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Turbulence Polyhedra

Jonah Berggren*[1]

[1]*Department of Mathematics, University of Kentucky, Lexington, Kentucky, USA*

Abstract. A framing on a DAG (directed acyclic graph) gives a notion of compatibility on its routes which induces a regular unimodular triangulation on the underlying flow polytope. We define framed turbulence charts and their turbulence polyhedra, generalizing DAGs by removing the assumptions of directedness and acyclicity. We obtain presentations for turbulence polyhedra using special walks called "trails" on the underlying turbulence chart and we obtain subdivisions (and in many cases triangulations) of the turbulence polyhedra induced by a notion of compatibility on its trails. As a special case, we obtain presentations and subdivisions for flow polyhedra of framed directed graphs (without assuming acyclicity). As a motivating special case, we define and study the turbulence polyhedron of the fringed quiver of a gentle algebra, providing new insights into g-vector fans of gentle algebras.

Keywords: flow polytopes, triangulation, gentle algebras

1 Introduction

Flow polytopes, which model the space of unit *flows* on a directed acyclic graph (DAG), are a fundamental object of combinatorial optimization and have relations to many fields such as representation theory [2], Grothendieck polynomials [8], and algebraic geometry [7]. Danilov, Karzanov, and Koshevoy [5] introduced *framings* on DAGs and defined a notion of pairwise compatibility on routes. The complex of *cliques*, or sets of pairwise compatible routes, of a framed DAG serves as a combinatorial model for a (regular unimodular) *DKK triangulation* of the associated flow polytope. Many important classes of polytopes and their canonical triangulations appear in this way, such as associahedra, generalized permutahedra [9], s-permutahedra [6], and many order polytopes [8].

Inspired by a recent connection between flow polytopes and the representation theory of gentle algebras [3], we will generalize the definition of a (resp. framed) DAG by doing away with the assumptions of (D)irectedness and (A)cyclicity. This results in a combinatorial object called a *(resp. framed) turbulence chart* $(G, \sim, R)$ giving rise to a *turbulence polyhedron* $\mathcal{F}_1(G, \sim)$. Generalizing routes of a DAG are *trails* of a turbulence chart, each of which is either a *route* (maximal walk) or a *band* (infinitely repeating cycle).

Just as we may obtain presentations and triangulations of flow polytopes in terms of routes on the underlying (framed) DAG, we may obtain presentations and subdivisions

*jrberggren@uky.edu. The author was partially supported by NSF grant DMS-2054255.

of turbulence polyhedra in terms of trails on the underlying (framed) turbulence chart. We first obtain a presentation of the turbulence polyhedron of a framed turbulence chart $(G, \sim, R)$ by characterizing its vertices as indicator vectors of *elementary routes* and showing that the indicator vectors of *elementary bands* form a minimal generating set for the recession cone of $\mathcal{F}_1(G, \sim, R)$. We then use the framing R to obtain a notion of pairwise compatibility on trails, giving rise to a *bundle complex* of pairwise compatible trails. We show that the bundle complex of $(G, \sim, R)$ serves as a combinatorial model for a subdivision of the turbulence polyhedron $\mathcal{F}_1(G, \sim, R)$ which we call the *bundle subdivision*. When $(G, \sim, R)$ is acyclic or gentle (see below), we obtain a unimodular triangulation called the *clique triangulation*. See Figure 1.

We isolate three independent properties which a turbulence chart may have: directability, acyclicity, and gentleness. A chart is acyclic if it has no bands, which is equivalent to boundedness of the turbulence polyhedron. Directable framed turbulence charts are those whose "flow" may be given a consistent direction. These are modelled by a new definition of "framed directed graphs," i.e., framed DAGs without the assumption of acyclicity. It is then a consequence of our presentation and subdivision results on turbulence polyhedra that the analogous presentation and subdivision results hold for framed directed graphs and their flow polyhedra.

Just as directable framed turbulence charts may be modelled by framed directed graphs, gentle framed turbulence charts are modelled by the fringed quiver of an associated gentle algebra. Fringed quivers were introduced [4, 10] and are equipped with a simplicial complex of pairwise compatible routes which describes the g-vector fan of the underlying gentle algebra. In fact, our bundle complex of gentle turbulence charts agrees with this story, and the addition of bands to the complex adds to the theory by giving some interpretation for the complement of the g-vector fan. Moreover, applying a quotient map of the turbulence polyhedron into the ambient space of the g-vector fan allows us to prove g-convexity of gentle algebras.

2 Background on Framed DAGs

We start by recalling some background on flow polytopes and framed DAGs. Let $G = (V, E)$ be a finite directed acyclic graph (DAG) with vertex set V and edge set E. For each $v \in V$, let $\mathrm{in}(v)$ and $\mathrm{out}(v)$ denote the set of incoming and outgoing edges of v, respectively. A vertex v is called a *source* if $\mathrm{in}(v) = \emptyset$, a *sink* if $\mathrm{out}(v) = \emptyset$, and *internal* otherwise. An edge $\alpha \in E$ is directed from its *tail* $t(\alpha)$ to its *head* $h(\alpha)$. The edge α is *internal* if it is between two internal vertices, and otherwise it is a *source edge* and/or a *sink edge*. A *route* of G is a maximal (directed) path in G.

Definition 2.1. A *flow* f on a DAG G is a function $f : E \to \mathbb{R}$ which preserves flow at each internal vertex, i.e., for every internal vertex v we have $\sum_{e\in \mathrm{in}(v)} f(e) = \sum_{e\in \mathrm{out}(v)} f(e)$.

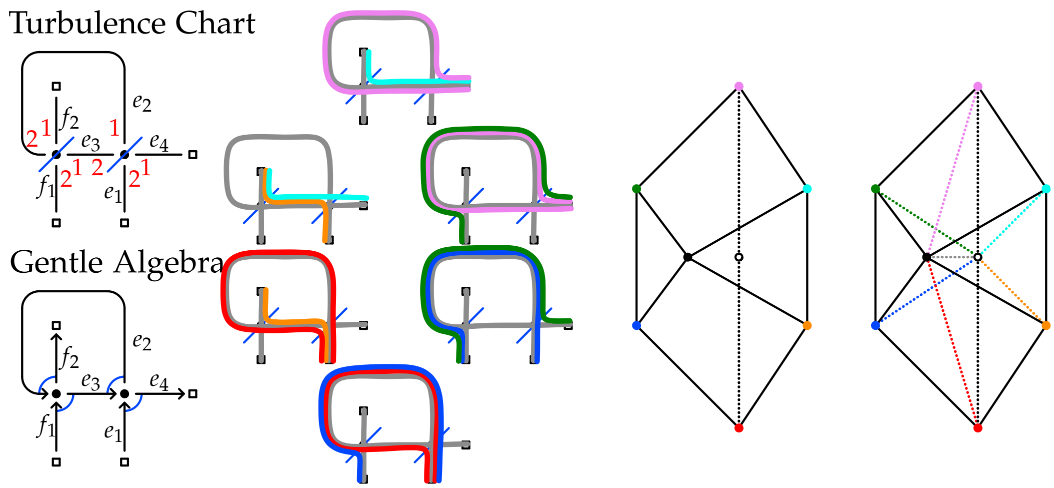

Figure 1: A turbulence chart with its fringed quiver and three-dimensional turbulence polyhedron. In the middle are its six maximal cliques, which give rise to six simplices making up the unimodular triangulation on the right.

The *(unit) flow polytope* $\mathcal{F}_1(G)$ is the space of *unit flows* on G; i.e., flows satisfying $x_e \geq 0$ for all edges $e \in E$ and $\sum_{\substack{v \text{ is a source} \\ e \in \text{out}(v)}} f(e) = 1$. The vertices of $\mathcal{F}_1(G)$ are precisely the indicator vectors of routes of G.

Definition 2.2. Let $G = (V, E)$ be a DAG. For each internal vertex v of G, assign a linear order to the edges in $\text{in}(v)$ and assign a linear order to the edges in $\text{out}(v)$. This assignment is called a *framing* of G, which we denote by R. If e is less than f in the linear order for R on $\text{in}(v)$, we write $e <_{R,\text{in}(v)} f$ (and similarly for $\text{out}(v)$).

To denote a framing, we label the internal half-edges of a DAG with integers. See Figure 2 for an example.

Definition 2.3. Two paths p and q of (G, R) are *incompatible* if without loss of generality p contains $\alpha_1 S \alpha_2$ and q contains $\beta_1 S \beta_2$, for some path S and some edges α_i, β_i with $\alpha_1 >_{R,\text{in}(v)} \beta_1$ and $\alpha_2 <_{R,\text{out}(w)} \beta_2$. Otherwise, they are *compatible*. A *clique* is a set of pairwise-compatible routes of (G, R). Cliques form the simplicial *clique complex*.

For example, in Figure 2, the route $\alpha_1\beta_2$ and the route $\alpha_2\beta_1$ are incompatible, as they share the unique internal vertex but $\alpha_2\beta_1$ enters this vertex with a higher edge and leaves with a lower edge compared to $\alpha_1\beta_2$.

Definition 2.4. Let P be a lattice polyhedron. A *subdivision* of P is a set $\mathcal{S}$ of lattice polyhedra such that $\cup_{Q \in \mathcal{S}} Q$ is a dense subset of P and for every choice of distinct $Q_1, Q_2 \in \mathcal{S}$, the set $Q_1 \cap Q_2$ is a common (possibly empty) proper face of Q_1 and Q_2. A *triangulation*

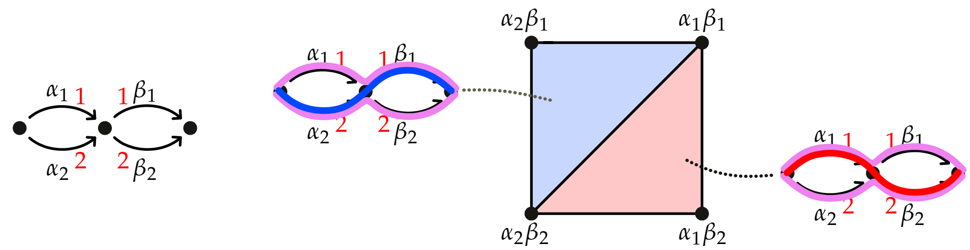

Figure 2: Shown is a framed DAG (framing in red), its two maximal cliques, and the corresponding triangulation of its flow polytope.

is a subdivision in which each cell is a simplex of dimension $\dim(P)$. A triangulation is *unimodular* if each cell has normalized volume 1.

Through the correspondence between routes of G and vertices of $\mathcal{F}_1(G)$, we may view maximal cliques of (G, R) as collections of vertices of $\mathcal{F}_1(G)$ which form a simplex of a regular unimodular triangulation:

Theorem 2.5 ([5]). *The set of maximal cliques of a framed DAG (G, R) forms a regular unimodular triangulation of the flow polytope $\mathcal{F}_1(G)$.*

The triangulation from Theorem 2.5 is called the *DKK triangulation* of (G, R). Figure 2 shows a framed DAG and its two maximal cliques, each of which is connected with the corresponding simplex of its DKK triangulation.

3 Turbulence Charts and Turbulence Polyhedra

We now give the main definitions and results of this abstract.

We call a vertex of an undirected graph G *internal* if it is incident to two or more edges (counting multiplicity), and *fringe* otherwise. An edge is *fringe* if it is incident to a fringe vertex, and otherwise is *internal*. A *half-edge* of G at a vertex v is a tuple (e, v), where e is an edge of G incident to a vertex v. We write that v is the vertex of (e, v). If e starts and ends at the same vertex, then we still consider e to be a part of two distinct half-edges so that every edge of G is a part of exactly two half-edges. We consider any undirected graph to have vertex set V, internal vertex set V_{int}, and edge set E.

Definition 3.1. A *turbulence chart* is a tuple $(G, \sim)$ where G is an undirected graph and $\sim := \{\sim_v : v \in V_{int}\}$ is the data of an equivalence relation $\sim_v$ splitting the half-edges of G at v into exactly two nonempty equivalence classes for every internal vertex $v \in V_{int}$.

We will see that we may obtain a turbulence chart from a directed graph by letting $\sim_v$ at each internal vertex separate the incoming half-edges from the outgoing half-edges, and then forgetting the orientations of edges (see the left of Figure 5). This operation respects the following definitions of the turbulence polyhedron and bundle complex.

Definition 3.2. A function $F : E \to \mathbb{R}_{\geq 0}$ is a *(nonnegative) flow* on $(G, \sim)$ if it satisfies **conservation of flow:** for any internal vertex v of $(G, \sim)$ with equivalence classes S_1 and S_2 of $\sim_v$, we have $\sum_{(e,v)\in S_1} F(e) = \sum_{(e,v)\in S_2} F(e)$. The flow F is **unit** if $\sum_{e \in E \text{ fringe}} F(e) = 2$. The *turbulence polyhedron* $\mathcal{F}_1(G, \sim)$ is the space of unit flows on $(G, \sim)$.

We are able to understand flow polytopes in terms of paths and routes on the underlying DAG; we now define the analogs on turbulence charts. It will be necessary to work with "oriented walks" along an unoriented graph – to do this, we need the notation of oriented edges.

Definition 3.3. Define an *oriented edge* of an undirected graph G as a tuple of the form $\tilde{e} := e^t e^h$, where e^t and e^h are the two (distinct) half-edges of one (unoriented) edge e. We consider $\tilde{e}$ to *start* at the vertex $t(\tilde{e})$ of e^t and *end* at the vertex $h(\tilde{e})$ of e^h. The *inverse edge* $\tilde{e}^{-1}$ is the oriented edge $e^h e^t$. A *string* on G of length $m \geq 0$ is a sequence $s = \tilde{e}_1 \tilde{e}_2 \dots \tilde{e}_m$, where each $\tilde{e}_j$ is an oriented edge of G and for any $j \in [m-1]$, we have $h(\tilde{e}_j) = t(\tilde{e}_{j+1})$ and the half-edges e_j^h and e_{j+1}^t are in different equivalence classes of $\sim_{h(\tilde{e}_j)}$. We say that s *starts* at $t(s) := t(\tilde{e}_1)$ and *ends* at $h(s) := h(\tilde{e}_m)$. The *inverse string* s^{-1} is the sequence $\tilde{e}_m^{-1} \tilde{e}_{m-1}^{-1} \dots \tilde{e}_1^{-1}$. For any $1 \leq a \leq b \leq m$, the string $\tilde{e}_a \tilde{e}_{a+1} \dots \tilde{e}_b$ is a *substring* of s.

See Figure 3 for an example of a turbulence chart (ignore the numbers for now). We draw turbulence charts so that internal vertices are dots, fringe vertices are boxes, and at each internal vertex v a blue line separates the two equivalence classes of $\sim_v$. Under these conventions, a string is a walk on the graph subject to the rule that each time one reaches an internal vertex, one may only continue by crossing the blue line at that vertex.

Definition 3.4. A *route* on $(G, \sim)$ is a string which starts and ends at fringe vertices. We consider a route p to be *equivalent* to p^{-1}. A *band* on $(G, \sim)$ is a string B such that $B^2 := B \circ B$ is a string and that B is not a power of any strictly smaller string B'. A *substring* of the band B is a substring of any power of its underlying string. Two bands B and B' are *equivalent* if (the underlying string of) B' is a substring of B^2 or $(B^{-1})^2$. A *trail* of $(G, \sim)$ is a route or band of $(G, \sim)$. The turbulence chart $(G, \sim)$ is *acyclic* if there are no bands of $(G, \sim)$.

On the middle and right of Figure 3, some routes and a band are depicted. Our depictions of trails do not include a direction and our depictions of bands do not mark a start point, as we only care about trails up to equivalence.

Recall that vertices of a flow polytope are precisely the indicator vectors of routes; we wish to give the analogous result for turbulence polyhedra. In order to do this, we first give the classes of bands and routes which we will use in our presentation.

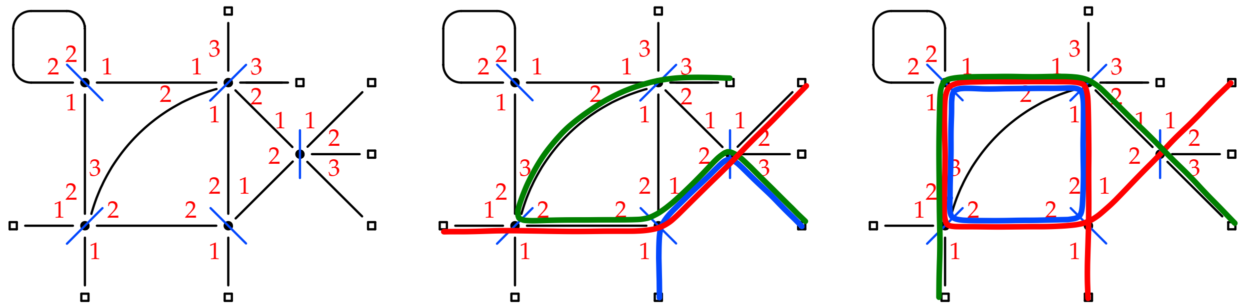

Figure 3: On the left is a framed turbulence chart. The middle is not a bundle because the red and green are incompatible, and the right shows a bundle.

Figure 4: Examples of elementary (blue) and nonelementary (red) routes and bands.

Definition 3.5. 1. A *tiara* is a string σ which does not use the same vertex twice except for $h(\sigma) = t(\sigma)$ such that $\sigma \circ \sigma$ is not a string (i.e., the first and final half-edges of σ are in the same equivalence class of $\sim_{h(\sigma)}$).

2. A string or route is *simple* if it does not use the same vertex twice.

3. A route is a *lollipop* if it is of the form $s\sigma s^{-1}$, s is a simple string and σ is a tiara and that s and σ intersect only at $h(s) = t(\sigma) = h(\sigma)$.

4. A band is *simple* if it does not use the same vertex twice, except that its head is the same as its tail.

5. A band is a *barbell* if up to cyclic equivalence it is of the form $s\sigma_1 s^{-1}\sigma_2$, where s is a simple string and σ_1 and σ_2 are tiaras such that s, σ_1, and σ_2 intersect only at the necessary vertices $h(s) = t(\sigma_2) = h(\sigma_2)$ and $t(s) = t(\sigma_1) = h(\sigma_1)$.

A route is *elementary* if it is simple or it is a lollipop. A band is *elementary* if it is simple or it is a barbell. See Figure 4 for a visual summary of these definitions.

We may use trails of $(G, \sim)$ to get at points of the turbulence polyhedron $\mathcal{F}_1(G, \sim)$:

Definition 3.6. Let $p = \tilde{e}_1\tilde{e}_2 \dots \tilde{e}_m$ be a route or band of $(G, \sim)$. The *indicator vector* $\mathcal{I}(p)$ is the vector in $\mathbb{R}^E$ such that the coordinate of an edge e is the number of indices $j \in [m]$ such that the underlying edge of $\tilde{e}_j$ is e. It is immediate that if p is a trail of $(G, \sim)$, then $\mathcal{I}(p) \in \mathcal{F}_1(G, \sim)$.

Theorem 3.7. *The map $p \mapsto \mathcal{I}(p)$ gives a bijection from elementary routes of $(G, \sim, R)$ to vertices of $\mathcal{F}_1(G, \sim)$. The map $B \mapsto \mathcal{I}(B)$ maps the elementary bands to a minimal generating set for the recession cone of $\mathcal{F}_1(G, \sim)$.*

Recall that the recession cone of a polyhedron $P \subseteq \mathbb{R}^n$ is the set $\mathrm{Rec}(P) = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{x} + \lambda\mathbf{y} \in P \text{ for all } \mathbf{x} \in P,\ \lambda \in \mathbb{R}_{\geq 0}\}$. Proving Theorem 3.7 amounts to showing that indicator vectors of nonelementary vertices may be obtained as convex combinations of indicator vectors of elementary vertices, and that indicator vectors of nonelementary bands may be obtained as nonnegative combinations of indicator vectors of elementary bands. See Example 3.13 and Example 3.14 for examples.

We now define framings on turbulence charts, generalizing framings on DAGs.

Definition 3.8. Let $(G, \sim)$ be a turbulence chart. Let v be an internal vertex of G and let S_1 and S_2 be the equivalence classes of half-edges at v given by $\sim_v$. Assign separate linear orders to S_1 and S_2. This data, ranging over all internal vertices of G, is called a *framing* R of $(G, \sim)$. We write $(G, \sim, R)$ to denote a *framed turbulence chart*. If h_1 and h_2 are half-edges in the same equivalence class of $\sim_v$ for some internal vertex v, we write $h_1 <_{R, \sim_v} h_2$ to represent that h_1 is lesser in the relevant order of R.

To denote a framing, we label the half-edges of a turbulence chart at internal vertices with red integers. We now use framings to define a notion of compatibility on trails.

Definition 3.9. Let v and w be internal vertices of G. Let $\tilde{e}_0$ and $\tilde{f}_0$ be oriented edges ending at v such that e_0^h and f_0^h are equivalent in $\sim_v$ and such that $e_0^h <_{R, \sim_v} f_0^h$. Let $\tilde{e}_{m+1}$ and $\tilde{f}_{m+1}$ be oriented edges starting at w such that e_{m+1}^t and f_{m+1}^t are equivalent in $\sim_w$ and such that $e_{m+1}^t <_{R, \sim_w} f_{m+1}^t$. Choose any s (which is either empty or a string) so that $\tilde{e}_0 s \tilde{e}_{m+1}$ is a string; then $\tilde{f}_0 s \tilde{f}_{m+1}$ is also a string. We say that $(\tilde{e}_0 s \tilde{e}_{m+1}, \tilde{f}_0 s \tilde{f}_{m+1})$ is an *incompatibility*. Two trails p and q are *incompatible* if without loss of generality there is a substring p' of p and q' of q such that (p', q') is an incompatibility. Otherwise, they are *compatible*. A trail is *rigid* if it is compatible with itself. A *clique* of $(G, \sim, R)$ is a set of pairwise compatible rigid routes (up to equivalence of routes). A *bundle* of $(G, \sim, R)$ is a set of pairwise compatible rigid trails (up to equivalence of trails). The *bundle complex* of $(G, \sim, R)$ is the simplicial complex of bundles of $(G, \sim, R)$.

In the middle of Figure 3, the red and blue routes are compatible, and the blue and green routes are compatible, but the red and green routes are incompatible because they share a segment which the green enters and leaves high relative to the red. On the right, all three trails are compatible and hence form a bundle. See the middle of Figure 1 for an example of the set of maximal cliques of a turbulence chart.

We now generalize the theory of DKK triangulations by using routes, bands, and indicator vectors to obtain subdivisions of turbulence polyhedra.

Definition 3.10. Let $\bar{\mathcal{K}}$ be a bundle of $(G, \sim, R)$ and let $\mathcal{K}$ be the clique consisting of all routes of $\bar{\mathcal{K}}$. A *(unit) $\bar{\mathcal{K}}$-bundle combination* is a linear combination $\sum_{p \in \bar{\mathcal{K}}} a_p \mathcal{I}(p)$ of indicator vectors of trails, such that each a_p is nonnegative and the sum of a_p over routes of $\bar{\mathcal{K}}$ is 1. If $\bar{\mathcal{K}} = \mathcal{K}$ is a clique, then it is a *clique combination*. The bundle combination is *positive* if a_p is nonzero for every $p \in \bar{\mathcal{K}}$. The *(unit) bundle space* $\Delta_1(\bar{\mathcal{K}})$ is the space of unit bundle combinations of $\bar{\mathcal{K}}$. If $\bar{\mathcal{K}} = \mathcal{K}$ is a band, then $\Delta_1(\mathcal{K})$ is a simplex and we call it the *clique simplex* of $\mathcal{K}$.

Any bundle space is a polyhedron, and is bounded if and only if it is a clique simplex.

Theorem 3.11. *Let $(G, \sim, R)$ be a turbulence chart. Any flow in $\mathcal{F}_1(G, \sim)$ has at most one description as a positive bundle combination, and all rational flows may be described as a bundle combination.*

Theorem 3.11 is proven by giving a constructive algorithm to decompose any rational flow as a bundle combination. We will see that when $(G, \sim, R)$ contains no bands, then every flow in $\mathcal{F}_1(G, \sim)$ may be described as a positive bundle combination, hence Theorem 3.11 generalizes the triangulation result Theorem 2.5 on flow polytopes. We now make explicit the subdivision given by Theorem 3.11. Define the *bundle subdivision* of $\mathcal{F}_1(G, \sim, R)$ as $\mathcal{S}_{bundle}(G, \sim, R) := \{\Delta_1(\bar{\mathcal{K}}) \ : \ \bar{\mathcal{K}} \text{ is a maximal bundle of } (G, \sim, R)\}$.

Corollary 3.12. *If $(G, \sim, R)$ is a turbulence chart, then the bundle subdivision is a subdivision of $\mathcal{F}_1(G, \sim, R)$ covering all rational points.*

Example 3.13. Consider the three-dimensional turbulence polytope of Figure 1. Routes of $(G, \sim, R)$ are determined by the sequence of edges used, so we refer to them as such without using half-edges. In the middle of the figure is shown the clique complex of $(G, \sim, R)$, and hence all rigid routes. The only non-elementary routes are the rigid route $e_1e_2e_3e_4$ and the non-rigid route $e_4e_2e_3e_1$ (whose indicator vector is the same as $\mathcal{I}(e_1e_2e_3e_4)$). Hence, by Theorem 3.7 taking indicator vectors of every *other* route gives all vertices of the turbulence polytope. The indicator vector $\mathcal{I}(e_1e_2e_3e_4)$ is the unique lattice point of $\mathcal{F}_1(G, \sim)$ which is not a vertex, depicted as an open vertex along the back edge of the turbulence polyhedron. The indicator vector of any colored route is drawn as a vertex of the same color, and the indicator vector of f_1f_2 is the black vertex. Shown also are the six maximal cliques, each of which gives a clique simplex of the turbulence polyhedron whose vertices are the indicator vectors of its routes (note that all simplices consist of the black vertex and open lattice point with two of the colored vertices).

Example 3.14. Consider the three-dimensional turbulence polyhedron of Figure 5.
Routes of $(G, \sim, R)$ are determined by the sequence of edges used, so we refer to them

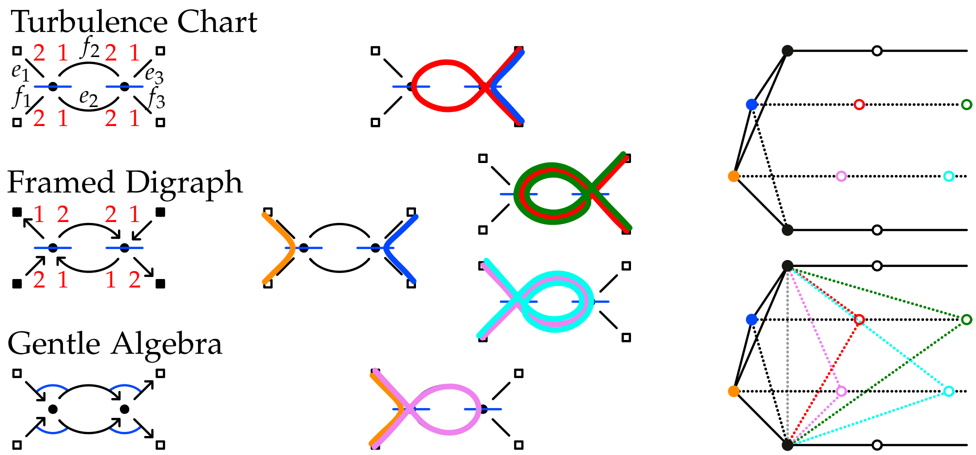

Figure 5: A turbulence chart with its fringed quiver, five maximal cliques, and (triangulated) turbulence polyhedron.

as such without using half-edges. The full list of maximal bundles of $(G, \sim, R)$ are

$$\{e_1e_2e_3,\ f_1f_2f_3,\ e_2f_2\},\quad \{e_1e_2e_3,\ f_1f_2f_3,\ e_1f_1,\ e_3f_3\},$$
$$\{e_1e_2e_3,\ f_1f_2f_3,\ e_1(e_2f_2)^m f_1,\ e_1(e_2f_2)^{m+1}f_1\} \text{ (for any } m \geq 0\text{), and}$$
$$\{e_1e_2e_3,\ f_1f_2f_3,\ e_3(e_2f_2)^m f_3,\ e_3(e_2f_2)^{m+1}f_3\} \text{ (for any } m \geq 0\text{).}$$

Note that all maximal bundles contain $\{e_1e_2e_3, f_1f_2f_3\}$ and that there is only one maximal bundle containing a band (namely, e_2f_2). In the middle of the figure are depicted five maximal cliques, with the caveat that $\{e_1e_2e_3, f_1f_2f_3\}$ are not drawn for readability. The four elementary routes of $(G, \sim, R)$ are $\{e_1f_1, e_2f_2, e_1e_2e_3, f_1f_2f_3\}$, which correspond through indicator vectors to the four vertices of $\mathcal{F}_1(G, \sim, R)$ (in particular, $\mathcal{I}(e_1e_2e_3)$ and $\mathcal{I}(f_1f_2f_3)$ are the top and bottom vertices of the polyhedron). The turbulence polyhedron goes on infinitely to the right, which is the direction given by the indicator vector of the unique (elementary) band $\mathcal{I}(e_2f_2)$. Hence, the presentation of $\mathcal{F}_1(G, \sim, R)$ given in Theorem 3.7 works in this case. Each maximal clique gives a full-dimensional clique simplex consisting of the two black vertices and two of the colored points. The maximal bundle $\{e_1e_2e_3, f_1f_2f_3, e_2f_2\}$ gives a bundle space whose vertices are at the black vertices and which continues infinitely to the right to make a two-dimensional strip. Note that every point of the turbulence polyhedron is contained in a maximal bundle space.

4 Special Cases: Acyclic, Directable, and Gentle

We now isolate three properties of turbulence charts which may be independently true or false: gentleness, directedness, and acyclicity. Directable turbulence charts may be modelled by framed directed graphs, and gentle turbulence charts may be modelled by gentle algebras. We remark that the class of gentle, acyclic, and directable turbulence charts are modelled by the amply framed DAGs of [3, 5]. Figure 1 is gentle, acyclic, but not directable. Figure 5 is gentle and directable, but not acyclic.

We first discuss acyclic turbulence charts. Recall that a turbulence chart is acyclic if it has no bands. It follows from Theorem 3.7 that a turbulence chart is acyclic if and only if its turbulence polyhedron is bounded. In this acyclic case, every bundle is actually a clique, and the bundle subdivision is a unimodular triangulation.

Proposition 4.1. *If $(G, \sim, R)$ is acyclic, then every point of $\mathcal{F}_1(G, \sim)$ is obtained uniquely as a clique combination and the bundle subdivision is a unimodular triangulation of $\mathcal{F}_1(G, \sim)$.*

4.1 Directable Turbulence Charts

We begin this subsection by discussing *framed directed graphs*. One may define a framing on a directed graph (which may not be acyclic) as in Definition 2.2. We say that a *band* of a directed graph is a directed cycle which is not a power of a strictly smaller directed cycle; we consider bands up to cyclic equivalence. A *trail* of a directed graph is a band or route. We may finally define compatibility of trails as in Definition 2.3, giving us the bundle complex of a framed directed graph. We define the *flow polyhedron* $\mathcal{F}_1(G, R)$ of a framed directed graph (G, R) as in Definition 2.1.

A turbulence chart $(G, \sim, R)$ is *directable* if the edges of G may be given orientations such that at each internal vertex v, the equivalence class $\sim_v$ separates the arrow heads from the arrow tails. The chart of Example 3.14 is directable, but that of Example 3.13 is not. If (G, R) is a framed directed graph, define a turbulence chart $(G', \sim', R')$ where G' is the underlying undirected graph of G, and at each internal vertex v the equivalence class $\sim'_v$ separates the half-edges into those coming from arrow heads (ordered by R' the same as R) and those coming from arrow tails (ordered by R' opposite to R).

We remark that the reversal of the framing R at arrow tails explains why routes of a framed DAG are incompatible when one enters low and leaves high relative to another, while trails of a framed turbulence chart are incompatible when one enters low and leaves low relative to another.

Proposition 4.2. *Notating as above, we have $\mathcal{F}_1(G, R) = \mathcal{F}_1(G', \sim')$, and there is a natural correspondence from trails of (G, R) to trails of $(G', \sim', R')$ which preserves compatibility. Any directable framed turbulence chart may be obtained from a framed directed graph in this way.*

Theorem 3.7, Theorem 3.11, and Corollary 3.12 give results about presentations and subdivisions of turbulence polyhedra in terms of the bundle complex of the turbulence chart. Proposition 4.2 allows us to translate these results verbatim into framed directed graph language, obtaining the vertices and recession cone of a flow polyhedron in terms of elementary routes and bands and obtaining a subdivision of a flow polyhedron induced by the bundle complex.

See Example 3.14; below the turbulence chart $(G, \sim, R)$ is pictured a framed DAG (G', R') which maps to it as in Proposition 4.2 (we could have alternatively chosen the framed DAG given by switching all 1's and 2's and reversing all orientations). The turbulence polyhedron $\mathcal{F}_1(G, \sim, R)$ is equal to the flow polyhedron $\mathcal{F}_1(G', R')$. The bundle complex of $(G, \sim, R)$ may also then be viewed as the bundle complex of the framed DAG (G', R'), hence the associated presentations and bundle subdivisions can be seen to come from the bundle complex of (G', R').

4.2 Gentle Turbulence Charts

Definition 4.3. A *fringed quiver* $\tilde{\Lambda} = (Q, I)$ is a directed graph Q such that every vertex of Q is either *internal*, in which case its degree is four and it is incident to exactly two incoming and two outgoing arrows, or *fringe*, in which case its degree is one. Moreover, Q is equipped with a set I of *relations* of Q of the form $\alpha\beta$, where $h(\alpha) = t(\beta)$. We require that if v is an internal vertex with incoming arrows $\{\alpha, \beta\}$ and outgoing arrows $\{\gamma, \delta\}$, then without loss of generality $\alpha\gamma$ and $\beta\delta$ are in I and $\{\alpha\delta, \beta\gamma\}$ are not.

See the bottom-left of Figures 1 and 5 for examples of fringed quivers. If $\alpha\beta \in I$, then we draw a blue arc between the end of α and the start of β to notate the relation. For example, in Figure 1 we have $I = \{e_1 f_2, f_1 e_2, f_2 e_3, e_2 f_3\}$.

Fringed quivers were defined in [4, 10] in order to study the τ-tilting theory of the underlying gentle algebra. They come equipped with routes and bands which are associated to certain modules over the underlying gentle algebra. Routes have a notion of pairwise compatibility wherein two routes p and q are incompatible if they share a common substring which p enters and leaves through arrow heads, and q enters and leaves through arrow tails. Compatibility of routes translates to τ-rigidity of modules.

Let $\tilde{\Lambda} = (Q, I)$ be a fringed quiver of a gentle algebra Λ. We define a turbulence chart $(G, \sim, R)$ as follows. The graph G is the underlying undirected graph of Q. For each internal vertex v of Q with relations $\alpha\beta$ and $\gamma\delta$, the equivalence relation $\sim_v$ separates $\{\alpha, \beta\}$ from $\{\gamma, \delta\}$. The framing R orders all half-edges coming from arrow heads as high, and all half-edges coming from arrow tails low. See Figures 1 and 5 for two examples. We say that a turbulence polyhedron is *gentle* if it arises in this way. Routes and bands of $(G, \sim, R)$ are in natural bijection with routes and bands of $\tilde{\Lambda}$. This bijection respects compatibility, realizing an isomorphism between the clique complex of $(G, \sim, R)$ and the clique complex of $\tilde{\Lambda}$. We may then consider Theorems 3.7 and 3.11 as presentation and subdivision results of the turbulence polyhedron of a gentle algebra through the combinatorics of its τ-tilting theory.

Theorem 4.4. *If $(G, \sim, R)$ is gentle, then the union of the simplices given by bundle spaces of cliques are dense in $\mathcal{F}_1(G, \sim, R)$. We call this the the* bundle subdivision *of $\mathcal{F}_1(G, \sim, R)$.*

Remark 4.5. There is a projection map from $\mathcal{F}_1(G, \sim, R)$ to the ambient space of the g-vector fan of Λ. We define its image to be the *g-polyhedron* of Λ, which extends the definition of g-polytopes [1] outside of the τ-tilting-finite case. In this way we are able to show g-convexity of both representation-finite and -infinite gentle algebras.

Remark 4.6. In the literature, the combinatorics of the g-vector fan of Λ is described by the clique complex of $\tilde{\Lambda}$. The addition of bands to our bundle complex then gives some interpretation for the complement of the g-vector fan of Λ.

We finish the extended abstract by speaking to our methods of proving the results of this paper. In fact, most of our technical work is done directly on fringed quivers of gentle algebras. We then show that general (bundle-subdivided) turbulence polyhedra are precisely faces of gentle (bundle-subdivided) turbulence polyhedra, and use this fact to port Theorem 3.7, Theorem 3.11, and Corollary 3.12 to general turbulence polyhedra.

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