic condition.

Let Q be a quiver with vertices a and b such that there is no edge between a and b. We define $Q_{a\to b}$ to be the quiver Q with an added directed edge $a \to b$, and $Q_{b\to a}$ is defined similarly.

Theorem 5.3. *Let a, b be two torically incomparable elements in the toric poset $[Q]$.*

(i) If a, b are in different connected components of the graph of $[Q]$, then $[Q_{a\to b}] = [Q_{b\to a}]$ and

$$\mathcal{L}_{\mathrm{tor}}([Q]) = \mathcal{L}_{\mathrm{tor}}([Q_{a\to b}]) = \mathcal{L}_{\mathrm{tor}}([Q_{b\to a}]).$$

(ii) If a, b are in the same connected component and $Q' \in [Q]$ is a representative where a, b are ordinary incomparable, then the sets $\mathcal{L}_{\mathrm{tor}}([Q'_{a\to b}])$ and $\mathcal{L}_{\mathrm{tor}}([Q'_{b\to a}])$ are disjoint subsets of $\mathcal{L}_{\mathrm{tor}}([Q])$, but the inclusion of the disjoint union

$$\mathcal{L}_{\mathrm{tor}}([Q'_{a\to b}]) \sqcup \mathcal{L}_{\mathrm{tor}}([Q'_{b\to a}]) \subseteq \mathcal{L}_{\mathrm{tor}}([Q])$$

may be proper.

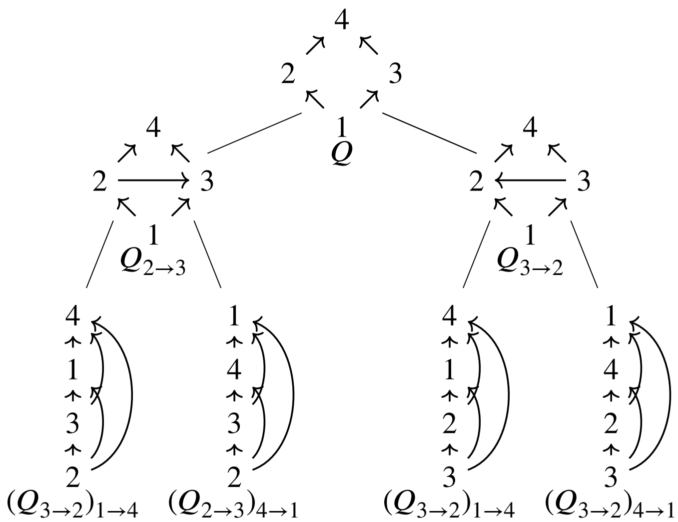
(iii) On the other hand, assume that a, b are distance two in the graph of the toric transitive closure $\overline{[Q]}$, say both adjacent to the vertex v. Then if one chooses $Q' \in \overline{[Q]}_v$, that is, Q' is a representative of $\overline{[Q]}$ with v a source, the inclusion in (ii) becomes an equality:

$$\mathcal{L}_{\mathrm{tor}}([Q]) = \mathcal{L}_{\mathrm{tor}}([Q'_{a\to b}]) \sqcup \mathcal{L}_{\mathrm{tor}}([Q'_{b\to a}]).$$

This result can be used to recursively compute $\mathcal{L}_{\mathrm{tor}}(\overline{[Q]})$ in terms of $\mathcal{L}_{\mathrm{tor}}(\overline{[Q_i]})$ for toric posets $[Q_i]$, each having more edges in their toric Hasse diagram than that of $[Q]$. Each of the latter toric posets has fewer toric total extensions, so they are easier to understand. We show that when this iterative process ends, our resulting toric posets are exactly the toric total orders $[w]$; that is, $\mathcal{L}_{\mathrm{tor}}([w]) = \{[w]\}$. Note that the transitive closure of a toric chain is a complete graph.

Corollary 5.4. *For a toric poset $[Q]$, iterative application of Theorem 5.3 gives a finite algorithm for finding $\mathcal{L}_{\mathrm{tor}}([Q])$, where the resulting toric posets correspond to toric total orders $[w]$. In other words, for every toric poset that is not a toric total order, either Theorem 5.3 part (i) or (iii) applies.*

Example 5.5. We calculate $\mathcal{L}_{\text{tor}}([Q])$ for the toric poset from Example 3.5, and illustrate Theorem 5.3 with the following tree. If node $\overline{[Q]}$ has children $\overline{[Q_i]}$, then $\mathcal{L}_{\text{tor}}(\overline{[Q]}) = \bigsqcup_{\overline{[Q_i]}} \mathcal{L}_{\text{tor}}(\overline{[Q_i]})$. For each of our toric posets, we draw one representative. Note that in this example $\overline{[Q]} = [Q]_{\text{Hasse}}$.



Reading the leaves left-to-right, $\mathcal{L}_{\text{tor}}([Q]) = \{[(1,4,2,3)], [(1,2,3,4)], [(1,4,3,2)], [(1,3,2,4)]\}$.

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