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Multigraded strong Lefschetz property for balanced simplicial complexes

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Abstract. Generalizing the strong Lefschetz property for an $\mathbb{N}$-graded algebra, we introduce multigraded strong Lefschetz property for an $\mathbb{N}^m$-graded algebra. We show that, for $\boldsymbol{a} \in \mathbb{N}_+^m$, the generic $\mathbb{N}^m$-graded Artinian reduction of the Stanley–Reisner ring of an $\boldsymbol{a}$-balanced homology sphere over a field of characteristic 2 satisfies the multigraded strong Lefschetz property. A corollary is the inequality $h_{\boldsymbol{b}} \leq h_{\boldsymbol{c}}$ for $\boldsymbol{b} \leq \boldsymbol{c} \leq \boldsymbol{a} - \boldsymbol{b}$ for the flag h-vector of an $\boldsymbol{a}$-balanced simplicial sphere. This can be seen as a common generalization of the unimodality of the h-vector of a simplicial sphere by Adiprasito and balanced generalized lower bound inequality by Juhnke-Kubitzke and Murai. Another combinatorial consequence is that a k-dimensional completely balanced simplicial complex which is a subcomplex of a simplicial $2k$-sphere satisfies $f_k \leq 2f_{k-1}$.

Keywords: Lefschetz property, Stanley–Reisner ring, balancedness, multigraded algebra, unimodality

1 Introduction

The f-vector and h-vector of simplicial complexes have been extensively studied in algebraic and topological combinatorics in the last decades. Here, for a $(d-1)$-dimensional simplicial complex Δ, the *f-vector* of Δ is a sequence $(f_{-1}, \ldots, f_{d-1})$, where f_i is the number of i-dimensional faces of Δ, and the *h-vector* $(h_0, \ldots, h_d)$ of Δ is defined by $\sum_{i=0}^d h_i t^i = \sum_{i=0}^d f_{i-1} t^i (1-t)^{d-i}$ using a variable t. For the study of f-vector and h-vector of simplicial complexes, Stanley–Reisner ring (or face ring) has been used. A recent breakthrough announced by Adiprasito [1] (see also [2, 11, 15]) is the hard Lefschetz theorem for the Stanley–Reisner ring of a simplicial (or homology) sphere, generalizing the work of Stanley [17] for the boundary complex of simplicial polytopes. Among many combinatorial consequences, this implies the celebrated g-conjecture, in particular generalized lower bound inequality (GLBI) asserting that the h-vector of a simplicial sphere is unimodal.

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Adding extra combinatorial constraints to topological (or homological) ones and then studying behaviors of f- and h-vectors have been of interest (see [5, 8, 18] for example). Juhnke-Kubitzke and Murai [8] investigated completely balanced (or $(1, \ldots, 1)$-balanced in the definition below) simplicial spheres, and showed (assuming Hard Lefschetz theorem [1]) that the Stanley–Reisner ring of its rank selected subcomplex possesses the top-heavy strong Lefschetz property (or dual-weak Lefschetz property). A corollary is a balanced GLBI asserting that the h-vector of a completely balanced simplicial $(d-1)$-sphere satisfies

$$\frac{h_i}{\binom{d}{i}} \leq \frac{h_{i+1}}{\binom{d}{i+1}} \text{ for } i < \frac{d}{2}. \tag{1.1}$$

In this extended abstract of the full preprint [14], we investigate simplicial complexes with a combinatorial constraint called $\boldsymbol{a}$-balancedness. For a positive integer vector $\boldsymbol{a} = (a_1, \ldots, a_m)$ with $|\boldsymbol{a}| := a_1 + \cdots + a_m = d$, a pair (Δ, κ) of a $(d-1)$-dimensional simplicial complex Δ and a map $\kappa : V(\Delta) \to [m] := \{1, \ldots, m\}$ is *$\boldsymbol{a}$-balanced* if each face of Δ contains at most a_j vertices of color j for each $j = 1, \ldots, m$. Stanley [16] initiated a research of $\boldsymbol{a}$-balanced simplicial complexes and showed that the Stanley–Reisner ring of an $\boldsymbol{a}$-balanced simplicial complex admits a system of parameters homogeneous in the fine $\mathbb{N}^m$-grading induced by the coloring κ.

With this in mind, we introduce multigraded strong Lefschetz property for an $\mathbb{N}^m$-graded algebra, generalizing strong Lefschetz property of an $\mathbb{N}$-graded algebra of [7]. We then show that the generic $\mathbb{N}^m$-graded Artinian reduction of the Stanley–Reisner ring of an $\boldsymbol{a}$-balanced homology sphere over a field of characteristic 2 satisfies the multigraded strong Lefschetz property (Theorem 5.1). Note that Theorem 5.1 is a common generalization of the above mentioned two algebraic results. Our proof of Theorem 5.1 relies on an anisotropy technique, in particular differential identity, over a field of characteristic 2 in [2, 3, 11, 15].

A corollary of Theorem 5.1 is a combinatorial result on flag h-vector. The *flag f-vector* of $\boldsymbol{a}$-balanced simplicial complex (Δ, κ) is an m-dimensional array $(f_{\boldsymbol{b}})_{\boldsymbol{0} \leq \boldsymbol{b} \leq \boldsymbol{a}}$ where $f_{\boldsymbol{b}}$ is the number of faces $\sigma \in \Delta$ with $|\sigma \cap \kappa^{-1}(j)| = b_j$ for $j = 1, \ldots, m$. Here $\boldsymbol{c} \leq \boldsymbol{d}$ denotes the component-wise inequality $c_i \leq d_i$ for all i. The *flag h-vector* of (Δ, κ) is an m-dimensional array $(h_{\boldsymbol{b}})_{\boldsymbol{0} \leq \boldsymbol{b} \leq \boldsymbol{a}}$ defined by

$$\sum_{\boldsymbol{0} \leq \boldsymbol{b} \leq \boldsymbol{a}} h_{\boldsymbol{b}} \boldsymbol{t}^{\boldsymbol{b}} = \sum_{\boldsymbol{0} \leq \boldsymbol{b} \leq \boldsymbol{a}} f_{\boldsymbol{b}} \boldsymbol{t}^{\boldsymbol{b}} (\boldsymbol{1} - \boldsymbol{t})^{\boldsymbol{a} - \boldsymbol{b}},$$

where $\boldsymbol{t} = (t_1, \ldots, t_m)$ is a vector of variables. Here, we denote $\boldsymbol{t}^{\boldsymbol{c}} = t_1^{c_1} \cdots t_m^{c_m}$ for $\boldsymbol{t} = (t_1, \ldots, t_m)$ and $\boldsymbol{c} = (c_1, \ldots, c_m) \in \mathbb{N}$.

Theorem 1.1. For an $\boldsymbol{a}$-balanced homology sphere (Δ, κ) over $\mathbb{F}_2$, we have $h_{\boldsymbol{b}} \leq h_{\boldsymbol{c}}$ for any $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{N}^m$ with $\boldsymbol{b} \leq \boldsymbol{c} \leq \boldsymbol{a} - \boldsymbol{b}$.

Note that Theorem 1.1 can be seen as a common generalization of GLBI and balanced GLBI. GLBI is the case of $m = 1$ in Theorem 1.1. Balanced GLBI (1.1) follows from the inequality $h_{i\boldsymbol{e}_j} \leq h_{(i+1)\boldsymbol{e}_j}$ for $\boldsymbol{a} = \mathbf{1} + 2i\boldsymbol{e}_j$ in Theorem 5.1 together with the averaging argument of Goff, Klee and Novik [6]. See [8] for details.

Another combinatorial corollary is related to the following balanced version of Grünbaum–Kalai–Sarkaria conjecture posed by Kalai–Nevo–Novik [10].

Conjecture 1.2 ([10, Conjecture 8.2, Proposition 8.3])**.** Let Δ be a k-dimensional completely balanced simplicial complex embedabble in $\mathbb{S}^{2k}$. Then, $f_k \leq 2f_{k-1}$ holds.

We prove the following significant partial result on Conjecture 1.2.

Theorem 1.3. Let Δ be a k-dimensional completely balanced simplicial complex such that there is a simplicial $2k$-sphere Γ with $\Delta \subset \Gamma$. Then, $f_k \leq 2f_{k-1}$ holds.

The derivation of Theorem 1.3 from Theorem 5.1 is analogous to [1], and we omit it from the extended abstract because of the length limit. The detailed argument for Theorem 1.3 can be found in the updated version of full preprint [14].

This extended abstract is organized as follows. After preliminaries are given in Section 2, Lee's formula for the evaluation map is recalled in Section 3. In Section 4, generic $\mathbb{N}^m$-graded Artinian reduction is defined, and differential formula for the evaluation map in multigraded setting is derived. In Section 5, we derive the main result Theorem 5.1 about multigraded strong Lefschetz property over a field of characteristic 2. In Section 6, we briefly discuss further results in the full preprint [14].

2 Preliminaries

We highlight some definitions and notations we use (see [19] for general reference). We denote the set of nonnegative (resp. positive) integers by $\mathbb{N}$ (resp. $\mathbb{N}_+$).

2.1 Simplicial complex and Stanley–Reisner ring

Throughout, by a simplicial complex, we always mean an abstract simplicial complex, i.e., a downward closed collection of subsets of a finite set. The vertex set of a simplicial complex Δ is denoted by $V(\Delta)$.

For an $\mathbb{N}^m$-graded module M and $\boldsymbol{b} \in \mathbb{N}^m$, we denote by $M_{\boldsymbol{b}}$ the submodule of all homogeneous elements of degree $\boldsymbol{b}$.

Let $\Bbbk$ be a field and let Δ be a simplicial complex. Let us denote by $\Bbbk[\boldsymbol{x}]$ the polynomial ring $\Bbbk[x_v : v \in V(\Delta)]$. The Stanley–Reisner ring of Δ over $\Bbbk$ is $\Bbbk[\Delta] = \Bbbk[\boldsymbol{x}]/I_\Delta$, where I_Δ is the ideal generated by $x_\tau := \prod_{v \in \tau} x_v$ over all $\tau \notin \Delta$. It is known that the

Stanley–Reisner ring of Δ has Krull dimension $\dim \Delta + 1$. For a $(d-1)$-dimensional simplicial complex Δ, a length d sequence of linear forms $\Theta = (\theta_1, \dots, \theta_d)$ of $\Bbbk[\Delta]$ is called a *linear system of parameters* (*l.s.o.p.* for short) for $\Bbbk[\Delta]$ if $\Bbbk[\Delta]/(\Theta) = \Bbbk[\Delta]/(\theta_1, \dots, \theta_d)$ is a finite dimensional $\Bbbk$-vector space. The resulting quotient algebra $\Bbbk[\Delta]/(\Theta)$ is called an *Artinian reduction* of $\Bbbk[\Delta]$ with respect to Θ.

For an $\boldsymbol{a}$-balanced simplicial complex (Δ, κ) with $\boldsymbol{a} \in \mathbb{N}^m$, the polynomial ring $\Bbbk[\boldsymbol{x}]$ naturally has the $\mathbb{N}^m$-grading, sometimes called the *fine grading*, defined by $\deg x_v = \boldsymbol{e}_{\kappa(v)}$, where $\boldsymbol{e}_j \in \mathbb{N}^m$ denotes the j-th unit coordinate vector. For an $\boldsymbol{a}$-balanced simplicial complex (Δ, κ), we say that a system of parameters Θ for $\Bbbk[\Delta]$ is $\mathbb{N}^m$-*graded* (or $\boldsymbol{a}$-colored) if each θ_i is homogeneous in the fine $\mathbb{N}^m$-grading of $\Bbbk[\Delta]$. Stanley [16, Theorem 4.1] showed that if $\Bbbk$ is an infinite field, every $\boldsymbol{a}$-balanced simplcial complex (Δ, κ) has an $\mathbb{N}^m$-graded l.s.o.p. Θ for $\Bbbk[\Delta]$, and $(\Bbbk[\Delta]/(\Theta))_{\boldsymbol{b}} = 0$ unless $\boldsymbol{0} \leq \boldsymbol{b} \leq \boldsymbol{a}$. Note that for an $\mathbb{N}^m$-graded l.s.o.p. Θ for the Stanley–Reisner ring of an $\boldsymbol{a}$-balanced simplicial complex, Θ contains exactly a_j elements of degree $\boldsymbol{e}_j$ for each j.

2.2 Homological properties

A simplicial complex Δ is called *Cohen–Macaulay* over $\Bbbk$ if there is an l.s.o.p. $(\theta_1, \dots, \theta_d)$ for $\Bbbk[\Delta]$ such that $\Bbbk[\Delta]$ is a free $\Bbbk[\theta_1, \dots, \theta_d]$-module. By Reisner's theorem, a simplcial complex Δ is Cohen–Macaulay over $\Bbbk$ if and only if it is pure and, for every face $\sigma \in \Delta$, $\widetilde{H}_i(\mathrm{lk}_\sigma(\Delta); \Bbbk) = 0$ for all $i \neq \dim \Delta - |\sigma|$ (see [19, Corollary II.4.2]). Here $\widetilde{H}_*(\Delta; \Bbbk)$ denotes the reduced simplicial homology group of Δ with coefficients in $\Bbbk$, and $\mathrm{lk}_\tau(\Delta) = \{\sigma \in \Delta : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\}$ denoted the *link* with respect to $\tau \in \Delta$. Note that for an $\boldsymbol{a}$-balanced Cohen–Macaulay complex (Δ, κ), the equality $\dim(\Bbbk[\Delta]/(\Theta))_{\boldsymbol{b}} = h_{\boldsymbol{b}}$ holds for $\boldsymbol{0} \leq \boldsymbol{b} \leq \boldsymbol{a}$.

For an $\mathbb{N}$-graded $\Bbbk[\boldsymbol{x}]$-module M, its *socle* is the submodule $\mathrm{Soc}(M) = \{a \in M : \mathfrak{m}a = 0\}$, where $\mathfrak{m} = (x_1, \dots, x_n)$ is the maximal graded ideal of $\Bbbk[\boldsymbol{x}]$. An $\mathbb{N}$-graded $\Bbbk$-algebra of Krull dimension zero is said to be *Gorenstein* if its socle is a one-dimensional $\Bbbk$-vector space. Note that an $\mathbb{N}$-graded finitely generated standard $\Bbbk$-algebra $A = A_0 \oplus \cdots \oplus A_d$ with $A_d \neq 0$ is Gorenstein if and only if $\dim A_d = 1$ and the multiplication map $A_i \times A_{d-i} \to A_d \xrightarrow{\cong} \Bbbk$ is a nondegenerate bilinear pairing for $i = 0, \dots, d$ [4, Lemma 36].

We say that a simplicial complex Δ is a *simplcial* $(d-1)$-*sphere* if its geometric realization is homeomorphic to $\mathbb{S}^{d-1}$. Let $\Bbbk$ be a field. A simplcial complex Δ is a *homology* $(d-1)$-*sphere* over $\Bbbk$ if $\widetilde{H}_*(\mathrm{lk}_\tau \Delta; \Bbbk) \cong \widetilde{H}_*(\mathbb{S}^{d-|\tau|-1}; \Bbbk)$ for every face $\tau \in \Delta$. If Δ is a homology sphere over $\Bbbk$, an Artinian reduction $A = \Bbbk[\Delta]/(\Theta)$ is Gorenstein with respect to any l.s.o.p. Θ [19, Theorem II.5.1].

A pure $(d-1)$-dimensional simplicial complex is *strongly connected* if for every pair of facets σ and τ of Δ, there is a sequence of facets $\sigma = \sigma_0, \sigma_1, \dots, \sigma_m = \tau$ such that $|\sigma_{i-1} \cap \sigma_i| = d-1$ for $i = 1, \dots, m$. A $(d-1)$-*pseudomanifold* (without boundary) is a strongly connected pure $(d-1)$-dimensional simplicial complex such that every $(d-2)$-

face is contained in exactly two facets. A $(d-1)$-pseudomanifold is *orientable* over $\Bbbk$ if $\widetilde{H}_{d-1}(\Delta;\Bbbk) \cong \Bbbk$, and such a pseudomanifold is said to be *oriented* if the facets are given an ordering such that the coefficients of nonzero $\mu \in \widetilde{H}_{d-1}(\Delta;\Bbbk)$ is constant over all oriented facets.

3 Lee's formula for the evaluation map

Let $\Bbbk$ be a field of arbitrary characteristic, and let Δ be a $(d-1)$-dimensional simplicial complex. Let $A = \Bbbk[\Delta]/(\Theta) = A_0 \oplus \cdots \oplus A_d$ be an Artinian reduction of $\Bbbk[\Delta]$ with respect to an l.s.o.p. Θ. Then, by [20, Corollary 3.2], A_d is linearly isomorphic to $\widetilde{H}_{d-1}(\Delta;\Bbbk)$. Thus, for a $(d-1)$-pseudomanifold Δ (without boundary) orientable over $\Bbbk$, A_d is a one-dimensional linear space. The linear isomorphism $\Psi : A_d \xrightarrow{\cong} \Bbbk$ which is determined unique up to the scaling is called the *evaluation map* (or *degree map*, *volume map*, *Brion's isomorphism*). Lee [12] gave an explicit description of the evaluation map Ψ with the appropriate scaling. We shall recall this formula below.

Let us prepare some conventions and notations used throughout. We always assume that $V(\Delta) = [n] := \{1,\ldots,n\}$ and let $\Bbbk[\boldsymbol{x}] = \Bbbk[x_1,\ldots,x_n]$. For a sequence $J = (v_1,\ldots,v_k)$ of vertices, we denote $x_J = x_{v_1}\cdots x_{v_k}$. We abbreviate the projection from $\Bbbk[\boldsymbol{x}]$ to an Artinian reduction A of $\Bbbk[\Delta]$ as long as it is not confusing. So, for example, the composite $\Bbbk[\boldsymbol{x}]_d \twoheadrightarrow A_d \xrightarrow{\cong} \Bbbk$ is also denoted as Ψ. An l.s.o.p. $\Theta = (\theta_1,\ldots,\theta_d)$ for $\Bbbk[\Delta]$ is identified with a map $p : V(\Delta) \to \Bbbk^d$ through the relation $\theta_k = \sum_{v\in V(\Delta)} p(v)_k x_v$ for $k = 1,\ldots,d$. The map p is called a *point configuration*. For an (oriented) facet $\sigma = [v_1,\ldots,v_d]$ of Δ, let $[\sigma] = \det\begin{pmatrix} p(v_1) & \cdots & p(v_d)\end{pmatrix}$.

We also need the following notations to state Lee's formula. Let v^* be a new vertex not in $V(\Delta)$ with an associated position[1] $p'(v^*) \in \Bbbk^d$, and for an (oriented) facet $\sigma = v_1,\ldots,v_d$, let $[\sigma - v_i + v^*]$ be the determinant of the matrix obtained by replacing the i-th column of the matrix $\begin{pmatrix} p(v_1) & \cdots & p(v_d)\end{pmatrix}$ with $p'(v^*)$.

Now we are ready to state Lee's formula.

Lemma 3.1. Let Δ be an orientable $(d-1)$-pseudomanifold over a field $\Bbbk$. Let A be an Artinian reduction of $\Bbbk[\Delta]$ with respect to Θ, and let $\Psi : A_d \to \Bbbk$ be the evaluation map. Then, under suitable normalization, for any length d sequence of vertices $J =$

[1]Here, $p'(v^*)$ has to be in sufficiently general position so that none of $[\sigma - v_i + v^*]$ vanishes. One may need to extend the field to choose such a vector.

$(v_1, \dots, v_d)$,

$$\Psi(x_J) = \sum_{\sigma\in\Delta:\text{ facet containing }\{J\}} \Psi(x_\sigma) \frac{\prod_{k=1}^{d}[\sigma + v^* - v_k]}{\prod_{v\in\sigma}[\sigma + v^* - v]}$$
$$= \sum_{\sigma\in\Delta:\text{ facet containing }\{J\}} \frac{1}{[\sigma]} \frac{\prod_{k=1}^{d}[\sigma + v^* - v_k]}{\prod_{v\in\sigma}[\sigma + v^* - v]} \tag{3.1}$$

holds. Here $\{J\}$ denotes the set $\{v_1, \dots, v_d\}$, and the sum is taken over all oriented facets of Δ containing $\{J\}$.

Throughout, we assume that the evaluation map Ψ is normalized so that (3.1) holds.

4 Generic Artinian reduction

4.1 Generic $\mathbb{N}^m$-graded Artinian reduction

For an $\boldsymbol{a}$-balanced simplicial complex (Δ, κ) with $\boldsymbol{a} \in \mathbb{N}_+^m$ and $|\boldsymbol{a}| = d$, we define the generic $\mathbb{N}^m$-graded Artinian reduction of $\mathbb{k}[\Delta]$ as follows. Fix a partition $\mathcal{I}_1 \sqcup \cdots \sqcup \mathcal{I}_m$ of $[d]$ with $|\mathcal{I}_j| = a_j$ for $j = 1, \dots, m$. Consider the set of new auxiliary indeterminates

$$\{p_{k,v} : k \in [d], v \in V(\Delta), k \in \mathcal{I}_{\kappa(v)}\}$$

and let $\widetilde{\mathbb{k}} = \mathbb{k}(p_{k,v})$ be the rational function field of these indeterminates with coefficients in $\mathbb{k}$. Define the $\mathbb{N}^m$-graded l.s.o.p. $\Theta = (\theta_1, \dots, \theta_d)$ by

$$\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_d \end{pmatrix} = \boldsymbol{P} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

where the (k, v)-th entry of the coefficient matrix $\boldsymbol{P}$ is $p_{k,v}$ if $k \in \mathcal{I}_{\kappa(v)}$ and 0 otherwise. The quotient $\mathbb{N}^m$-graded algebra $A = \widetilde{\mathbb{k}}[\Delta]/(\Theta)$ is called the *generic* $\mathbb{N}^m$*-graded Artinian reduction* of $\mathbb{k}[\Delta]$ (with respect to a coloring κ). Note that, when $m = 1$, the generic $\mathbb{N}$-graded Artinian reduction coincides with the generic Artinian reduction in the sense of [15]. We remark that to be consistent with the definition of [15], $A = \widetilde{\mathbb{k}}[\Delta]/(\Theta)$ is called the generic $\mathbb{N}^m$-graded Artinian reduction of $\mathbb{k}[\Delta]$, not of $\widetilde{\mathbb{k}}[\Delta]$, though A is the Artinian reduction of $\widetilde{\mathbb{k}}[\Delta]$ in the usual sense. By [16, Theorem 4.1], as an $\mathbb{N}^m$-graded algebra, A is decomposed into $\mathbb{N}^m$-homogeneous components as $A = \bigoplus_{\boldsymbol{0}\le\boldsymbol{b}\le\boldsymbol{a}} A_{\boldsymbol{b}}$. The homogeneous decomposition as $\mathbb{N}$-graded algebra is denoted as $A = \bigoplus_{i=0}^{d} A_i$.

4.2 Differential formula in characteristic 2

In the generic $\mathbb{N}$-graded Artinian reduction, the right-hand-side of (3.1) in Lee's formula is a rational function of new indeterminates $p_{k,v}$. Papadakis–Petrotou [15] considered taking partial derivative of (3.1) with respect to new indeterminates $p_{k,v}$, and they prove a remarkable formula in characteristic 2. This formula is later generalized by Karu–Xiao [11, Theorem 4.1]. We recall this formula.

In this subsection, we assume that the field $\Bbbk$ is of characteristic 2. Then automatically every pseudomanifold is orientable. For a pseudomanifold Δ, let $A = \Bbbk[\Delta]/(\Theta)$ be the generic ($\mathbb{N}$-graded) Artinian reduction of $\Bbbk[\Delta]$, where $\widetilde{\Bbbk} = \Bbbk(p_{kv} : k \in [d], v \in V(\Delta))$. For a length d sequence $I = (v_1, \dots, v_d)$ of vertices, define the differential operator ∂_I by $\partial_{p_{1,v_1}} \circ \cdots \circ \partial_{p_{d,v_d}}$, where $\partial_{p_{k,v}}$ denoted the (formal) partial derivative with respect to $p_{k,v}$. Under these notations, the following holds.

Theorem 4.1. [11, Theorem 4.1] Let Δ be a $(d-1)$-pseudomanifold, and let $\Bbbk$ be a field of characteristic 2. Let $A = \widetilde{\Bbbk}[\Delta]/(\Theta)$ be the generic $\mathbb{N}$-graded Artinian reduction of $\Bbbk[\Delta]$, where $\widetilde{\Bbbk} = \Bbbk(p_{kv} : 1 \le k \le d,\ v \in V(\Delta))$. Let $\Psi : A_d \to \widetilde{\Bbbk}$ be the evaluation map normalized as in Lemma 3.1. Then, for any length d sequences I and J of vertices,

$$\partial_I \Psi(x_J) = \Psi(\sqrt{x_I x_J})^2$$

holds. Here, for a monomial x_L, define its square root $\sqrt{x_L}$ by x_K if there is a monomial x_K with $x_K^2 = x_L$ and 0 otherwise.

We generalize the formula in Theorem 4.1 in the setting of generic $\mathbb{N}^m$-graded Artinian reduction by a simple trick of substitution. Let (Δ, κ) be an $\boldsymbol{a}$-balanced pseudomanifold and let $A = \widetilde{\Bbbk}[\Delta]/(\Theta)$ be the generic $\mathbb{N}^m$-graded Artinian reduction of $\Bbbk[\Delta]$, where $\widetilde{\Bbbk} = \Bbbk(p_{kv})$. We call a length d sequence of vertices $I = (v_1, \dots, v_d)$ (possibly with repetition) *κ-transversal* if $k \in \mathcal{I}_{\kappa(v_k)}$ for $k = 1, \dots, d$. Note that $I = (v_1, \dots, v_d)$ is a κ-transversal sequence if and only if there exist corresponding variables $p_{1,v_1}, \dots, p_{d,v_d}$. Note also that for every degree $\boldsymbol{a}$ monomial x_J in $\Bbbk[\boldsymbol{x}]$, J can be reordered into a κ-transversal sequence. For a κ-transversal sequence $I = (v_1, \dots, v_d)$, define the differential operator ∂_I by $\partial_{p_{1,v_1}} \circ \cdots \circ \partial_{p_{d,v_d}}$. The following differential formula for the map Ψ holds.

Lemma 4.2. Let (Δ, κ) be an $\boldsymbol{a}$-balanced $(d-1)$-pseudomanifold for $\boldsymbol{a} \in \mathbb{N}_+^m$ with $|\boldsymbol{a}| = d$ and let $\Bbbk$ be a field of characteristic 2. Let $A = \widetilde{\Bbbk}[\Delta]/(\Theta)$ be the generic $\mathbb{N}^m$-graded Artinian reduction of $\Bbbk[\Delta]$. Let $\Psi : A_{\boldsymbol{a}} \to \widetilde{\Bbbk}$ be the evaluation map normalized as in Lemma 3.1. Then, for any κ-transversal sequence I and any length d sequence J of vertices,

$$\partial_I \Psi(x_J) = \Psi(\sqrt{x_I x_J})^2$$

holds. Here, for a monomial x_L, define its square root $\sqrt{x_L}$ by x_K if there is a monomial x_K with $x_K^2 = x_L$ and 0 otherwise.

Proof. When $m = 1$, this formula coincides with the formula in Theorem 4.1. For general case, the identity is obtained by substituting $p_{k,v}$ to 0 in the formula in Theorem 4.1 for all indeterminates corresponding to pairs (k, v) with $k \notin \mathcal{I}_{\kappa(v)}$. Note that this substitution is a valid one since the denominator of (3.1) does not vanish after the substitution by the Kind–Kleinschmidt's criterion on l.s.o.p. for Stanley–Reisner ring [19, Lemma III.2.4]. □

Lemma 4.2 can be readily strengthened as below.

Corollary 4.3. Let (Δ, κ), d, A, Ψ be as in Lemma 4.2. For a κ-transversal sequence I, an element $g \in A_i$ with $i \leq \frac{d}{2}$, and a length $d - 2i$ sequence J of vertices,

$$\partial_I \Psi(g^2 x_J) = \Psi(g\sqrt{x_I x_J})^2$$

holds.

Proof. Writing $g = \sum_K \lambda_K x_K$ $(\lambda_K \in \widetilde{\Bbbk})$[2], we have

$$\begin{aligned}
\partial_I \Psi(g^2 x_J) &= \partial_I \Psi\left(\sum_K \lambda_K^2 x_K^2 x_J\right) && \text{(by characteristic 2)} \\
&= \sum_K \partial_I(\lambda_K^2 \Psi(x_K^2 x_J)) && \text{(by linearlity of } \Psi, \partial_I) \\
&= \sum_K \lambda_K^2 \partial_I \Psi(x_K^2 x_J) && \text{(by } \partial_{p_{k,v}}(f^2 g) = f^2 \partial_{p_{k,v}}(g) \text{ for } f, g \in \widetilde{\Bbbk} \text{ in char. 2)} \\
&= \sum_K \lambda_K^2 \Psi(x_K \sqrt{x_I x_J})^2 && \text{(by Lemma 4.2)} \\
&= \Psi(g\sqrt{x_I x_J})^2.
\end{aligned}$$

□

5 Multigraded strong Lefschetz property via anisotropy

Throughout this section, we assume that $\Bbbk$ is a field of characteristic 2 and (Δ, κ) is an $\boldsymbol{a}$-balanced homology sphere over $\mathbb{F}_2$ for $\boldsymbol{a} \in \mathbb{N}_+^m$ with $|\boldsymbol{a}| = d$. Let $A = \widetilde{\Bbbk}[\Delta]/(\Theta)$ be the generic $\mathbb{N}^m$-graded Artinian reduction of $\Bbbk[\Delta]$, where $\widetilde{\Bbbk} = \Bbbk(p_{kv})$. By Gorensteiness, the multiplication map $A_i \times A_{d-i} \to A_d \xrightarrow{\Psi} \widetilde{\Bbbk}$ is a nondegenerate for each $0 \leq i \leq d$. Hence, the multiplication map $A_{\boldsymbol{b}} \times A_{\boldsymbol{a}-\boldsymbol{b}} \to A_{\boldsymbol{a}} \xrightarrow{\Psi} \widetilde{\Bbbk}$ is nondegenerate for each $\boldsymbol{b} \in \mathbb{N}^m$ with $\boldsymbol{b} \leq \boldsymbol{a}$. We call this property as *multigraded Poincaré duality*.

Let $\Bbbk$ be a field and let $A = \bigoplus_{\boldsymbol{0} \leq \boldsymbol{b} \leq \boldsymbol{a}} A_{\boldsymbol{b}}$ be an Artinian Gorenstein standard $\mathbb{N}^m$-graded $\Bbbk$-algebra[3] with $A_{\boldsymbol{0}} \cong A_{\boldsymbol{a}} \cong \Bbbk$. We say that A has the *multigraded strong Lefschetz*

[2]Recall that we are abbreviating the projection from the polynomial ring to A.
[3]An $\mathbb{N}^m$-graded algebra is *standard* if it is generated by $A_{\boldsymbol{e}_1} \oplus \cdots \oplus A_{\boldsymbol{e}_m}$.

property (as an $\mathbb{N}^m$-graded algebra) if there is an element $\ell_j \in A_{\boldsymbol{e}_j}$ for each $j = 1, \dots, m$ such that the multiplication map

$$\times \boldsymbol{\ell}^{\boldsymbol{a}-2\boldsymbol{b}} : A_{\boldsymbol{b}} \to A_{\boldsymbol{a}-\boldsymbol{b}}$$

is an isomorphism for all $\boldsymbol{b} \in \mathbb{N}^m$ with $\boldsymbol{b} \le \frac{\boldsymbol{a}}{2}$, where $\boldsymbol{e}_j \in \mathbb{N}^m$ is the j-th unit coordinate vector. We prove the following.

Theorem 5.1. Let $\Bbbk$ be a field of characteristic 0 or 2 and let (Δ, κ) be an $\boldsymbol{a}$-balanced homology sphere over $\mathbb{F}_2$. Then the generic $\mathbb{N}^m$-graded Artinian reduction $A = \widetilde{\Bbbk}[\Delta]/(\Theta)$ of the Stanley–Reisner ring $\Bbbk[\Delta]$ has the multigraded strong Lefschetz property.

Here $\widetilde{\Bbbk}$ is a purely transcendental field extension of $\Bbbk$ resulted in the generic $\mathbb{N}^m$-graded Artinian reduction.

Our proof of Theorem 5.1 relies on anisotropy technique used in [2, 3, 11, 15]. For a vector space W over a field $\Bbbk$, a bilinear form $\varphi : W \times W \to \Bbbk$ is *anisotropic* if $\varphi(u, u) \neq 0$ holds for any nonzero $u \in W$. Note that a bilinear form is anisotropic if and only if the restriction $\varphi|_{W' \times W'}$ is nondegenerate for any nonzero subspace W' of W. We prove the following combination of anisotropy and multigraded strong Lefschetz property in a field of characteristic 2 with the explicit Lefschetz elements.

Theorem 5.2. Let (Δ, κ) be an $\boldsymbol{a}$-balanced homology sphere over $\mathbb{F}_2$, and let $\Bbbk$ be a field of characteristic 2. Let $A = \widetilde{\Bbbk}[\Delta]/(\Theta)$ be the generic $\mathbb{N}^m$-graded Artinian reduction of $\Bbbk[\Delta]$. Define $\ell_j = \sum_{v \in \kappa^{-1}(j)} x_v \in A_{\boldsymbol{e}_j}$ for $j = 1, \dots, m$. Then, for any $\boldsymbol{b} \in \mathbb{N}^m$ with $\boldsymbol{b} \le \frac{\boldsymbol{a}}{2}$, the bilinear form $\mathcal{Q} : A_{\boldsymbol{b}} \times A_{\boldsymbol{b}} \to \widetilde{\Bbbk}$ defined by

$$\mathcal{Q}(g, h) = \Psi(gh\boldsymbol{\ell}^{\boldsymbol{a}-2\boldsymbol{b}})$$

is anisotropic, where $\Psi : A_{\boldsymbol{a}} \to \widetilde{\Bbbk}$ is the evaluation map.

Toward the proof of Theorem 5.2, we first prove an auxiliary lemma, which can be seen as the combination of a multigraded version of weak Lefschetz property and anisotropy.

Lemma 5.3. Let (Δ, κ), $\boldsymbol{a}$, $\Bbbk$, A, ℓ_j be as in Theorem 5.2. Let S be a (possibly empty) subset of $[m]$ and let $\boldsymbol{e}_S = \sum_{j \in S} \boldsymbol{e}_j \in \mathbb{N}^m$ be the characteristic vector of S. For $\boldsymbol{b} \in \mathbb{N}^m$ with $2\boldsymbol{b} + \boldsymbol{e}_S \le \boldsymbol{a}$, define the bilinear form $\mathcal{Q}' : A_{\boldsymbol{b}} \times A_{\boldsymbol{b}} \to A_{2\boldsymbol{b}+\boldsymbol{e}_S}$ by

$$\mathcal{Q}'(g, h) = gh\boldsymbol{\ell}^{\boldsymbol{e}_S}.$$

Then $\mathcal{Q}'(g, g) \neq 0$ for any nonzero $g \in A_{\boldsymbol{b}}$,.

Proof. Suppose that g is a nonzero element of $A_{\boldsymbol{b}}$. As $A_{\boldsymbol{a}-\boldsymbol{b}}$ is generated by monomials, by multigraded Poincaré duality of A, there is a monomial x_K of degree $\boldsymbol{a}-\boldsymbol{b}$ such that $gx_K \neq 0$ in $A_{\boldsymbol{a}}$. Its square x_K^2 is of degree $2\boldsymbol{a}-2\boldsymbol{b}$, where $2\boldsymbol{a}-2\boldsymbol{b} \geq \boldsymbol{a}+\boldsymbol{e}_S$ by assumption. Hence there is a κ-transversal sequence I and a set of vertices $U^* \in V_S := \prod_{j\in S}\kappa^{-1}(j)$ and a length $d-2|\boldsymbol{b}|-|S|$ sequence of vertices J satisfying $x_K^2 = x_I x_{U^*} x_J$[4].

Now we have the following identity:

$$\begin{aligned} \partial_I \Psi(\mathcal{Q}'(g,g)x_J) &= \sum_{U\in V_S} \partial_I \Psi(g^2 x_U x_J) && \text{(by linearity of } \Psi\text{, } \partial_I \text{ and } \boldsymbol{\ell}^{\boldsymbol{e}_S} = \sum_{U\in V_S} x_U) \\ &= \sum_{U\in V_S} \Psi(g\sqrt{x_I x_U x_J})^2 && \text{(by Corollary 4.3)} \\ &\overset{(*)}{=} \Psi(g\sqrt{x_I x_{U^*} x_J})^2 = \Psi(gx_K)^2 && (5.1) \end{aligned}$$

Here, in $(*)$, we are using the fact that, by the definition of square root, for a fixed monomial $x_I x_J$, there is a unique squarefree monomial $x_{U'}$ with $\sqrt{x_I x_{U'} x_J} \neq 0$. By our choice of U^*, this is achieved by $x_{U'} = x_{U^*}$. As monomials x_U for $U \in V_S$ are all distinct and squarefree, the equality $(*)$ holds. Now, gx_K is a nonzero element in $A_{\boldsymbol{a}}$ and Ψ is an isomorphism, so we have $\Psi(gx_K)^2 \neq 0$. Hence, by the identity (5.1), $\partial_I \Psi(\mathcal{Q}'(g,g)x_J)$ must be nonzero. Therefore $\mathcal{Q}'(g,g)$ is nonzero. □

Now we are ready to prove Theorem 5.2.

Proof of Theorem 5.2. Suppose that $\mathcal{Q}(g,g) = 0$ holds for $g \in A_{\boldsymbol{b}}$. As Ψ is an isomorphism, we have $g^2\boldsymbol{\ell}^{\boldsymbol{a}-2\boldsymbol{b}} = 0$. By applying Lemma 5.3 for

$$g\prod_{j\in[m]} \ell_j^{\left\lfloor \frac{a_j-2b_j}{2} \right\rfloor}$$

and $S = \{j \in [m] : a_j - 2b_j \text{ is odd}\}$, we have

$$g\prod_{j\in[m]} \ell_j^{\left\lfloor \frac{a_j-2b_j}{2} \right\rfloor} = 0. \tag{5.2}$$

By multiplying g to both sides of (5.2), we obtain

$$g^2\prod_{j\in[m]} \ell_j^{\left\lfloor \frac{a_j-2b_j}{2} \right\rfloor} = 0.$$

By repeating in this way, we can reduce the power of ℓ_js' and we eventually obtain $g = 0$. □

[4] Since, for any degree $\boldsymbol{a}$ monomial x_L, L can be reordered into κ-transversal sequence, the desired decomposition $x_K^2 = x_I x_U x_J$ is obtained by assigning variables in greedy way.

Now Theorem 5.1 for characteristic 2 is immediate.

Proof of Theorem 5.1 for characteristic 2. Suppose that the field $\Bbbk$ is of characteristic 2. Define the Lefschetz elements ℓ_j for $j = 1, \ldots, m$ as in Theorem 5.2. Then, Theorem 5.2 implies that the linear map $\times \boldsymbol{\ell}^{\boldsymbol{a}-2\boldsymbol{b}} : A_{\boldsymbol{b}} \to A_{\boldsymbol{a}-\boldsymbol{b}}$ is injective for every $\boldsymbol{b} \le \frac{\boldsymbol{a}}{2}$. By multigraded Poincaré duality of A, we have $\dim A_{\boldsymbol{b}} = \dim A_{\boldsymbol{a}-\boldsymbol{b}}$, and thus the map is an isomorphism. □

Theorem 1.1 is readily obtained as a corollary of Theorem 5.1.

Proof of Theorem 1.1. By Theorem 5.1 over a field $\Bbbk$ of characteristic 2, the composite

$$A_{\boldsymbol{b}} \xrightarrow{\times \boldsymbol{\ell}^{\boldsymbol{c}-\boldsymbol{b}}} A_{\boldsymbol{c}} \xrightarrow{\times \boldsymbol{\ell}^{\boldsymbol{a}-\boldsymbol{b}-\boldsymbol{c}}} A_{\boldsymbol{a}-\boldsymbol{b}}$$

is an isomorphism. So, the linear map $\times \boldsymbol{\ell}^{\boldsymbol{c}-\boldsymbol{b}} : A_{\boldsymbol{b}} \to A_{\boldsymbol{c}}$ is injective. Thus, $h_{\boldsymbol{b}} = \dim A_{\boldsymbol{b}} \le \dim A_{\boldsymbol{c}} = h_{\boldsymbol{c}}$ holds. □

6 Further results in the full preprint

We end this extended abstract by listing the further contents of the full paper [14]. We give a proof of Theorem 5.1 for a field of characteristic 0. We further generalize Theorem 5.1 to manifolds and simplicial cycles and doubly Cohen–Macaulay complexes. This is a multigraded generalization of the almost strong Lefschetz property of manifolds [4, Section 8] and the strong Lefschetz property of simplicial cycles (after Gorensteinification) [2, Theorem I] and the top-heavy strong Lefschetz property for doubly Cohen–Macaulay complexes [2, Corollary 3.2]. A combinatorial corollary, we obtain a generalization of Theorem 1.1 to the flag h''-vector of manifolds (without boundary), which is a common generalization of manifold GLBI [13] and balanced manifold GLBI [9].

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References

[1] K. Adiprasito. "Combinatorial Lefschetz theorems beyond positivity". 2018. arXiv:1812.10454.

[2] K. Adiprasito, S. A. Papadakis, and V. Petrotou. "Anisotropy, biased pairings, and the Lefschetz property for pseudomanifolds and cycles". 2021. arXiv:2101.07245.

[3] K. Adiprasito, S. A. Papadakis, V. Petrotou, and J. K. Steinmeyer. "Beyond positivity in Ehrhart Theory". 2022. arXiv:2210.10734.

[4] K. Adiprasito and G. Yashfe. "The partition complex: an invitation to combinatorial commutative algebra". *Surveys in combinatorics 2021, London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, 2021, pp. 1–42. DOI.

[5] S. a. R. Gal. "Real root conjecture fails for five- and higher-dimensional spheres". *Discrete Comput. Geom.* **34**.2 (2005), pp. 269–284. DOI.

[6] M. Goff, S. Klee, and I. Novik. "Balanced complexes and complexes without large missing faces". *Ark. Mat.* **49**.2 (2011), pp. 335–350. DOI.

[7] T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi, and J. Watanabe. *The Lefschetz properties*. Vol. 2080. Lecture Notes in Mathematics. Springer, Heidelberg, 2013, xx+250 pp. DOI.

[8] M. Juhnke-Kubitzke and S. Murai. "Balanced generalized lower bound inequality for simplicial polytopes". *Selecta Math. (N.S.)* **24**.2 (2018), pp. 1677–1689. DOI.

[9] M. Juhnke-Kubitzke, S. Murai, I. Novik, and C. Sawaske. "A generalized lower bound theorem for balanced manifolds". *Math. Z.* **289**.3-4 (2018), pp. 921–942. DOI.

[10] G. Kalai, E. Nevo, and I. Novik. "Bipartite rigidity". *Trans. Amer. Math. Soc.* **368**.8 (2016), pp. 5515–5545. DOI.

[11] K. Karu and E. Xiao. "On the anisotropy theorem of Papadakis and Petrotou". *Algebr. Comb.* **6**.5 (2023), pp. 1313–1330. DOI.

[12] C. W. Lee. "P.L.-spheres, convex polytopes, and stress". *Discrete Comput. Geom.* **15**.4 (1996), pp. 389–421. DOI.

[13] I. Novik and E. Swartz. "Applications of Klee's Dehn-Sommerville relations". *Discrete Comput. Geom.* **42**.2 (2009), pp. 261–276. DOI.

[14] R. Oba. "Multigraded strong Lefschetz property for balanced simplicial complexes". 2024. arXiv:2408.17110.

[15] S. Papadakis and V. Petrotou. "The characteristic 2 anisotropicity of simplicial spheres". 2020. arXiv:2012.09815.

[16] R. P. Stanley. "Balanced Cohen-Macaulay complexes". *Trans. Amer. Math. Soc.* **249**.1 (1979), pp. 139–157. DOI.

[17] R. P. Stanley. "The number of faces of a simplicial convex polytope". *Adv. in Math.* **35**.3 (1980), pp. 236–238. DOI.

[18] R. P. Stanley. "On the number of faces of centrally-symmetric simplicial polytopes". *Graphs Combin.* **3**.1 (1987), pp. 55–66. DOI.

[19] R. P. Stanley. *Combinatorics and commutative algebra*. Second. Vol. 41. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1996, x+164 pp. DOI.

[20] T.-S. Tay and W. Whiteley. "A homological interpretation of skeletal ridigity". *Adv. in Appl. Math.* **25**.1 (2000), pp. 102–151. DOI.