Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #57, 12 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

Positivity Phenomena for Lattice Paths at $q = -1$

Sylvie Corteel*1, Alexander Lazar†2, and Anna Vanden Wyngaerd‡2

1 *IRIF, Université de Paris-Cité, Paris, France*
2 *Département de Mathématique, Université Libre de Bruxelles, Brussels, Belgium*

Abstract. The *valley delta square* conjecture proposes that the coefficients of the symmetric function $\frac{[n-k]_q}{[n]_q}\Delta_{e_{n-k}}\omega(p_n)$ can be expressed in terms of a certain class of *decorated square paths* with respect to the bistatistic $(\mathsf{dinv}, \mathsf{area})$. Inspired by recent positivity results of Corteel, Josuat-Vergès, and Vanden Wyngaerd, we study the evaluation of this enumerator at $q = -1$. By considering a cyclic group action on the decorated square paths, we show that $\left\langle \frac{[n-k]_q}{[n]_q}\Delta_{e_{n-k}}\omega(p_n), h_1^n \right\rangle\Big|_{q=-1}$ is 0 whenever $n - k$ is even, and is a positive polynomial related to the Euler numbers when $n - k$ is odd. We also show that the combinatorics of this enumerator is closely connected to that of the Dyck path enumerator for $\langle \Delta'_{e_{n-k-1}} e_n, h_1^n\rangle$ considered by Corteel–Josuat Vergès–Vanden Wyngaerd.

Keywords: Square paths, delta conjecture

1 Introduction

The symmetric function ∇e_n, also known as the Frobenius characteristic of the ring of diagonal coinvariants [11], has been the subject of a multitude of papers in algebraic combinatorics and related fields in the last three decades. In [8], the authors proposed a combinatorial formula of this remarkable function in terms of *decorated Dyck paths*, which was proved a decade later in [1]. This result is known as the *shuffle theorem*. Many special cases of this formula lead to interesting combinatorics: $\langle \nabla e_n, e_n\rangle$, $\langle \nabla e_n, e_{n-d}h_d\rangle$, $\langle \nabla e_n, h_{1^n}\rangle$ are the q,t-Catalan numbers, q,t-Schröder numbers and q,t-parking functions, respectively (see [6, 7]).

Many generalizations and analogues to the shuffle formula have been studied. In [9], the authors provide conjectural formulas for the symmetric function $\Delta'_{e_{n-k-1}} e_n$, in terms of *decorated Dyck paths*, which reduces to the shuffle theorem when $k = 0$. In this work, we consider the Hilbert series of the *valley version* of the *Delta conjecture*, which can be stated as follows

$$\langle \Delta'_{e_{n-k-1}} e_n, h_1^n\rangle = \sum_{P \in \mathsf{stLD}(n)^{\bullet k}} t^{\mathsf{area}(P)} q^{\mathsf{dinv}(P)},$$

*corteel@irif.fr.
†alexander.leo.lazar@ulb.be.
‡anna.vanden.wyngaerd@ulb.be.

where the sum is over *valley-decorated standardly labeled Dyck paths*.

Another result related to this story is the *square theorem*, which gives a formula for $\nabla \omega p_n$ in terms of *square paths* (conjectured in [18] and proved in [20]). A "Delta generalization" of the square theorem was proposed in [4]. Its Hilbert series is:

$$\frac{[n-k]_q}{[n]_q} \langle \Delta_{e_{n-k}} \omega(p_n), h_1^n \rangle = \sum_{P \in \mathsf{stLSQ}(n)^{\bullet k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)}, \tag{1.1}$$

where the sum is over *valley-decorated standardly labeled square paths*.

A number of elegant results have been found by setting one of the q, t-variables to 0 or 1 [15, 19, 14]. In [3], the authors study

$$D_{n,k} := \sum_{P \in \mathsf{stLD}(n)^{\bullet K}} (-1)^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)},$$

i.e. the specialization of the combinatorial enumerator of the valley Delta conjecture at $q = -1$, which has nicer properties than one might expect. They prove that

$$\sum_{k=0}^{n-1} D_{n,k} z^k = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathsf{inv}_3(\sigma)} z^{\mathsf{monot}(\sigma)}, \tag{1.2}$$

where inv_3 and monot are certain permutation statistics. When $z = 0$, the shuffle theorem implies that

$$\langle \nabla e_n, h_{1^n} \rangle|_{q=-1} = D_{n,0} = t^{\lfloor n^2/4 \rfloor} E_n(t),$$

where $E_n(t)$ is a known t-analog of the Euler numbers introduced in [12] and further studied in [2, 17]: it t-counts alternating permutations[1] with respect to $31-2$ patterns[2]. This polynomial identity is itself a t-refinement of a more classical identity relating the Euler numbers E_n to an alternating sum over the set of parking functions (c.f. [3]).

Inspired by [3], in this work we study

$$S_{n,k} := \sum_{P \in \mathsf{stLSQ}(n)^{\bullet k}} (-1)^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)},$$

i.e. the square paths enumerator in the valley delta square conjecture at $q = -1$, and find that it *also* has well-behaved combinatorics by providing two cancellation-free interpretations of the sum: one in terms of permutation enumeration (Theorem 4.5) and another in terms of equivalence classes of lattice paths (Theorem 5.5).

Moreover, we show that $S_{n,k} = 0$ when $n-k$ is even and that

$$\sum_{k=0}^{n-1} S_{n,k} z^k = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathsf{revmaj}(\sigma)} z^{\mathsf{parity-dec}(\sigma)}, \tag{1.3}$$

[1] A permutation σ is said to be *alternating* if $\sigma_1 > \sigma_2 < \sigma_3 > \cdots$.

[2] A $31-2$-pattern of a permutation $\sigma \in \mathfrak{S}_n$ is a pair (i, j) such that $1 < i+1 < j \leq n$ and $\sigma_{i+1} < \sigma_j < \sigma_i$.

where parity − dec is a combinatorial quantity (the number of decorations resulting from the parity decorating algorithm, see Proposition 4.6). The $z = 0$ case, combined with the square theorem, becomes

$$\langle \nabla \omega p_n, h_{1^n} \rangle|_{q=-1} = S_{n,0} = \begin{cases} [n]_t t^{\lfloor (n-1)^2/4 \rfloor} E_{n-1}(t), & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even} \end{cases},$$

and specializing to $t = 1$ yields

$$\sum_{P \in \mathsf{stLSQ}(n)} (-1)^{\mathsf{dinv}(P)} = \begin{cases} nE_{n-1}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases},$$

which are the coefficients of $\frac{x^n}{n!}$ in the Taylor series of $\frac{x}{\cos(x)}$ (OEIS sequence A009843).

We also discuss the strong connection between the valley Delta conjecture and the valley Delta square conjecture at $q = -1$. By Theorems 4.11 and 4.12 in [4], we have

$$\sum_{k=0}^{n-1} \frac{[n-k]_q}{[n]_q} \Delta_{e_{n-k}} \omega(p_n) \Bigg|_{q=-1} = \sum_{k=0}^{n-1} \Delta'_{e_{n-k-1}} e_n \Bigg|_{q=-1} = \nabla e_n|_{q=0}$$

Taking the scalar product with h_λ yields the multinomial coefficient $\begin{bmatrix} n \\ \lambda \end{bmatrix}_q$. When $\lambda = 1^n$, we establish on the combinatorial side that

Proposition 1.1.

$$\sum_{k=0}^{n-1} S_{n,k} = \sum_{k=0}^{n-1} D_{n,k} = [n]_t!$$

by exhibiting an explicit bijection of the combinatorial enumerators of both sides. Moreover, we show that we have the following recursive relationship:

Theorem 1.2. *For all $n, k \in \mathbb{N}$, we have*

$$S_{n,k} = \begin{cases} [n]_t (D_{n-1,k} + D_{n-1,k-1}), & \text{if } n-k \text{ is odd} \\ 0, & \text{if } n-k \text{ is even} \end{cases},$$

where $D_{-1,k} = D_{n,-1} = 0$.

The rest of the abstract is organized as follows: in Section 2, we define all the necessary combinatorics of lattice paths to state the Valley Delta square conjecture. In Section 3 we recall the notion of *schedule numbers*, which are a crucial tool in our proofs. Section 4 contains the motivation and statement of one enumerative formula for $S_{n,k}$, and a proof of the relationship between the enumerators of decorated Dyck paths and square paths at $q = -1$. In Section 5 we discuss a cyclic group action on the decorated labeled square paths, which yield the other formula for $S_{n,k}$. We think this group action will also be of independent interest.

(a) An element of $\mathsf{LSQ}(7)^{\bullet 3}$.

(b) An element of $\mathsf{stLSQ}(7)^{\bullet 3}$.

Figure 1: Labeled square paths.

2 Labeled lattice paths

In this section we give the definitions necessary to state the lattice path results in the Introduction. These definitions were introduced in [9, 4].

Definition 2.1. A *square* is any area-1 square in the plane whose vertices have integer coordinates. We will always consider steps of paths to be either left or top edges of squares, so we can refer to *the* square of a step without ambiguity. We refer to the collection of squares that have two vertices on the line $y = x + i$ as the *i-th diagonal*. The 0-diagonal is sometimes referred to as the *main diagonal*. The *bottom diagonal* of a path is the lowest diagonal that intersects with the path.

Definition 2.2. A *square path* of size n is a lattice path in the plane from $(0, 0)$ to (n, n), consisting of unit north and east steps, such that the final step is an east step. A square path is called a *Dyck path* if it stays weakly above the main diagonal.

Definition 2.3. Given a square path π, a *labeling* of π is a word w of positive integers whose i-th letter labels the i-th vertical step of π such that, when the label is placed in the square of its step, the labels appearing in squares of the same column are increasing from bottom to top. Such a pair (π, w) is called a *labeled square path*. The set of labeled square (respectively, Dyck) paths of size n is denoted by $\mathsf{LSQ}(n)$ (respectively $\mathsf{LD}(n)$).

A labeling of a path of size n is said to be *standard* if its labels are exactly $[n]$. The set of standardly labeled square (respectively, Dyck) paths is denoted $\mathsf{stLSQ}(n)$ ($\mathsf{stLD}(n)$).

Definition 2.4. The *area word* of a square path of size n is the word a of n integers whose i-th letter equals j if the starting point of the i-th vertical step lies on the line $y = x + j$. The *shift* of a square path π of area word a, is the absolute value of the minimum letter of its area word. Notice that the shift of a Dyck path is always 0. The *area* of a square path π with shift s is the number of whole squares between the paths and the line $y = x - s$.

Example 2.5. The path in Figure 1a has area word $(-1, -2, -1, 0, 0, 0, 0)$, shift 2, area 10.

Definition 2.6. Given $P := (\pi, w) \in \mathsf{LSQ}(n)$, the i-th vertical step of P is called a *contractible valley* if it is preceded by a horizontal step and the following holds: after replacing the two steps ┘ with ┌ (and accordingly shifting the i-th label one cell to the left), we still get a valid labeled path where labels are increasing in each column.

Definition 2.7. A *valley decorated labeled square (respectively, Dyck) path* is a triple (π, w, dv) where $(\pi, w) \in \mathsf{LSQ}(n)$ (respectively $\mathsf{LD}(n)$) and dv is some subset of the contractible valleys of (π, w). The elements of dv are called *decorations*, and we visualize them by drawing a $\bullet$ to the left of these contractible valleys. The set $\mathsf{LD}(n)^{\bullet k}$ denotes the decorated labeled Dyck paths with exactly k decorations.

Example 2.8. The path in Figure 1a has 4 contractible valleys: the vertical steps of indices $1, 2, 6, 7$. Three of them (2, 6 and 7) have been decorated.

Definition 2.9. Let $P := (\pi, w, dv) \in \mathsf{LSQ}(n)^{\bullet k}$ with area word a and (i, j) a pair of indices of vertical steps with $1 \leq i < j \leq n$. These steps are said to *attack each other*, or to be in an *attack relation* if either $a_i = a_j, w_i < w_j$ and $i \notin dv$; or $a_i = a_j + 1, w_i > w_j$ and $i \notin dv$. The set of such pairs of indices is denoted by $\mathsf{Attack}(P)$. An attack relation of the first kind is referred to as *primary dinv* and of the second kind as *secondary dinv*.

Definition 2.10. Given a path $P \in \mathsf{LSQ}(n)^{\bullet k}$ with area word a, we define its *dinv* to be

$$\mathsf{dinv}(P) := \#\mathsf{Attack}(P) + \#\{i \mid a_i < 0\} - k.$$

The second term of this sum is the number of labels in negative diagonals and is referred to as *tertiary* or *bonus* dinv.

Example 2.11. The attack relations for the path in Figure 1a are $(1, 2), (5, 6)$ and $(5, 7)$ (the first is secondary and the other two primary dinv). There are 3 labels under the line $x = y$, so 3 units of tertiary dinv. There are 3 decorated valleys so $\mathsf{dinv} = 3 + 3 - 3 = 3$.

k	$S_{1,k}$	$D_{1,k}$	$S_{2,k}$	$D_{2,k}$	$S_{3,k}$	$D_{3,k}$
0	1	1	0	t	$t^3 + t^2 + t$	$t^3 + t^2$
1			$t + 1$	1	0	$t^2 + 2t$
2					$t^2 + t + 1$	1

Table 1: Values of $S_{n,k}$ and $D_{n,k}$ for small n, k.

3 Schedule numbers

In our work, we make extended use of *schedule numbers*, which allow us to factor the q, t-enumerators of Dyck and square paths. They were introduced in [13] and have proven to be a very useful tool (see [21, 10, 16]).

Definition 3.1. Given $P \in \mathsf{stLSQ}(n)^{\bullet k}$ with shift s, set ρ_i to be the labels of P that lie in the $(i-s)$-th diagonal, written in decreasing order. Add a $\bullet$-symbol on top of each label that belongs to a decorated step. We define the *diagonal word* of P to be the word $\mathsf{dw}(P) := \rho_0\rho_1 \cdots$. The *shifted diagonal word* of P is the pair $\mathsf{sdw}(P) = (\mathsf{dw}(P), s)$.

Example 3.2. The diagonal word of the path in Figure 1b is $4\overset{\bullet}{1}65\overset{\bullet}{3}\overset{\bullet}{2}7$.

The diagonal word of such a path is a *decorated permutation.*

Definition 3.3. The set of decorated permutations $\mathfrak{S}_n^{\bullet}$ is the set of permutations of n where some letters are decorated with a $\bullet$. The subset where exactly k letters are decorated is denoted by $\mathfrak{S}_n^{\bullet k}$.

For all $P \in \mathsf{stLSQ}(n)^{\bullet k}$, $\mathsf{dw}(P) \in \mathfrak{S}_n^{\bullet k}$, but not every decorated permutation is the diagonal word of a path. We recall the following classical definition.

Definition 3.4. Given a permutation σ of n, its *major index* is

$$\mathsf{maj}(\sigma) = \sum_{i:\sigma_i > \sigma_{i+1}} i.$$

We denote by $\mathsf{revmaj}(\sigma)$ the major index of the permutation $\sigma_n\sigma_{n-1}\cdots\sigma_1$.

Proposition 3.5. *All paths with a common diagonal word σ share the same area,* $\mathsf{revmaj}(\sigma)$. *Paths with the same shifted diagonal word have the same bonus* dinv.

Definition 3.6. Given a decorated permutation $\sigma \in \mathfrak{S}_n^{\bullet}$ such that $\sigma = \rho_0\rho_1\cdots\rho_l$ and the ρ_i are the decreasing runs of σ. Given a shift $s \in \{0, \dots, l-1\}$, we call a run ρ_i *negative, zero* or *positive* with respect to s, if $i < s$, $i = s$ or $i > s$, respectively.

Let $\tilde{\rho}_i$ be the subword of undecorated letters of ρ_i. For $c \in \sigma$, define its *(shifted) schedule number* to be

$$w_{\sigma,s}(c) = \begin{cases} \#\{d \in \tilde{\rho}_i \mid d > c\} + 1, & \rho_i \text{ is zero and } c \text{ is undecorated} \\ \#\{d \in \tilde{\rho}_i \mid d > c\} + \#\{d \in \tilde{\rho}_{i-1} \mid d < c\}, & \rho_i \text{ is positive and } c \text{ is undecorated} \\ \#\{d \in \tilde{\rho}_i \mid d < c\} + \#\{d \in \tilde{\rho}_{i+1} \mid d > c\}, & \rho_i \text{ is negative or } c \text{ is decorated} \end{cases}$$

For ease of notation, $w_{\sigma,s}(c) = 0$ for any $s \in \mathbb{N}$ with $s \geq l$. For a path $P \in \mathsf{stLSQ}(n)^{\bullet k}$, its *schedule word* $\mathsf{sched}(P)$ is the word whose letters are $w_{\mathsf{sdw}(P)}(c)$ for $c \in \mathsf{sdw}(P)$.

Example 3.7. Let σ be the diagonal word of the path in Figure 1b. Then

$$\begin{array}{rl} c & 4\overset{\bullet}{1}65\overset{\bullet}{3}\overset{\bullet}{2}7 \\ w_{\sigma,1}(c) & 2212112 \end{array}$$

The following is a specialization of [16, Theorem 5] to standardly labeled paths.

Theorem 3.8. *For all $n, s, k \in \mathbb{N}$ and $\sigma \in \mathfrak{S}_n^{\bullet k}$ we have*

$$\sum_{\substack{P \in \mathsf{stLSQ}(n)^{\bullet k} \\ \mathsf{sdw}(P)=(\sigma,s)}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} = t^{\mathsf{revmaj}(\sigma)} q^{u(\sigma,s)} \prod_{c\in[n]} [w_s(c)]_q$$

where $u(\sigma, s)$ is the number of undecorated letters in negative runs.

Corollary 3.9. *Given $(\sigma, s) \in \mathfrak{S}_n^{\bullet k} \times \mathbb{N}$*

$$\prod_{c\in[n]} w_{\sigma,s}(c) = \#\{P \in \mathsf{stLSQ}(n)^{\bullet k} \mid \mathsf{sdw}(P) = (\sigma, s)\}.$$

4 Enumeration of decorated square paths at $q = -1$

Setting $q = -1$ in Theorem 3.8, we notice that if $(\sigma, s) \in \mathfrak{S}_n^{\bullet} \times \mathbb{N}$ has an even schedule number, the sum over paths with shifted diagonal word (σ, s) evaluates to 0. The following lemma tells us the same is true if (σ, s) has at least one schedule > 1.

Lemma 4.1. *Let $(\sigma, s) \in \mathfrak{S}_n \times \mathbb{N}$ such that $w_{\sigma,s}(c) > 0$ for all $c \in \sigma$. Then there exists a $j \in \mathbb{N}$ such that $\{w_{\sigma,s}(c) \mid c \in \sigma\} = [j]$.*

By Lemma 4.1 and Theorem 3.8 we conclude that when evaluating (1.1) at $q = -1$, we are left only with paths whose schedule word is 1^n:

$$S_{n,k} = \sum_{\substack{P \in \mathsf{stLSQ}(n)^{\bullet k} \\ \mathsf{sched}(P)=1^n}} (-1)^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)}. \tag{4.1}$$

Unfortunately, this formula is not yet cancellation-free. In view of Corollary 3.9, we know that a path whose schedule word is 1^n is entirely determined by its shifted diagonal word (σ, s). By Theorem 3.8, we know that the dinv of such a path is entirely accounted for by $u(\sigma, s)$. Thus, we may reformulate (4.1) as follows:

$$S_{n,k} = \sum_{\substack{(\sigma,s) \in \mathfrak{S}_n^{\bullet k} \\ w_{\sigma,s}(\sigma_i)=1 \ \forall i}} (-1)^{u(\sigma,s)} t^{\mathsf{revmaj}(\sigma)}. \tag{4.2}$$

A permutation σ appears on the right-hand side once for each shift for which its schedules are 1. We can show that each σ has a net contribution of 1. This motivates the following definition.

Definition 4.2. A permutation $\sigma \in \mathfrak{S}_n^{\bullet k}$ is an *alternating dinv representative* (ADR) if, for some shift s, (σ, s) has all (shifted) schedule numbers equal to 1. It is called a *Dyck* alternating dinv representative if this shift may be 0. Denote the sets of such representatives by $\mathsf{ADR}_{n,k}$ and $\mathsf{DADR}_{n,k}$, respectively.

Example 4.3. For example, $\sigma = \dot{7}84\dot{2}3561$ is an *alternating dinv representative*, since for shift 2 and 3 all the schedule numbers equal 1. It is not a Dyck alternating dinv representative because $w_{\sigma,0}(8) = 0$. See Example 5.6 for more details.

In [3], the authors established an enumeration formula for $D_{n,k}$ in terms of DADRs.

Theorem 4.4. *For all $n, k \in \mathbb{N}$, we have*

$$D_{n,k} = \sum_{\sigma \in \mathsf{DADR}_{n,k}} t^{\mathsf{revmaj}(\sigma)}$$

Theorem 4.5 (Cancellation Theorem – word formulation)**.** *For all $n, k \in \mathbb{N}$, we have*

$$S_{n,k} = \begin{cases} \sum_{\sigma \in \mathsf{ADR}_{n,k}} t^{\mathsf{revmaj}(\sigma)} & \text{if } n-k \text{ is odd} \\ 0 & \text{if } n-k \text{ is even.} \end{cases}$$

The following is proved via an explicit bijection, and gives a correspondence between the combinatorics at $q = -1$ of the Hilbert series of the valley Delta conjecture and the valley Delta square conjecture. Along with Theorem 4.5, it yields Formula (1.3).

Proposition 4.6. *For each permutation $\sigma \in \mathfrak{S}_n$, there exists exactly one Dyck ADR with underlying permutation σ; and exactly one ADR with underlying permutation σ and an odd number of undecorated letters.*

The next result allows us to prove Proposition 1.1. The proof of Theorem 1.2 uses related techniques, but is somewhat more involved.

Corollary 4.7. *There exists a bijection*

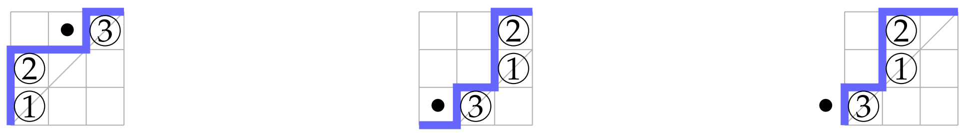
$$\phi : \bigsqcup_{\substack{k \in [n-1] \\ n-k \text{ is odd}}} \mathsf{ADR}_{n,k} \to \bigsqcup_{k \in [n-1]} \mathsf{DADR}_{n,k}.$$

The underlying permutations of σ and $\phi(\sigma)$ coincide, so $\mathsf{revmaj}(\sigma) = \mathsf{revmaj}(\phi(\sigma))$.

We can describe ϕ explicitly as follows. Let σ be an ADR with an odd number of undecorated letters. If the first decreasing run of σ contains no undecorated letters, remove the decoration from σ_1; if it contains exactly one undecorated letter, do nothing; and if it contains two undecorated letters, decorate σ_1.

Example 4.8. For $n = 3$ the bijection is given by the following correspondence

σ	123	231	$\dot{1}\dot{3}2$	312	$\dot{2}\dot{1}3$	$\dot{3}\dot{2}1$
$\phi(\sigma)$	123	231	$1\dot{3}2$	$\dot{3}12$	$2\dot{1}3$	$\dot{3}\dot{2}1$.



(a) $P = \psi_3(P)$ (b) $\psi_1(P)$ (c) $\psi_2(P)$

Figure 2: An element of $\mathsf{LSQ}(3)^{\bullet 1}$ and its images by the ψ_i. $\mathcal{CC}(P) = \{P, \psi_1(P)\}$.

5 Cutting cycles

Definition 5.1. Given $P \in \mathsf{LSQ}(n)^{\bullet k}$ and $i \in [n]$, split P into two decorated labeled subpaths $P_1 P_2$ where P_1 ends with the i-th east step of P. Define $\psi_i(P)$ to be $P_2 P_1$.

Note that $\psi_i(P)$ is not always a valid decorated labeled square path. In Figure 2, we draw a path $P \in \mathsf{LSQ}(3)^{\bullet 1}$ and its images by the ψ_i. We see that $\psi_2(P) \notin \mathsf{LSQ}(3)^{\bullet 1}$ because its first step is not a contractible valley, but is decorated. If the step following the i-th horizontal step of $P \in \mathsf{stLSQ}(n)^{\bullet k}$ is not a decorated valley then $\psi_i(P) \in \mathsf{LSQ}(n)^{\bullet k}$.

Definition 5.2. For any path $P \in \mathsf{LSQ}(n)^{\bullet k}$ its *cutting cycle* is the subset of the $\psi_i(P)$ which are valid decorated square paths; i.e., $\mathcal{CC}(P) := \{\psi_i(P) \mid i \in [n]\} \cap \mathsf{LSQ}(n)^{\bullet k}$.

Observations 5.3.

1. The cutting cycles partition $\mathsf{LSQ}(n)^{\bullet k}$.
2. For all $P \in \mathsf{LSQ}(n)^{\bullet k}$, $\#\mathcal{CC}(P) = n - k$.
3. Elements in the same cutting cycle have the same diagonal word and area.

Definition 5.4. Define an equivalence relation $\sim$ on $\mathsf{LSQ}(n)^{\bullet k}$ by $P \sim Q$ iff $Q \in \mathcal{CC}(P)$. For $n, k \in \mathbb{N}$,

$$\mathcal{S}_{n,k} := \{P \in \mathsf{stLSQ}(n)^{\bullet k} \mid \mathsf{sched}(P) = 1^n\} / \sim .$$

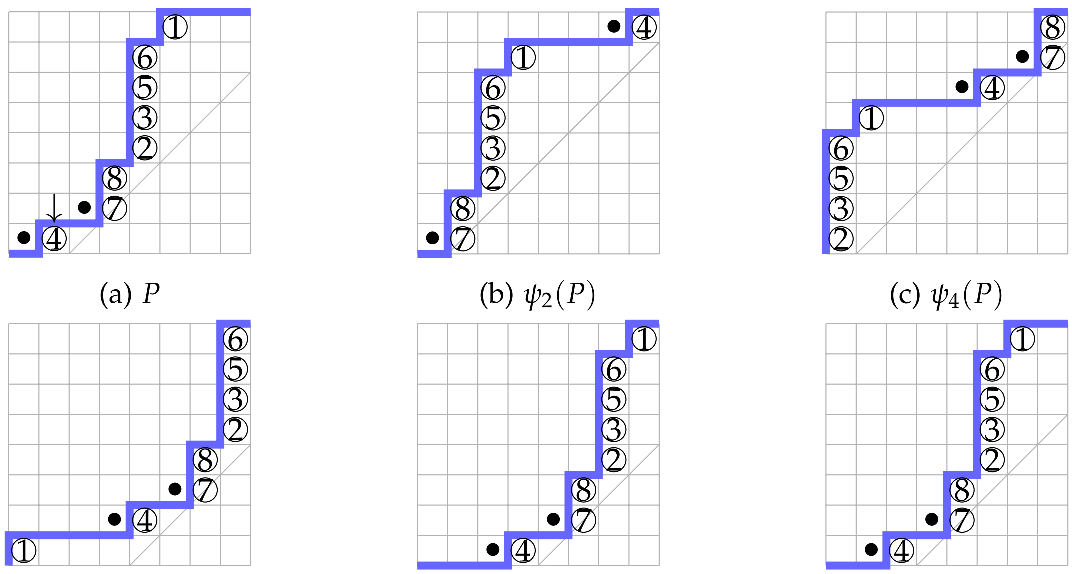
For $C \in \mathcal{S}_{n,k}$ we denote by $\mathsf{area}(C)$ and $\mathsf{dw}(C)$ the area and diagonal word of C.

Theorem 5.5 (Cancellation Theorem – path formulation)**.** *We have*

$$S_{n,k} = \begin{cases} \sum_{C \in \mathcal{S}_{n,k}} t^{\mathsf{area}(C)} & \text{if } n-k \text{ is odd} \\ 0 & \text{if } n-k \text{ is even} \end{cases}.$$

Example 5.6. In Figure 3, we show the cutting cycle of a schedule 1^n path $P \in \mathcal{S}_{8,2}$. Here we compute the schedule numbers for all possible shifts: there are 5 runs in the diagonal word $\dot{7}84\dot{2}3561$, so the nonzero schedules occur for shifts between 0 and 4:

c	$\dot{7}84\dot{2}3561$	c	$\dot{7}84\dot{2}3561$	c	$\dot{7}84\dot{2}3561$
$w_{\sigma,0}(c)$	10111111	$w_{\sigma,1}(c)$	11121111	$w_{\sigma,2}(c)$	11111111
$w_{\sigma,3}(c)$	11111111	$w_{\sigma,4}(c)$	11111112		



(a) P (b) $\psi_2(P)$ (c) $\psi_4(P)$ (d) $\psi_5(P)$ (e) $\psi_6(P)$ (f) $\psi_7(P)$

Figure 3: The equivalence class of a path P of schedule word 1^n

We confirm that P (see Figure 3a) has shift 2 and schedule word 1^n. There are no paths with diagonal word $\dot{7}84\dot{2}3561$ and shift 0, since one of the schedule numbers is 0. In P's cutting cycle there are 2 paths of shift 1, 1 of shift 3, and 2 of shift 4.

Lemma 5.7. *Take* $P \in \mathsf{stLSQ}(n)^{\bullet k}$ *such that* $\mathsf{sched}(P) = 1^n$. *There is an ordering* $Q_0, \dots, Q_{n-k-1}$ *of the elements of* $\mathcal{CC}(P)$ *such that* $\mathsf{dinv}(Q_i) = i$.

Corollary 5.8. *If* $P \in \mathsf{stLSQ}(n)^{\bullet k}$ *such that* $\mathsf{sched}(P) = 1^n$ *then*

$$\sum_{\substack{Q \in \mathcal{CC}(P) \\ \mathsf{sched}(Q)=1^n}} (-1)^{\mathsf{dinv}(Q)} = \begin{cases} 1 & \text{if } n-k \text{ is odd} \\ 0 & \text{if } n-k \text{ is even} \end{cases}.$$

Proposition 5.9. *Theorem 5.5 is equivalent to Theorem 4.5.*

6 Open questions and future work

In [16], the authors formulate a "modified" version of the valley Delta square conjecture, using the novel Theta operators from [5]: $\Theta_{e_k} \nabla \omega(p_{n-k})$ is given in terms of the subset of labelled decorated square paths that have at least one non-decorated vertical step on its lowest diagonal. The symmetric functions of both versions are related as follows:

$$\frac{[n]_t}{[n-k]_t} \Theta_{e_k} \nabla \omega(p_{n-k}) = \frac{[n-k]_q}{[n]_q} \Delta_{e_{n-k}} \omega(p_n).$$

When $q = -1$, the Hilbert series of $\Theta_{e_k} \nabla \omega(p_{n-k})$ seems to also always be t-positive and to equal 0 whenever $n-k$ is even. When trying to adapt the arguments of this paper to the "modified" setting, the combinatorics seem to behave less nicely. In particular, the sum over k does not yield $[n]_t!$.

Inspired by Theorem 1.2, we have the following conjectural formula.

Conjecture 6.1. *For all positive integers n and partitions λ of $n-1$, we have*

$$\left\langle \frac{[n-k]_q}{[n]_q} \Delta_{e_{n-k}} \omega(p_n), h_{(\lambda,1)} \right\rangle \Big|_{q=-1} = \begin{cases} [n]_t \left(\left\langle \Delta'_{e_{(n-1)-k-1}} e_{n-1} + \Delta'_{e_{(n-1)-(k-1)-1}} e_{n-1}, h_\lambda \right\rangle \right) \Big|_{q=-1} & \text{if } n-k \text{ is odd} \\ 0 & \text{if } n-k \text{ is even.} \end{cases}$$

Computational evidence suggests that setting $q = -1$ yields t-positive results for many other polynomials related to the shuffle theorem and Delta conjectures (as in [3]).

The following line of questions was generously suggested by an anonymous referee:

Question 6.2. *Suppose that $\langle F, h_1^n \rangle|_{q=-1}$ is t-positive. Can it be realized as the graded Euler characteristic of a complex which arises from the $\mathfrak{S}_n$-module whose Frobenius characteristic is F?*

Acknowledgements

Lazar and Vanden Wyngaerd were partially supported by ARC "From Algebra to Combinatorics, and Back". The authors wish to thank the referees for their close reading and thoughtful comments.

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