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The monopole-dimer model on high-dimensional cylindrical, toroidal, Möbius and Klein grids

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Abstract. The dimer (monomer-dimer) model deals with weighted enumeration of perfect matchings (matchings). The monopole-dimer model is a signed variant of the monomer-dimer model whose partition function is a determinant. In 1999, Lu and Wu [12] evaluated the partition function of the dimer model on two-dimensional grids embedded on a Möbius strip and a Klein bottle. While the partition function of the dimer model has been known for the two-dimensional grids with different boundary conditions, we present a similar product formula for the partition function of the monopole-dimer model on higher dimensional cylindrical and toroidal grid graphs. We also evaluate the same for the three-dimensional Möbius and Klein grid graphs and show that the formula does not generalise for the higher dimensions. Further, we present a relation between the product formula for the three-dimensional cylindrical and Möbius grid.

Keywords: Monopole-dimer model, Loop-vertex model, Determinantal formula, Boustrophedon labelling, Möbius strip, Klein bottle, Boundary conditions, Grid graphs.

1 Introduction

The *dimer model* is the study of the physical process of adsorption of diatomic molecules (like oxygen) on a solid surface. Its partition function can be interpreted as enumerating weighted perfect matchings in an edge-weighted graph. Kasteleyn [8] solved the problem completely for planar graphs, by showing that the partition function of the dimer model can be written as a Pfaffian of a certain adjacency matrix built using a special class of orientations called Pfaffian orientations on the graph. One immediate consequence of Kasteleyn's result is that the Pfaffian is unaffected by the choice of orientation on the planar graph. For the case of two-dimensional grid graphs $Q_{m,n}$, Kasteleyn [9] and Temperley–Fisher [4, 14] independently gave an explicit product formula for the partition function. For example, when m and n are even, horizontal (resp. vertical) edges have weight a (resp. b), the partition function of the dimer model on $Q_{m,n}$ can be written

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as

$$\prod_{i=1}^{m/2}\prod_{j=1}^{n/2}\left(4a^2\cos^2\frac{i\pi}{m+1}+4b^2\cos^2\frac{j\pi}{n+1}\right). \tag{1.1}$$

This formula is remarkable because although each factor is a degree-two polynomial in a and b with coefficients that may not be rational, the product turns out to be a polynomial with nonnegative integer coefficients. In particular, when $a = b = 1$, the result is an integer.

A similar product formula for the weighted enumeration of the perfect matchings has been given by McCoy and Wu [13] for the cylindrical and toroidal boundary conditions and by Lu and Wu [12] for the Möbius and Klein boundary conditions. Tesler [15] showed that the partition function of the dimer model on graphs embedded on non-orientable surfaces can be enumerated as a linear combination of some Pfaffians. Brankov and Priezzhev [3] gave explicit expressions for the free energy of the dimer model on finite quadratic lattices embedded on a Möbius strip.

Efforts have been made to generalise and extend the dimer model while preserving this elegant structure. The natural physical generalisation is the *monomer-dimer model*, which represents adsorption of a gas cloud consisting of both monoatomic and diatomic molecules. In the more abstract sense, it is the weighted enumeration of all matchings in a graph. Heilmann and Lieb [6] showed that the monomer-dimer model does not exhibit phase transitions. However, this problem is known to be computationally difficult to handle [7] and the partition function associated with it lacks a simple formula. A lower bound for the partition function of the monomer-dimer model for d-dimensional grid graphs has been obtained by Hammersley–Menon [5] by generalising the method of Kasteleyn and Temperley–Fisher.

In another direction, a signed variant of the monomer-dimer model called the *loop-vertex model* has been introduced by Ayyer [2] for oriented graphs. Configurations of the loop-vertex model can be thought of as superposition of two monomer-dimer configurations (matchings) having monomers (unmatched vertices) at the same locations. Consequently, loop-vertex configurations consist of even loops and isolated vertices. The loop-vertex model is less physical because of the presence of signs in their weights. On the other hand, the partition function here can be expressed as a determinant. Ayyer also provided an orientation-independent interpretation of this model called the monopole-dimer model for planar graphs. This interpretation has been extended by the author and Ayyer [1] for the Cartesian product of planar graphs.

The aim of this paper is to generalise the product formulas for the partition function of the dimer model on two-dimensional grids embedded on different surfaces to higher dimensions. We begin by introducing some notations and background results in Section 2. We define high-dimensional cylindrical and toroidal grid graphs and compute the partition function of the monopole-dimer model on them in Section 3. We also give the product formula for the partition function of the monopole-dimer model on the three-dimensional grids with Möbius and Klein boundary condition in Theorems 4.3 and 5.1. We show that the formulas do not hold for higher dimensions by providing counterexample in Example 4.4. Further, we establish a relationship between three-dimensional grids with cylindrical and Möbius boundary conditions in Theorem 4.5.

2 Background

A *graph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is the set of *vertices* of G and $E(G)$ is a collection of two-element subsets of $V(G)$, known as *edges*. We will only consider finite undirected labelled graphs. Unless stated otherwise, graphs will have the natural labelling $\{1, 2, \ldots, |V(G)|\}$. An *orientation* on a graph G is the assignment of arrows to its edges. A graph G with an orientation $\mathcal{O}$ is called an *oriented graph* and will be denoted as $(G, \mathcal{O})$. A *canonical orientation* on a labelled graph is obtained by orienting its edges from a lower labelled vertex to a higher labelled vertex. Recall that a *planar* graph is a graph which can be drawn in such a way that no edges will cross each other. Such an embedding of a planar graph is referred as a *plane graph* and it divides the whole plane into regions, each of which is called a *face*.

Definition 2.1. *Let G_1 and G_2 be two graphs. The Cartesian product of G_1 and G_2 is the graph denoted $G_1 \square G_2$ with vertex set $V(G_1) \times V(G_2)$ and edge set*

$$\left\{ ((u_1, u_2), (u_1', u_2')) \middle| \begin{array}{r} \text{either } u_1 = u_1' \text{ and } (u_2, u_2') \in E(G_2) \\ \text{or } u_2 = u_2' \text{ and } (u_1, u_1') \in E(G_1) \end{array} \right\}.$$

The above definition generalises to the *Cartesian product of k graphs* $G_1, \ldots, G_k$, denoted $G_1 \square \cdots \square G_k$. Throughout the text, we will denote the *path* graph and the *cycle* graph on n vertices as P_n and C_n, respectively. P_n and C_n are associated with canonical orientation, unless stated otherwise. We write $Q_{n_1,\ldots,n_d}$ for the d-dimensional grid graph which is the Cartesian product $P_{n_1} \square \cdots \square P_{n_d}$.

Definition 2.2 ([1, Definition 3.5])**.** *The* oriented Cartesian product *of naturally labeled and oriented graphs $(G_1, \mathcal{O}_1), \ldots, (G_k, \mathcal{O}_k)$ is the graph $G_1 \square \cdots \square G_k$ with orientation $\mathcal{O}$ given as follows. For each $i \in [k]$, if $u_i \to u_i'$ in $\mathcal{O}_i$, then $\mathcal{O}$ gives orientation $(u_1, \ldots, u_i, \ldots, u_k) \to (u_1, \ldots, u_i', \ldots, u_k)$ if $u_{i+1} + u_{i+2} + \cdots + u_k + (k - i) \equiv 0 \pmod 2$ and $(u_1, \ldots, u_i', \ldots, u_k) \to (u_1, \ldots, u_i, \ldots, u_k)$ otherwise.*

The graph in Figure 1 can be thought of as an oriented Cartesian product of path P_3 with itself which is naturally labeled consecutively from one leaf to another.

Definition 2.3. *An orientation on a plane graph G is said to be* Pfaffian *if it satisfies the property that each simple loop enclosing a bounded face has an odd number of clockwise oriented edges. This property is also known as the* clockwise-odd property*.*

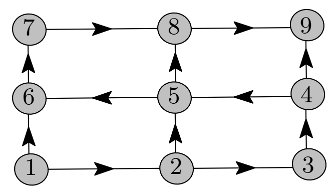


Figure 1: A Pfaffian orientation on grid graph, $Q_{3,3}$ induced from boustrophedon labelling.

For example, the orientation in Figure 1 is a Pfaffian orientation. Kasteleyn has shown that every plane graph possesses a Pfaffian orientation [8]. A *dimer covering* (*perfect matching*) is a collection of edges in the graph G such that each vertex is covered in exactly one edge. The set of all dimer coverings of G will be denoted as $\mathcal{M}(G)$. Let G be an edge-weighted graph on $2n$ vertices with edge-weight w_e for $e \in E(G)$. Then the *dimer model* is the collection of all dimer covers together with the weight of each dimer covering $M \in \mathcal{M}(G)$ given by $w(M) = \prod_{e \in M} w_e$. The *partition function* of the dimer model on G is then defined as

$$Z_G := \sum_{M \in \mathcal{M}(G)} w(M),$$

which is basically the weighted enumeration of perfect matchings in G. Kasteleyn has shown that Z_G, for a plane graph G, can be written as Pfaffian of a skew-symmetric matrix defined using a Pfaffian orientation on the plane graph G.

McCoy and Wu obtained a product formula for the two-dimensional grid graphs embedded on a cylinder and a torus similar to the one by Kasteleyn and Temperley–Fisher's formula in (1.1) for the two-dimensional grid graphs.

Theorem 2.4 ([13]). *The partition function of the dimer model on the two-dimensional grid graph $Q_{2m,2n}$ where horizontal (resp. vertical) edges have weight a (resp. b) with cylindrical and toroidal boundary conditions is given by*

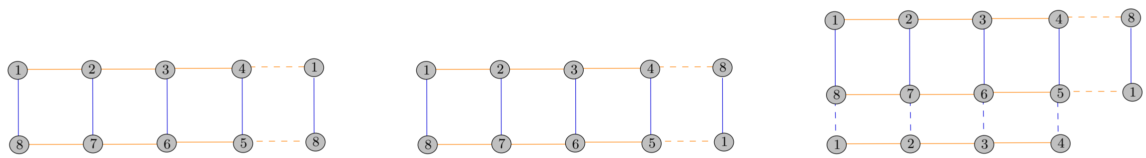
$$Z^{Cyl}_{Q_{2m,2n}} = \prod_{i=1}^{m}\prod_{j=1}^{n}\left(4a^2\sin^2\frac{(2i-1)\pi}{2m} + 4b^2\cos^2\frac{j\pi}{2n+1}\right), \tag{2.1}$$

and

$$Z^{Tor}_{Q_{2m,2n}} = \prod_{i=1}^{m}\prod_{j=1}^{n}\left(4a^2\sin^2\frac{(2i-1)\pi}{2m} + 4b^2\sin^2\frac{(2j-1)\pi}{2n}\right), \tag{2.2}$$

respectively.

Lu and Wu have obtained the similar closed-form expressions for the partition function of the dimer model on $2m \times 2n$ grids embedded on non-orientable surfaces like Möbius strip and Klein bottle.



(A) 4×2 grid on a cylinder (B) 4×2 grid on a Möbius strip (C) 4×2 grid on a Klein bottle

Figure 2: 2D grid with different boundary conditions

Theorem 2.5 ([12]). *Let $Q_{2m,2n} = P_{2m} \square P_{2n}$ denote the two-dimensional grid graph with horizontal (resp. vertical) edges having weight a (resp. b). The partition function of the dimer model on $Q_{2m,2n}$ embedded on a Möbius strip and on a Klein bottle is given by*

$$Z^{Möb}_{Q_{2m,2n}} = \prod_{i=1}^{m} \prod_{j=1}^{n} \left(4a^2 \sin^2 \frac{(4i-1)\pi}{4m} + 4b^2 \cos^2 \frac{j\pi}{2n+1} \right), \tag{2.3}$$

and

$$Z^{Klein}_{Q_{2m,2n}} = \prod_{i=1}^{m} \prod_{j=1}^{n} \left(4a^2 \sin^2 \frac{(4i-1)\pi}{4m} + 4b^2 \sin^2 \frac{(2j-1)\pi}{2n} \right), \tag{2.4}$$

respectively.

Figure 2 shows a two-dimensional grid embedded on different surfaces.

In this work, we will generalise Theorems 2.4 and 2.5 for a more general model called the monopole-dimer model. Let us first recall the loop-vertex model [2]. Loops in the configurations will refer to directed cycles in the graph. We assume all the weights are real and positive. Let G be a simple weighted graph on n vertices with an orientation $\mathcal{O}$, vertex-weights $x(v)$ for $v \in V(G)$ and edge-weights $a_{v,v'} \equiv a_{v',v}$ for $(v, v') \in E(G)$. A *loop-vertex configuration* C of G is a subgraph of the graph G consisting of

- directed loops of even length (with length ≥ 4),
- doubled edges (which can be thought of as loops of length 2),
- isolated vertices,

with the condition that each vertex of G is either covered in exactly one loop or is an isolated vertex. We will denote the set of all loop vertex configurations of G as $\mathcal{L}(G)$. Figure 3 shows a loop-vertex configuration on the grid graph $Q_{3,3}$ (see Figure 1).

The *sign* of an edge $(v, v') \in E(G)$, is defined as

$$\operatorname{sgn}(v, v') := \begin{cases} 1 & \text{if } v \to v' \text{ in } \mathcal{O}, \\ -1 & \text{if } v' \to v \text{ in } \mathcal{O}. \end{cases} \tag{2.5}$$

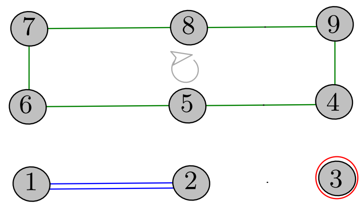


Figure 3: A loop-vertex configuration on $Q_{3,3}$ consisting of a doubled edge at $(1,2)$, a directed cycle (456789) and an isolated vertex at 3.

Let $\ell = (v_0, v_1, \ldots, v_{2k-1}, v_{2k} = v_0)$ be a directed even loop in G. The *weight* of the loop ℓ is given by

$$w(\ell) := -\prod_{i=0}^{2k-1} \operatorname{sgn}(v_i, v_{i+1})\, a_{v_i,v_{i+1}}. \tag{2.6}$$

A loop-vertex configuration, $C = (\ell_1, \ldots, \ell_j; v_1, \ldots, v_k)$ consisting of loops $\ell_1, \ldots, \ell_j$ and isolated vertices $v_1, \ldots, v_k$, is given the weight

$$w(C) = \prod_{i=1}^{j} w(\ell_i) \prod_{i=1}^{k} x(v_i). \tag{2.7}$$

Then the *loop-vertex model* on the oriented graph $(G, \mathcal{O})$ is the collection $\mathcal{L}(G)$ where the weight of each configuration, $C \in \mathcal{L}(G)$ is assigned a weight as specified in (2.7). The *(signed) partition function* of the loop-vertex model is defined as

$$\mathcal{Z}_{G,\mathcal{O}} := \sum_{C \in \mathcal{L}(G)} w(C).$$

Theorem 2.6 ([2, Theorem 2.5])**.** *The partition function of the loop-vertex model on $(G, \mathcal{O})$ is*

$$\mathcal{Z}_{G,\mathcal{O}} = \det(\mathcal{K}_{G,\mathcal{O}}),$$

where $\mathcal{K}_{G,\mathcal{O}}$ is a generalised adjacency matrix of $(G, \mathcal{O})$ defined as:

$$\mathcal{K}_{G,\mathcal{O}}(v, v') = \begin{cases} x(v) & \text{if } v = v', \\ a_{v,v'} & \text{if } v \to v' \text{ in } \mathcal{O}, \\ -a_{v,v'} & \text{if } v' \to v \text{ in } \mathcal{O}, \\ 0 & \text{if } (v, v') \notin E(G). \end{cases} \tag{2.8}$$

We will use $\mathcal{K}_G$ instead of $\mathcal{K}_{G,\mathcal{O}}$ whenever the underlying orientation is clear.

Remark 2.7. *In the case of oriented Cartesian product of plane graphs each with a Pfaffian orientation, the loop-vertex model is known as the* monopole-dimer model *and the weight of a loop $\ell = (v_0, v_1, \ldots, v_{2k-1}, v_{2k} = v_0)$ can be written* independent of the orientation *[1, Theorem 3.8].*

The author and Ayyer [1, Theorem 6.1] have extended the product formula in (1.1) for the monopole-dimer model on the higher dimensional grids, we present a particular case of their result.

First, let us define the *boustrophedon labelling*. Recall that P_n denotes the path graph on n vertices and $Q_{n_1,\dots,n_d}$ is the d-dimensional grid graph with side lengths $n_1,\dots,n_d$ which can be regarded as the Cartesian product of $P_{n_1},\dots,P_{n_d}$ denoted as $P_{n_1}\square\cdots\square P_{n_d}$. We will associate the boustrophedon labelling L_d (defined inductively) on $Q_{n_1,\dots,n_d}$ as follows:

For $d=1$, label L_1 is $1,2,\dots,n_1$. For $d>1$, $Q_{n_1,\dots,n_d}$ consists of n_d copies of $(d-1)$-dimensional grid graph $Q_{n_1,\dots,n_{d-1}}$. Successive copies (with successive last coordinate $1,2,\dots,n_d$) are labelled consecutively as $L_{d-1}, L'_{d-1}, L_{d-1}, L'_{d-1},\dots$ where L'_{d-1} represents the labelling of $(d-1)$-dimensional grid graph in reverse order of L_{d-1}. Figure 4 shows this labelling on the graph $Q_{4,2,2}$. Any snake-like labelling like the one above is a called

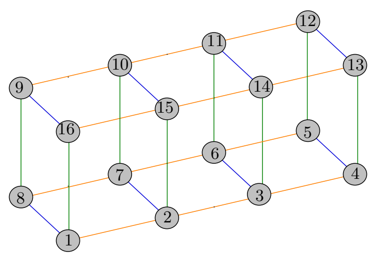


Figure 4: The boustrophedon labelling on $P_4\square P_2\square P_2$.

a *boustrophedon labelling*.

Theorem 2.8 ([1, Theorem 6.1])**.** *Let G be the d-dimensional grid graph $Q_{2m_1,\dots,2m_d}$ with boustrophedon labelling. Let $(G,\mathcal{O})$ be obtained from G by orienting the edges from a lower-labelled vertex to a higher-labelled vertex. Let the vertex weights be x for all vertices of G, and edge weights be $a_1,\dots,a_d$ for the edges along the different coordinate axes. Then the partition function of the monopole-dimer model on G is given by*

$$\mathcal{Z}_G \equiv \mathcal{Z}_{m_1,\dots,m_d} = \prod_{i_1=1}^{m_1}\cdots\prod_{i_d=1}^{m_d}\left(x^2+\sum_{q=1}^{d}4a_s^2\cos^2\frac{i_q\pi}{2m_q+1}\right)^{2^{d-1}}. \tag{2.9}$$

The monopole-dimer model reduces to the so-called double-dimer model [10, 11] when vertex weights are zero for all the vertices and there is only one plane graph in the Cartesian product. When $x=0$ and $d=2$ in (2.9), the partition function of the monopole-dimer model is the square of the partition function of the dimer model. In the following sections, we will extend the product formulas in Theorems 2.4 and 2.5 for the monopole-dimer model on higher dimensional grids with different boundary conditions. This attempt parallels the approach in Theorem 2.8 for higher dimensional grids. Recall the definition of the *Kronecker product* of two matrices.

Definition 2.9. *Let $A = (a_{i,j})$ be an $m \times n$ matrix and $B = (b_{i,j})$ be a $p \times q$ matrix, then the* Kronecker product, *$A \otimes B$, is an $mp \times nq$ block matrix defined as*

$$\begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

If A and B are square matrices of order n and p, respectively, then

$$\det A \otimes B = (\det A)^p (\det B)^n.$$

3 High-dimensional cylindrical and toroidal grid graphs

Let us now delve into the discussion regarding the partition function of the monopole-dimer model on higher dimensional grids with cylindrical and toroidal boundary conditions.

Definition 3.1. *We define an ℓ-*cylindrical grid *denoted $Q^{\ell}_{n_1,\dots,n_d}$ as the graph $C_{n_1} \square \cdots \square C_{n_\ell} \square P_{n_{\ell+1}} \square \cdots \square P_{n_d}$. For $\ell = 1$ ($\ell = d$), we call it a* cylindrical (toroidal) grid *and use the notation $Q^{Cyl}_{n_1,\dots,n_d}$ ($Q^{Tor}_{n_1,\dots,n_d}$).*

We sometimes refer to $Q^{\ell}_{n_1,\dots,n_d}$ as the d-dimensional grid $Q_{n_1,\dots,n_d}$ with cylindrical, toroidal and mixed boundary conditions depending on whether ℓ is $1, d$ or in between, respectively. Note that $Q^{\ell}_{n_1,\dots,n_d}$ with canonical orientation induced from boustrophedon labelling can be regarded as the oriented cartesian product of $C_{n_1}, \dots, C_{n_\ell}, P_{n_{\ell+1}}, \dots, P_{n_d}$. Thus, the loop-vertex model on an ℓ-cylindrical grid is nothing but the monopole-dimer model.

Theorem 3.2. *Let G be the ℓ-cylindrical grid graph $Q^{\ell}_{2m_1,\dots,2m_d}$ with boustrophedon labelling in d dimension. Let $(G, \mathcal{O})$ be obtained from G by orienting the edges from a lower-labelled vertex to a higher-labelled vertex. Let the vertex weights be x for all vertices of G, and edge weights be $a_1, \dots, a_d$ for the edges along the different coordinate axes. Then the partition function of the monopole-dimer model on G is given by*

$$\mathcal{Z}^{Mix}_{2m_1,\dots,2m_d} = \prod_{i_1=1}^{m_1} \cdots \prod_{i_d=1}^{m_d} \left(x^2 + \sum_{s=1}^{\ell} 4a_s^2 \sin^2 \frac{(2i_s - 1)\pi}{2m_s} + \sum_{t=\ell+1}^{d} 4a_t^2 \cos^2 \frac{i_t \pi}{2m_t + 1} \right)^{2^{d-1}}.$$

Figure 5 shows a three-dimensional grid graph with boustrophedon labelling and cylindrical boundary conditions. Using [1, Corollary 3.9], $\mathcal{Z}^{\mathrm{Cyl}}_G$ remains independent of the various Pfaffian orientations on $C_{2n_1}, P_{2n_2}, \dots, P_{2n_d}$. The formula for cylindrical and toroidal grids can be obtained from Theorem 3.2 using $\ell = 1$ and $\ell = d$, respectively.

For the rest of the paper, we will focus on higher dimensional Möbius and Klein grid graphs.

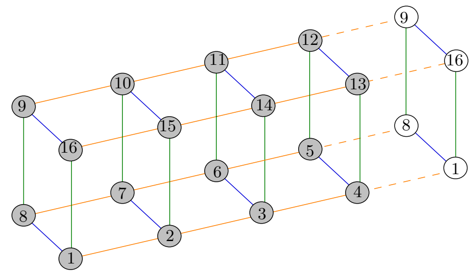


Figure 5: The boustrophedon labelling on the cylindrical grid $Q^{\text{Cyl}}_{4,2,2}$.

4 High-dimensional Möbius grid graphs

In this section, we will extend the product formula (2.3) for three-dimensional grids and show that the formula doesn't extend to higher dimensions in the obvious way.

Definition 4.1. *Let $Q_{n_1,\ldots,n_d}$ be the d-dimensional grid graph, we add an edge between the vertices $(1, k_2, \ldots, k_d)$ and $(n_1, n_2 - k_2 + 1, \ldots, n_d - k_d + 1)$ for all $1 \le k_i \le n_i$ $(2 \le i \le d)$ to obtain $Q_{n_1,\ldots,n_d}$ with* Möbius boundary condition *along the first direction. We call these edges as* dashed edges *and the remaining as* solid edges. *We call this new graph as d-dimensional* Möbius grid graph *and denote it as $Q^{\text{Möb}}_{n_1,\ldots,n_d}$.*

Let $G = Q^{\text{Möb}}_{n_1,\ldots,n_d}$ be the d-dimensional Möbius grid graph with boustrophedon labelling. Orient the solid edges from lower-labelled vertex to higher-labelled vertex, orient the dashed edge at 1 outward and the remaining dashed edges such that each two-dimensional square satisfies the clockwise-odd property. Let us denote the resulting oriented graph as $(G, \mathcal{O})$. We will always orient the edges coming from the Möbius boundary condition as described above. Figure 6 shows such an orientation over the Möbius grid graph $Q^{\text{Möb}}_{4,2,2}$.

Definition 4.2. *We define the* monopole-dimer model *on the d-dimensional Möbius grid graph G as the loop-vertex model on G with the above orientation $\mathcal{O}$.* The partition function *of the monopole-dimer model is then the partition function of the loop-vertex model.*

Theorem 4.3. *Let G be the three-dimensional Möbius grid graph $Q^{\text{Möb}}_{2m_1,2m_2,2m_3}$ with boustrophedon labelling. Let the vertex weights be x for all vertices of G, and edge weights be a_1, a_2 and a_3 for the edges along the x-,y- and z- coordinate axes respectively. Then the partition function of the monopole-dimer model on G is given by*

$$\mathcal{Z}^{\text{Möb}}_{2m_1,2m_2,2m_3} = \prod_{i_1=1}^{m_1}\prod_{i_2=1}^{m_2}\prod_{i_3=1}^{m_3}\left(x^2 + 4a_1^2\sin^2\frac{(4i_1-1)\pi}{4m_1} + 4a_2^2\cos^2\frac{i_2\pi}{2m_2+1} + 4a_3^2\cos^2\frac{i_3\pi}{2m_3+1}\right)^4.$$

The product formula in Theorem 4.3 remains unchanged even if one starts by orienting the dashed edge at 1 inward and the remaining dashed edges such that each two-dimensional

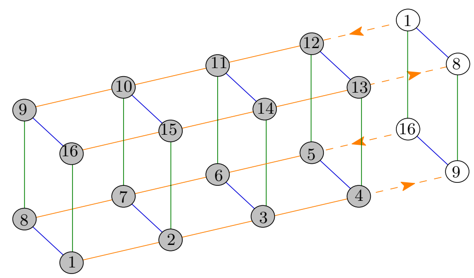


Figure 6: The three-dimensional Möbius grid graph $Q^{\text{Möb}}_{4,2,2}$.

square satisfies the clockwise-odd property. Note that the oriented d-dimensional Möbius grid $Q^{Möb}_{2m_1,\dots,2m_d}$ can be regarded as the oriented Cartesian product of $P_{2m_1},\dots,P_{2m_d}$ (oriented from one leaf to another) together with some additional dashed edges oriented in the specified way. We believe that the partition function of the monopole-dimer model remains unchanged regardless of the orientation on the path graphs. That's the reason for naming it as the monopole-dimer model.

The idea of the proof is to compute the determinant of the generalised adjacency matrix defined in (2.8), which can be written as a sum of four terms; each is a Kronecker product of three matrices. We use some unitary transforms to reach a stage where the partition function is the product of the determinant of some 2×2 block matrices. We now provide an example showing that the formula does not generalise for higher dimensions.

Example 4.4. *Let $G = Q^{Möb}_{2,2,2,2}$ be the four-dimensional Möbius grid . The solid edges are oriented from lower labelled vertex to higher labelled vertex and dashed edges are oriented as described in the paragraph just below Definition 4.1. Let the vertex weight be 0 for all the vertices and edge weights be a_1, a_2, a_3 and a_4 for the edges along the different coordinate axes. Then, the partition function of the monopole-dimer model on G is*

$$(4a_1^2 + a_2^2 + a_3^2 + a_4^2)^4 (a_2^2 + a_3^2 + a_4^2)^4$$

which is not an 8^{th} power. This leads us to conclude that the product formula in (2.3) does not generalise to higher dimensions.

The generalized relationship between the partition function of the monopole-dimer model on the three-dimensional Möbius and cylindrical grid graphs, akin to the relationship between the partition function of the dimer model on two-dimensional grids embedded on a cylinder and a Möbius strip [12, (24)] is given by the following result.

Theorem 4.5. *Let $\mathcal{Z}^{Cyl}_{4n_1,2n_2,2n_3}$ and $\mathcal{Z}^{Möb}_{2n_1,2n_2,2n_3}$ be the partition function of the monopole-dimer model on the three-dimensional Möbius grid $Q^{Möb}_{4n_1,2n_2,2n_3}$ and cylindrical grid $Q^{Cyl}_{2n_1,2n_2,2n_3}$ with boustrophedon labelling, respectively. Then*

$$\mathcal{Z}^{Cyl}_{4n_1,2n_2,2n_3} = \left(\mathcal{Z}^{Möb}_{2n_1,2n_2,2n_3}\right)^2. \tag{4.1}$$

5 High-dimensional Klein grid graphs

A d-dimensional Klein grid is defined similar to a d-dimensional Möbius grid by starting with $P_{n_1} \square C_{n_2} \square \cdots \square C_{n_d}$ instead of $Q_{n_1,\ldots,n_d} = P_{n_1} \square P_{n_2} \square \cdots \square P_{n_d}$. In other words, a d-dimensional grid is considered to have *Klein boundary conditions* if it exhibits Möbius boundary conditions along the first direction and cylindrical boundary conditions along the remaining directions. We define the *monopole-dimer model* on the d-dimensional Klein grid graph G similar to one defined for the Möbius grid.

Figure 7 shows an orientation over the Klein grid graph $Q^{\text{Klein}}_{4,2,2}$.

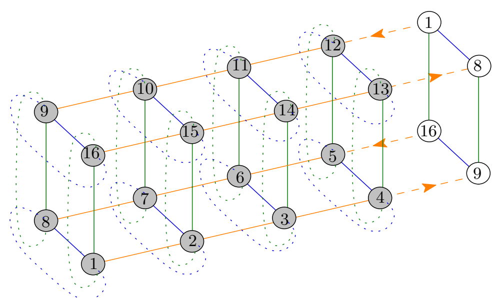


Figure 7: The three-dimensional Klein grid graph $Q^{\text{Klein}}_{4,2,2}$.

Theorem 5.1. *Let $G = Q^{Klein}_{2m_1,2m_2,2m_3}$ be the three-dimensional Klein grid graph. Let vertex weights be x for all vertices of G, and edge weights be a_1, a_2 and a_3 for the edges along the x-,y- and z- coordinate axes respectively. Then the partition function of the monopole-dimer model on $(G, \mathcal{O})$ is given by*

$$\mathcal{Z}^{Klein}_{2m_1,2m_2,2m_3} = \prod_{i_1=1}^{m_1} \prod_{i_2=1}^{m_2} \prod_{i_3=1}^{m_3} \left(x^2 + 4a_1^2 \sin^2 \frac{(4i_1-1)\pi}{4m_1} + 4a_2^2 \sin^2 \frac{(2i_2-1)\pi}{2m_2} + 4a_3^2 \sin^2 \frac{(2i_3-1)\pi}{2m_3} \right)^4.$$

We conclude by raising a pertinent question of whether is it possible to define the non-orientable boundary conditions such that the product formulas for higher dimensions still hold while preserving the relationship in (4.1).

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