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Expansion Formulae for SL_3 Fock–Goncharov Cluster Algebras

Ziyao Fu* [1], Ni Tang†[2], Sylvester W. Zhang‡[3], and Yunchi Zheng§[4]

[1] *School of Mathematical Sciences, Peking University, Beijing, China*
[2] *Department of Mathematics, Johns Hopkins University, MD, US*
[3] *School of Mathematics, University of Minnesota, MN, US*
[4] *School of Mathematical Sciences, Nankai University, Tianjin, China*

Abstract. We investigate the cluster algebra of Fock–Goncharov coordinates of the moduli space of decorated SL_3-local systems $\mathcal{A}_{\mathrm{SL}_3,S}$ when S is a marked disk. In particular, we give explicit expansion formula for the cluster variables with coefficients in terms of dimer covers (or perfect matchings) of certain subgraphs of Goncharov's A_2 plabic graphs. We describe the poset structures on the said dimer covers, and show that the corresponding rank functions compute the F-polynomials of the corresponding cluster variables.

1 Introduction

Cluster algebras were introduced by Fomin and Zelevinsky [4, 5], motivated by dual canonical basis of quantum groups. Loosely speaking, a cluster algebra is a commutative ring with a distinguished set of variables called *cluster variables*, which are grouped into equal-sized subsets called *clusters*. Two clusters that differ by one entry are connected by an involution called *mutation*, of the form $\{x_1, \cdots, x_i, \cdots, x_n\} \leftrightarrow \{x_1, \cdots, x_i', \cdots, x_n\}$ where the product of $x_i x_i'$ satisfies a binomial exchange relation.

Structures of cluster algebras are often found in the coordinate rings of geometric objects (such as the Grassmannian) where the coordinates correspond to clusters and mutations are change of coordinate transformations. One of the examples of cluster algebras is the Teichmüller space $T(S) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2)/\mathrm{PSL}_2$ of a surface S. Robert Penner [10] introduced moduli spaces $\tilde{T}(S)$ that are principal $\mathbb{R}_+$-bundles over $T(S)$, known as the decorated Teichmüller spaces. Specifically, by attaching horocycles at ideal points on S, Penner introduced the notion of λ-lengths for each pair of ideal points, which coordinatize $\tilde{T}(S)$. The relations among the λ-lengths coordinates are cluster mutations.

*fzy1003@stu.pku.edu.cn
†ntang15@jh.edu.
‡swzhang@umn.edu
§2111565@mail.nankai.edu.cn

Meanwhile, Thurston's lamination [13] gives the *shear coordinates* of the usual Teichmüller space, which are cross-ratios of Penner's λ-lengths.

The duality between Penner's λ-lengths and Thurston's shear coordinates is generalized by Fock and Goncharov [3] to a much higher realm. For certain algebraic groups G and its Langlands dual $\check{G}$, they introduced pairs of positive spaces $(\mathcal{A}^+_{G,S}, \mathcal{X}^+_{\check{G},S})$ called higher Teichmüller spaces (a.k.a. character varieties). When $G = \mathrm{SL}_2$ and $\check{G} = \mathrm{PGL}_2$, these spaces recover Penner's decorated Teichmüller spaces and Thurston's laminations respectively. When $G = \mathrm{SL}_n$, they also admit a cluster structure as explained in [3].

In the language of cluster algebras, the $\mathcal{A}$-coordinates correspond to cluster variables and the $\mathcal{X}$-coordinates correspond to coefficients. Let $\mathcal{A}_{\mathrm{SL}_n,m}$ denoted the cluster algebra (with principal coefficients) of the Fock–Goncharov coordinates of $\mathcal{A}^+_{\mathrm{SL}_n,S}$ when S is a disk with m marked points on its boundary. In this paper, we investigate the combinatorics of cluster variables in $\mathcal{A}_{\mathrm{SL}_3,m}$ that correspond to the Fock–Goncharov coordinates. In particular, we give explicit combinatorial formulas using dimer covers (a.k.a. perfect matchings) on a weighted version of Goncharov's A_2-webs [6]. We conjecture that our construction generalize to general SL_n as well.

The plan of the paper is as follows. In Section 2 we review necessary background on cluster algebras and quiver mutations, and define the cluster algebra $\mathcal{A}_{\mathrm{SL}_3,m}$. In Section 3 we introduce a weighted version Goncharov's A_2 plabic graphs, and state our main theorem. We then conclude by giving an example for our main result.

2 Background

In this section, we introduce background on cluster algebras and introduce the Fock–Goncharov cluster algebra $\mathcal{A}_{\mathrm{SL}_3,m}$.

2.1 Cluster algebras and quiver mutations.

The definition of a cluster algebra $\mathcal{A}$ begins with its ground ring. Let $(\mathbb{P}, \bigoplus, \cdot)$ be a semifield, an abelian multiplicative group together with a binary operation $\bigoplus$ which is commutative, associative and distributive with respect to multiplication. Additionally, the multiplicative group $\mathbb{P}$ is torsion-free, and therefore is the group ring $\mathbb{Z}\mathbb{P}$, which will be the ground ring for $\mathcal{A}$. Let $\mathcal{F}$ be the field of rational functions in n independent variables, with coefficients in $\mathbb{Q}\mathbb{P}$. Here n is the rank of the cluster algebra $\mathcal{A}$.

Particularly, throughout this article, we will take $(\mathbb{P}, \cdot)$ to be a free abelian group (written multiplicatively) on n variables $\{y_1, y_2, ..., y_n\}$ and the addition $\bigoplus$ defined as follows:

$$\prod_j y_j^{a_j} \oplus \prod_j y_j^{b_j} = \prod_j y_j^{\min(a_j, b_j)}. \tag{2.1}$$

A seed is a triplet (x, y, Q) containing the following data.

- A cluster $x = (x_1, x_2, ..., x_n)$ is an n-tuple of elements of $\mathcal{F}$ forming a free basis,
- A coefficient tuple $y = \{y_1, y_2, ..., y_n\}$ is an n-tuple of elements of $\mathbb{P}$,
- A quiver Q is a directed graph with n vertices without loops or 2-cycles.

A cluster algebra $\mathcal{A} = \mathcal{A}(x, y, Q)$ is determined by the choice of an initial seed (x, y, Q). We call x the initial cluster and y the initial coefficient tuple of $\mathcal{A}$. We will occasionally denote the coefficient y_i associated to the cluster variable x_i as $\mathbf{y}_{x_i}$.

Now the cluster algebra $\mathcal{A}$ is a $\mathbb{Z}\mathbb{P}$-subalgebra of $\mathcal{F}$ generated by cluster variables, which can be produced from initial seed by *mutation*. The mutation μ_k in direction k, where $k = 1, 2, ...n$, transforms a seed (x, y, Q) into a new seed (x', y', Q') as follows:

- $x' = x \backslash \{x_k\} \cup \{x'_k\}$, where

$$x_k x'_k = \frac{1}{y_k \bigoplus 1} \left(y_k \prod_{i \to k} x_i + \prod_{i \leftarrow k} x_i \right). \tag{2.2}$$

- $y' = (y'_1, y'_2, ..., y'_n)$, where

$$y'_j = \begin{cases} y_k^{-1}, & \text{if } j = k, \\ y_j \prod_{k \to j} y_k (y_k \bigoplus 1)^{-1} \prod_{k \leftarrow j} (y_k \bigoplus 1) & \text{if } j \neq k. \end{cases} \tag{2.3}$$

- The quiver Q' is obtained from Q by a mutation at the vertex k, that is
  - for every path $i \to k \to j$, add one arrow $i \to j$,
  - reverse all arrows at k,
  - delete all 2-cycles.

Finally, the cluster algebra $\mathcal{A}$ is the $\mathbb{Z}\mathbb{P}$-subalgebra of $\mathcal{F}$ generated by the set of cluster variables obtained by mutating in all directions from the initial seed.

We now review the case of $\mathcal{A}_{\mathrm{SL}_2,m}$, where clusters are in bijection with triangulations of an m-gon. The initial quiver is constructed by placing a vertex at each edge of the triangulation T, and inside each face of T we put arrows to form clock-wise oriented triangles. See Figure 1 for example.

A mutation corresponds to a quadrilateral flip in the triangulation, which turns out to be

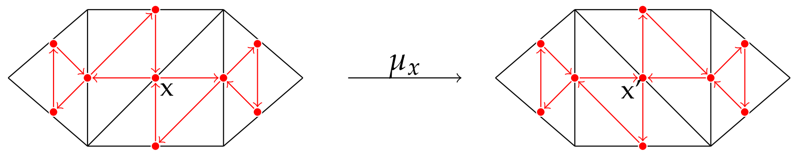

Figure 1: SL_2 quiver associated to a triangulation. Here $m = 6$.

identical to mutating the corresponding vertex in the quiver, as depicted in Section 2.1. Note that one cannot flip the boundary edges, therefore we call those variables "frozen".

2.2 Cluster algebra $\mathcal{A}_{SL_3,m}$.

We now construct the cluster algebra $\mathcal{A}_{SL_3,m}$. Similar to the SL_2 case, the initial data is given by a triangulation T of a m-gon P. Define the initial quiver $Q_3(T)$ as follows.

- Place two vertices on each arc of the triangulation T and one vertex in the interior of each triangle of T. The corresponding cluster variables are called *edge variables* and *face variables* respectively.
- Attach arrows to the vertices with clockwise oriented triangles as depicted in Figure 2.

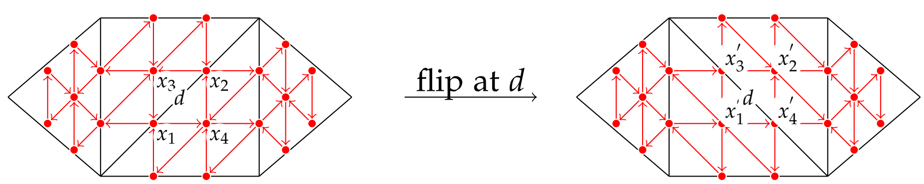

Figure 2: A 6-gon P with triangulation T and the corresponding quiver $Q_3(T)$ (3-triangulation). A flip at the diagonal d is obtained by first mutating x_1 and x_2, and then mutating x_3 and x_4.

Different from the SL_2 case, the quadrilateral flips are given by a sequence of four mutations. In particular, to flip a diagonal d, we first mutate the two vertices sitting on d (in any order since they commute, and so do the following two), and then mutate the two vertices sitting on the two triangles adjacent to d. The result of these mutations is exactly the quiver associated to the triangulation after the flip. See Figure 2 for illustration.

3 Plabic Graphs and Dimer Covers

We now recall the definition of Goncharov's A_2-plabic graph and describe a weighted version, which will be central to our main formula.

3.1 Construction of weight A_2 plabic graphs

Let P be a polygon and T be a triangulation on P, and $Q_3(T)$ the corresponding quiver. Recall that the 3-triangulation $\hat{T}$ (following Fock–Goncharov) associated to T is obtained by separating each triangle in T into 4 smaller triangles (see Figure 3). We further define the weight of edges on $\hat{T}$ as shown in the middle of Figure 3.

The plabic graph Γ is defined as follows. For each vertex of $\hat{T}$, we place a white vertex. And for each triangle of T that is being divided into four smaller ones in $\hat{T}$, we place a black vertex in each of the three small triangles except for the internal one. We then add black-to-white edges in each of the non-internal small triangles, making every black vertex trivalent. The weight of edges in the plabic graph is set to be the weight of the opposite edge in $\hat{T}$, see Figure 3 for illustration. Finally, we remove all the degree one vertices, and call the resulting weighted graph the weighted plabic graph associated to $Q_3(T)$.

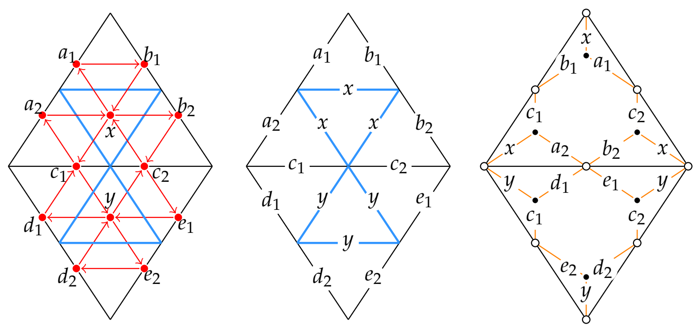

Figure 3: From left to right: the quiver $Q_3(T)$, the weighted 3-triangulation $\hat{T}$, the weighted plabic graph Γ. The weight of edges in Γ are labeled. And the label of a face of Γ is the cluster variable corresponding to the quiver vertex sitting inside the face.

Note that the plabic graph Γ_T is dual to the quiver $Q_3(T)$, so that every face of Γ_T corresponds to a vertex of $Q_3(T)$. Therefore each quadrilateral face or hexagonal face of Γ has a cluster variable sitting inside, which we define to be the label of that face, denoted $\text{label}(F)$.

Remark 1. In SL_2 case, all the clusters obtained from mutation of non-frozen vertices corresponds to Fock–Goncharov coordinates (i.e. triangulations). However, the SL_3 quiver $Q_3(T)$ is not of finite type in general. In fact, the only clusters that corresponds to triangulations are obtained by iteratively applying the special sequence of four mutations. We shall restrict ourselves in that case, and only allow such mutations. In particular, our main result will only provide formulae for clusters that are Fock–Goncharov coordinates.

3.2 Plabic Subgraphs

To establish the combinatorial expansion formula for cluster variables, we shall define a subgraph Γ_x of the weighted plabic graph Γ for each edge or face variable x of other triangulations, on which there is a one-to-one correspondence of dimer covers and expansion terms of x.

For each arc (i,j) on the polygon P, we denote x_{ij} and x_{ji} the two cluster variables corresponding to (i,j), where x_{ij} is the one that is placed closer to the vertex i. For each triangle (i,j,k) on P, we denote x_{ijk} it corresponding face variable. Note that for face variables x_{ijk} we do not distinguish the ordering of indices.

In what follows, we define plabic subgraphs that will be used to compute certain edge and face variables.

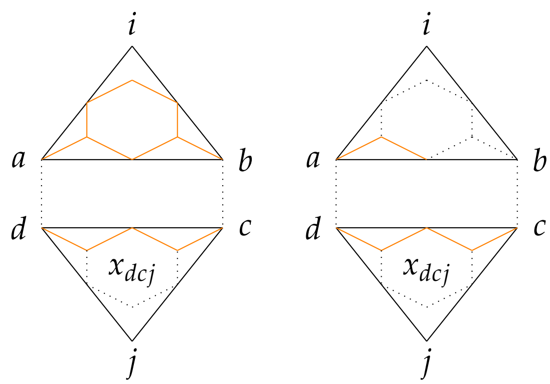

Figure 4: Left: The plabic subgraph $\Gamma_{x_{ij}}$. Right: The plabic subgraph $\Gamma_{x_{iaj}}$. Here only the first and last triangle are shown, and dashed edges being removed to form the plabic subgraph.

Definition 1. Consider a triangulation as described in Figure 4.

- For an edge variable x_{ij}, define the corresponding plabic subgraph Γ_{ij} to be the subgraph of Γ with a hexagonal face near vertex j removed. Define the *label* of $\Gamma_{x_{ij}}$ by

$$\mathrm{label}(\Gamma_{x_{ij}}) := x_{dcj} \prod_{F \in \text{faces of } \Gamma_{x_{ij}}} x_F. \tag{3.1}$$

- For a face variable x_{iaj} such that (i,a) is an edge in T, we define Γ_{ija} to be the subgraph of Γ with 3 less faces: two hexagonal faces inside (i,a,b) and (j,d,c), and one quadrilateral face near vertex b. The label of $\Gamma_{x_{ija}}$ is defined by

$$\mathrm{label}(\Gamma_{x_{ija}}) := x_{dcj} \prod_{F \in \text{faces of } \Gamma_{x_{iaj}}} x_F. \tag{3.2}$$

3.3 Dimer Covers and their weights and heights

A *dimer cover* (or *perfect matching*) M of a bipartite graph G is a collection of edges such that every vertex in G is incident to exactly on edge in M. We denote $\mathrm{D}(G)$ the set of all dimer covers of G. For weighted graphs, we define the weight of a dimer cover M to be simply the product of all edge weights in M. To account for the coefficients in our cluster algebras, we shall introduce an additional statistics called *height* on dimer covers of the plabic subgraphs, which relies on the notion of the *minimal dimer cover*.

Proposition 1. *Let G be a plabic subgraph. There exists a unique dimer cover $M_0 \in \mathrm{D}(G)$ such that every boundary edge of M_0 is oriented counterclockwise from white to black, and exactly half the boundary edges of G are included in M_0. We call M_0 the* minimal dimer cover *or* minimal matching*. See Figure 5 for example.*

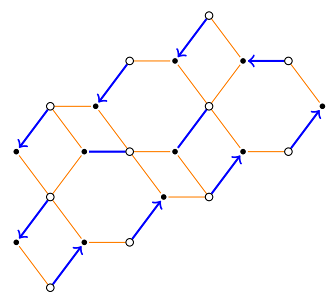

Figure 5: Example of a minimal dimer cover. The white-to-black counterclockwise orientation of boundary edges are shown.

Definition 2. Let G be a plabic subgraph, we define the height of $M \in \mathrm{D}(G)$ as follows. Consider the union $\overline{M} = M \cup M_0$ where M_0 is the minimal dimer cover defined above. Pictorially $\overline{M}$ is obtained by super-imposing M on top of M_0. Note that $\overline{M}$ is a double dimer cover which contains only doubled edges and cycles. We then define the height of M to be

$$\mathrm{ht}(M) = \prod_{f \in \text{ cycles of } \overline{M}} \mathbf{y}_f$$

where the product is over the faces of G that are surrounded by a cycle of $\overline{M}$, and $\mathbf{y}_f$ is the coefficient corresponding to the cluster variable sitting in f. See Figure 6 for example.

3.4 Statement of the Main Theorem

Theorem 1. *With above notation, let x be a cluster variable corresponding to a face or an edge, and let Γ_x be the corresponding plabic subgraph. Then the expansion of x in terms of the initial cluster is:*

$$x = \frac{1}{\mathrm{label}(\Gamma_x)} \sum_{M \in \mathrm{D}(\Gamma_x)} \mathrm{wt}(M)\,\mathrm{ht}(M). \tag{3.3}$$

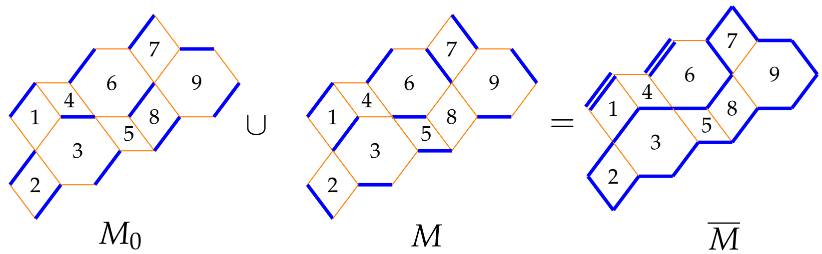

Figure 6: Superimposing a dimer cover M and the minimal dimer cover M_0. Face labels indicate the corresponding cluster variables and their coefficient. The height is $\mathrm{ht}(M) = \mathbf{y}_2\,\mathbf{y}_3\,\mathbf{y}_5\,\mathbf{y}_7\,\mathbf{y}_8\,\mathbf{y}_9$.

Example 1. Let P be a quadrilateral and T be the triangulation in Figure 7. To obtain the expansion formula for the edge variable x_{ij}, we figure out the subgraph $\Gamma_{x_{ij}}$ and its label. Then we find all dimer covers on this subgraph as well as their weight and height.

By definition we have $\mathrm{label}(\Gamma_{x_{ij}}) = xyc_1c_2$, and Theorem 1 implies that

$$x_{ij} = \frac{1}{xyc_1c_2}(xb_1yc_1e + x^2d_1c_1e_1\,\mathbf{y}_{c_1} + x^2e_2d_2c_2\,\mathbf{y}_y\,\mathbf{y}_{c_2} + a_2xyd_2c_2\,\mathbf{y}_y\,\mathbf{y}_{c_1}\,\mathbf{y}_{c_2}) \tag{3.4}$$

$$= \frac{b_1e}{c_2} + \frac{xd_1e_1\,\mathbf{y}_{c_1}}{c_2y} + \frac{xe_2d_2\,\mathbf{y}_y\,\mathbf{y}_{c_2}}{yc_1} + \frac{a_2d_2\,\mathbf{y}_y\,\mathbf{y}_{c_1}\,\mathbf{y}_{c_2}}{c_1}. \tag{3.5}$$

Remark 2. We note that a more common approach to obtain graph theoretic formulas for generalizations of cluster variables is a different method called "snake graphs" (see [9, 8, 1]). In SL_2 case, our method recovers an unpublished result of Caroll and Price [2].

Remark 3. The work of Muller–Speyer [7] and Postnikov [11] combined gives an expansion formula for Fock–Goncharov $\mathcal{X}$-coordinates as generating functions of certain plabic graphs with weights of Plücker variables associated to the faces. The plabic graphs we use are the same as theirs, except that the Plücker weights are attached to the edges. The Muller–Speyer results work in the scope for any SL_n. For the $\mathcal{A}$-variables, the expansion formula turns out to be more difficult for SL_n of $n \geq 4$, for reasons we will discuss in the following remark.

Remark 4. We conjecture that the same construction of plabic subgraphs gives an expansion formula for $\mathcal{A}_{\mathrm{SL}_n}$ for $n \geq 4$. Our proof for the SL_3 case, however, does not generalize to higher n. The main difficulty is that for higher n, the mutation sequence for one flip involves mutating at a vertex twice, which corresponds to dividing by a non-monomial.

3.5 Poset structure on Dimer covers

Recall that in a cluster algebra with principal coefficients, a cluster variable is a Laurent polynomial in the initial cluster variables $\{x_1, x_2, ..., x_n\}$ with coefficients in $\{y_1, y_2, ..., y_n\}$.

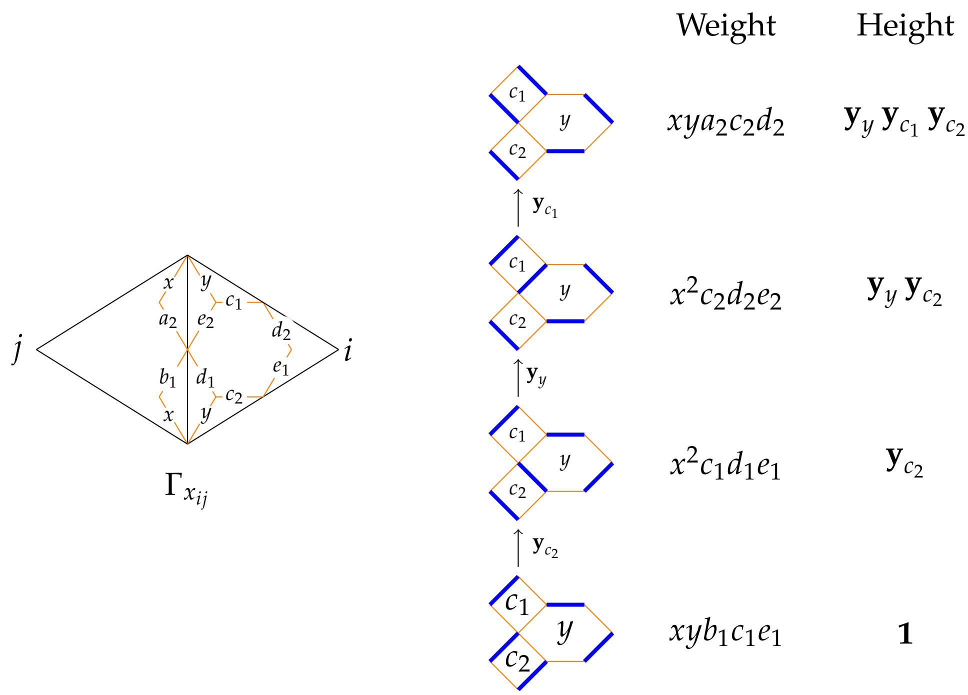

Figure 7: Left: the plabic subgraph Γ_{ij}. Right: the poset of all dimer covers on $\Gamma_{x_{ij}}$ and their corresponding weight and height. Every covering relation on the poset corresponds to toggling a face i, which is equivalent to multiplying the height by $\mathbf{y}_i$.

The corresponding F-polynomial is defined by specializing all the x_i's to be 1. In terms of plabic graphs, the F-polynomial is simply the sum of heights of all dimer covers. Our main result Theorem 1 gives a poset structure on dimer covers which can be used to compute the F-polynomial explicitly.

For G a plabic subgraph, define a poset $\mathcal{P}_{\mathrm{D}(G)}$ on $\mathrm{D}(G)$ as follows. For $M_1, M_2 \in \mathrm{D}(G)$, we have $M_1 < M_2$ if $\mathrm{ht}(M_1)$ is divisible by $\mathrm{ht}(M_2)$. This poset can also be constructed inductively. Fix M_0 as the minimal element, then every covering relation of the poset is given by *toggling* on faces (as illustrated in Figure 8). This perspective is detailed in the following theorem.

Theorem 2. *Let $M \in \mathrm{D}(G)$, its height can be computed via $\mathcal{P}_{\mathrm{D}(G)}$ as follows. Take any chain from M_0 to M which corresponds to a sequence of toggles, then* $\mathrm{ht}(M)$ *is the product of the y-coefficients of the faces being toggled. Note that the result does not depend on the specific choice of chains.*

Remark 5. The poset $\mathcal{P}_{\mathrm{D}(G)}$ is a distributive lattice via an argument of Propp [12], and its subposet consisting of join-irreducibles is isomorphic to a specific part of the quiver.

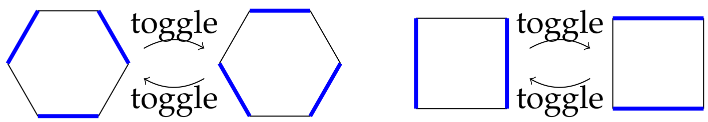

Figure 8: Illustration of toggle on a hexagon face and on a square face.

4 Example of Main Theorem

Consider the hexagon P, a triangulation T and its quiver $Q_3(T)$ as depicted in Figure 9. We wish to calculate the edge variable x_{ij}. Its plabic subgraph Γ_{ij} and the minimal matching M_0 is depicted in Figure 9. The poset of dimer covers of Γ_{ij} is shown in Figure 10, and by Theorem 1, the cluster variable x_{ij} is given by

$$
\begin{aligned}
x_{ij} = \frac{1}{xyzwc_1c_2e_1e_2g_1g_2} & (xyzwb_1c_1e_1^2e_2g_1i_1 + xyz^2b_1c_1e_1e_2h_1g_1i_1\,\mathbf{y}_8 + x^2zwc_1d_1e_1^2g_1i_1\,\mathbf{y}_2 \\
& + xyz^2b_1c_1e_1e_2h_2g_2i_2\,\mathbf{y}_8\,\mathbf{y}_9 + x^2z^2c_1d_1e_1e_2h_1g_1i_1\,\mathbf{y}_2\,\mathbf{y}_8 + x^2zwc_2d_2e_1e_2^2g_1i_1\,\mathbf{y}_2\,\mathbf{y}_3 \\
& + xyzwb_1c_1e_1e_2f_2g_2h_2\,\mathbf{y}_7\,\mathbf{y}_8\,\mathbf{y}_9 + x^2z^2c_1d_1e_1e_2g_2h_2i_2\,\mathbf{y}_2\,\mathbf{y}_8\,\mathbf{y}_9 + x^2z^2c_2d_2e_2^2g_1h_1i_1\,\mathbf{y}_2\,\mathbf{y}_3\,\mathbf{y}_8 \\
& + xyzwa_2c_2d_2e_1e_2g_1i_1\,\mathbf{y}_1\,\mathbf{y}_2\,\mathbf{y}_3 + x^2zwc_1d_1e_1e_2f_2g_2h_2\,\mathbf{y}_2\,\mathbf{y}_7\,\mathbf{y}_8\,\mathbf{y}_9 \\
& + x^2z^2c_2d_2e_2^2g_2h_2i_2\,\mathbf{y}_2\,\mathbf{y}_3\,\mathbf{y}_8\,\mathbf{y}_9 + x^2yzc_2e_2^2g_1g_2h_1i_1\,\mathbf{y}_2\,\mathbf{y}_3\,\mathbf{y}_5\,\mathbf{y}_8 \\
& + xyz^2a_2c_2d_2e_2g_1h_1i_1\,\mathbf{y}_1\,\mathbf{y}_2\,\mathbf{y}_3\,\mathbf{y}_8 + x^zwc_2d_2e_2^2f_2g_2h_2\,\mathbf{y}_2\,\mathbf{y}_3\,\mathbf{y}_7\,\mathbf{y}_8\,\mathbf{y}_9 \\
& + x^2yzc_2e_2^2g_2^2h_2i_2\,\mathbf{y}_2\,\mathbf{y}_3\,\mathbf{y}_5\,\mathbf{y}_8\,\mathbf{y}_9 + xyz^2a_2c_2d_2e_2g_2h_2i_2\,\mathbf{y}_1\,\mathbf{y}_2\,\mathbf{y}_3\,\mathbf{y}_8\,\mathbf{y}_9 \\
& + xy^2za_2c_2e_2g_1g_2h_1i_1\,\mathbf{y}_1\,\mathbf{y}_2\,\mathbf{y}_3\,\mathbf{y}_5\,\mathbf{y}_8 + x^2ywc_2e_2^2f_2g_2^2h_2\,\mathbf{y}_2\,\mathbf{y}_3\,\mathbf{y}_5\,\mathbf{y}_7\,\mathbf{y}_8\,\mathbf{y}_9 \\
& + xyzwa_2c_2d_2e_2f_2g_2h_2\,\mathbf{y}_1\,\mathbf{y}_2\,\mathbf{y}_3\,\mathbf{y}_7\,\mathbf{y}_8\,\mathbf{y}_9 + xy^2za_2c_2e_2g_2^2h_2i_2\,\mathbf{y}_1\,\mathbf{y}_2\,\mathbf{y}_3\,\mathbf{y}_5\,\mathbf{y}_8\,\mathbf{y}_9 \\
& + x^2ywc_2e_1e_2f_1g_1g_2h_2\,\mathbf{y}_2\,\mathbf{y}_3\,\mathbf{y}_5\,\mathbf{y}_6\,\mathbf{y}_7\,\mathbf{y}_8\,\mathbf{y}_9 + xy^2wa_2c_2e_2f_2g_2^2h_2\,\mathbf{y}_1\,\mathbf{y}_2\,\mathbf{y}_3\,\mathbf{y}_5\,\mathbf{y}_7\,\mathbf{y}_8\,\mathbf{y}_9 \\
& + xy^2wa_2c_2e_1f_1g_1g_2h_2\,\mathbf{y}_1\,\mathbf{y}_2\,\mathbf{y}_3\,\mathbf{y}_5\,\mathbf{y}_6\,\mathbf{y}_7\,\mathbf{y}_8\,\mathbf{y}_9 \\
& + xyzwa_2c_1c_2e_1g_1g_2h_2\,\mathbf{y}_1\,\mathbf{y}_2\,\mathbf{y}_3\,\mathbf{y}_4\,\mathbf{y}_5\,\mathbf{y}_6\,\mathbf{y}_7\,\mathbf{y}_8\,\mathbf{y}_9).
\end{aligned}
$$

The monomials in the above expression are in correspondence to the poset in Figure 10 from bottom to top and left to right.

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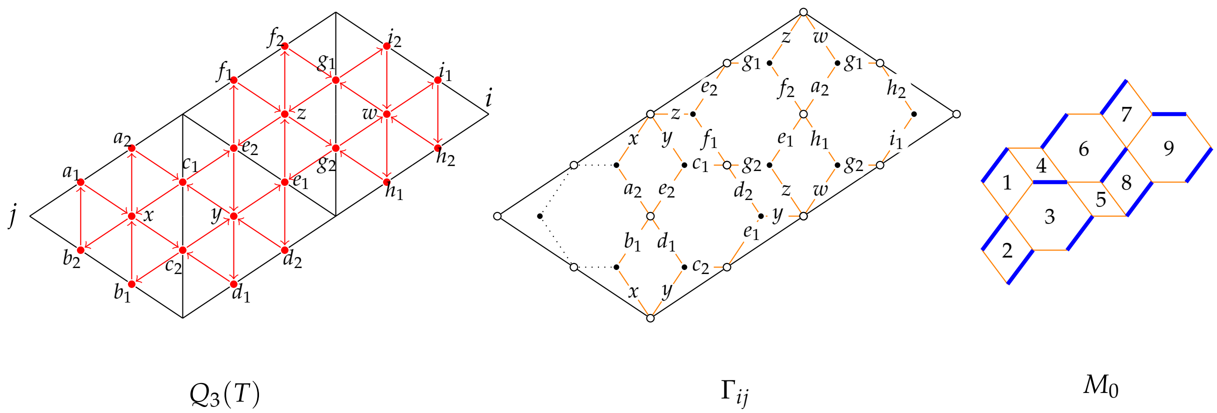

Figure 9: Left: Quiver $Q_3(T)$ of the hexagon P with triangulation T. Middle: plabic subgraph Γ_{ij} with edge weights. Right: the minimal matching.

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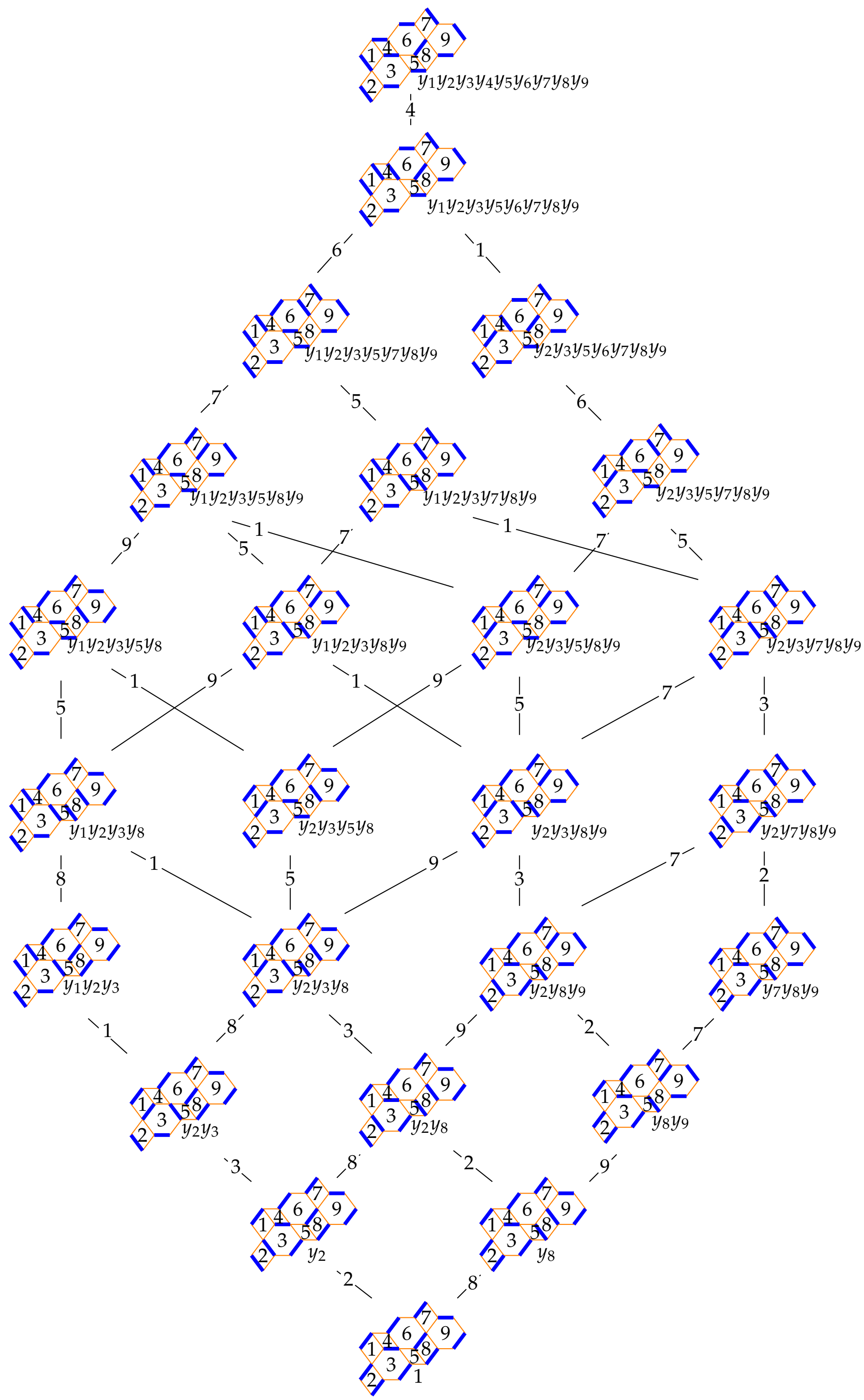

Figure 10: The Hasse diagram for the poset of dimer covers of Γ_{ij}, where covering relations are indicated by edges labeled by the face being toggled.