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Rook matroids and log-concavity of P-Eulerian polynomials

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Abstract. We establish the ultra-log-concavity of P-Eulerian polynomials for naturally labeled posets P of width two. This takes a step towards resolving a log-concavity conjecture of Brenti (1989) and completes the story of the Neggers–Stanley conjecture in this special case. We do so by introducing *rook matroids*, the bases of which are certain restricted rook placements on a skew Ferrers board. The associated generating polynomial of these rook placements is ultra-log-concave. We exhibit a bijection between bases of the rook matroid and linear extensions of a width two poset from which the main result follows. Along the way, we study the structure theory of rook matroids and note that they form a subclass of transversal matroids and positroids. They also enjoy a strong correspondence with lattice path matroids.

Keywords: Matroid, transversal matroid, rook placements, Lorentzian polynomials, Neggers–Stanley conjecture.

1 Introduction

In 1972, Heilmann and Lieb established a landmark result in the context of the application of the geometry of polynomials in algebraic combinatorics, by proving that the matching polynomial of a graph is a real-rooted [25]. Shortly thereafter, Nijenhuis reproved the same result in the special case of bipartite graphs, albeit in the language of rook placements on boards [30]. Contemporarily, Stanley introduced the notion of (P, ω)-Eulerian polynomials—modeled after the ubiquitous Eulerian polynomials—that are denoted by $W_{P,\omega}$ and defined as the generating polynomial of the descent statistic of the set of linear extensions of the the labeled poset (P, ω). Following Neggers [29], he conjectured that this polynomial has only real roots [36]. This conjecture came to be known as the Neggers–Stanley conjecture, or simply the Poset conjecture; much work thereafter went in establishing this conjecture in special cases [34, 19, 39, 23]. The Poset conjecture was eventually disproved in the early 2000s—first by Brändén who treated

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Stanley's formulation [13] and then by Stembridge who gave a counterexample to Neggers's original statement [38]. The question then turned to determining the strongest distributional properties for this family of polynomials. Reiner and Welker [32] showed that in the case of naturally labeled, graded posets, one can associate a simplicial polytopal sphere to P the h-polynomial of which coincides with the P-Eulerian polynomial of P. The g-theorem for polytopes then implies that W_P is symmetric and unimodal. Brändén later extended that result by showing gamma-positivity of W_P for the larger class of sign-graded posets [15].

In the last two decades, the technology with which one can probe the distributional properties—real-rootedness, log-concavity and unimodality—of combinatorially defined polynomials has flourished. On the one hand, the geometry of polynomials has offered sophisticated multivariate techniques including the theory of stable polynomials, the apotheosis of which is the work of Brändén, Borcea and Liggett that established a powerful theory of negative dependence [11, 12, 10]. On the other hand, the perspective of Hodge theory has provided deeper algebro-geometric reasons behind these properties and also served as guiding motivation for the current frontier of the subject [26, 1, 4, 5, 33]. The theory of Lorentzian polynomials [17, 18] represents a synthesis of these two approaches; its advantage is its ability to streamline negative dependence results in disparate contexts — from convex and algebraic geometry to matroid theory and knot theory — into the language of polynomials and linear algebra. One such example of this is recent work establishing the log-concavity of the Alexander polynomial of special alternating links [24]. The work presented here continues that trend by settling the log-concavity part of the Neggers–Stanley conjecture in the case of naturally labeled width two posets. The polynomial in question turns out to have an interpretation as the generating polynomial of certain restricted rook placements, and in the spirit of Heilmann–Lieb, we probe the distributional properties of this polynomial. Our main tool is the construction of a a new matroid arising from non-nesting rook placements on a skew-shaped board. We show that the bases of this matroid are in bijection with linear extensions of a width two poset. Along the way, we highlight the rich structural properties of the rook matroid and describe their relation to other well-studied classes of matroids including transversal matroids, lattice path matroids, and positroids. This is an extended abstact of the preprint [2], where full proofs and further results can be found.

2 Rook matroids

2.1 Structural properties

Let λ/μ be a skew shape contained in an $r \times c$-rectangle. Throughout we draw and refer to Young diagrams λ and μ in terms of the English convention, that is as a leftjustified array of boxes. We index the rows of λ/μ by $\{1,2,\ldots,r\}$ and the columns by $\{r+1,\ldots,r+c\}$, and we set

$$E_{\lambda/\mu} := \{1,2,\ldots,r\} \cup \{r+1,\ldots,r+c\}.$$

Two rooks in a non-attacking rook placement on a board form a *nesting* if one rook lies South-East of another. Hereafter, a *non-nesting rook placement* (or just rook placement, when it is clear that the setting is a non-nesting one) on a board B is a non-attacking rook placement such that no pair of rooks forms a nesting.

Given a non-nesting rook placement ρ on λ/μ, we associate to it the set $R(\rho) \cup C(\rho)$ where $R(\rho)$ is the set of row indices occupied by ρ and $C(\rho)$ is the set of column indices that are *not* occupied by ρ. Note that for each non-nesting rook placement ρ, we have that $|R(\rho) \cup C(\rho)| = c$, the number of columns of the shape. We gather these sets into a single collection: $\mathcal{R}_{\lambda/\mu} := \{R(\rho) \cup C(\rho) : \rho \in \mathrm{NN}_{\lambda/\mu}\}$, where $\mathrm{NN}_{\lambda/\mu}$ is the set of non-nesting rook placements on λ/μ. We note that the correspondence $\rho \mapsto R(\rho) \cup C(\rho)$ is bijective.

Given a skew shape, we may define a set system as follows. For $j \in [c]$, let A_j be the set of row indices occupied by column $r+j$ together with the index $r+j$ itself; that is

$$A_j := \{i \in [r] : (i, r+j) \in \lambda/\mu\} \cup \{r+j\}.$$

Then $\mathcal{A}_{\lambda/\mu} := (A_1, \ldots, A_c)$ is a set system on $E_{\lambda/\mu}$ which defines a transversal matroid, which we also denote by $\mathcal{A}_{\lambda/\mu}$.

Example 2.1. Consider the skew shape $\lambda/\mu = 77553/42$. The corresponding set system is

$$\begin{array}{llll} A_1 = \{3,4,5,6\} & A_2 = \{3,4,5,7\} & A_3 = \{2,3,4,5,8\} & A_4 = \{2,3,4,9\} \\ A_5 = \{1,2,3,4,10\} & A_6 = \{1,2,11\} & A_7 = \{1,2,12\}. & \end{array}$$

Figure 1: The skew shape λ/μ and non-nesting rook placement ρ on λ/μ with $R(\rho) \cup C(\rho) = \{1,2,4,5,8,9,12\}$, a transversal of $\mathcal{A}_{\lambda/\mu}$.

Theorem 2.2. *The set $\mathcal{R}_{\lambda/\mu}$ is the set of maximal partial transversals of the set system $\mathcal{A}_{\lambda/\mu}$. In particular, $\mathcal{R}_{\lambda/\mu}$ is the set of bases of a transversal matroid on the ground set $E_{\lambda/\mu}$.*

Identifying the resulting matroid by its set of bases, we call $\mathcal{R}_{\lambda/\mu}$ the *rook matroid* on λ/μ; for any matroid M, we say that M is a rook matroid if it is isomorphic to a rook matroid of the form $\mathcal{R}_{\lambda/\mu}$. For example, the uniform matroid $U_{k,n}$ corresponds to the rook matroid on the $(n-k)\times k$ rectangle.

Remark 2.3. It is natural to ask if the skew shape assumption is necessary in the above theorem. Indeed, it is necessary; in upcoming work [27] it is shown that skew shaped boards are the only boards on which the the set of non-nesting rook placements forms the bases of a matroid. For a concrete example of this, consider the board B in Figure 2. If $\mathrm{NN}(B)$ denotes the set of non-nesting rook placements on B, then the collection of would-be bases of the rook matroid on B is (with same order as in Figure 2):

$$\{R(\rho)\cup C(\rho) : \rho \in \mathrm{NN}(B)\} = \{456, 146, 256, 246, 245, 346, 126, 234\}.$$

However, this does not satisfy the basis-exchange axiom: $6 \in \{1,4,6\} \setminus \{2,3,4\}$ but neither of $\{1,4,2\}$ or $\{1,4,3\}$ are contained in the collection above.

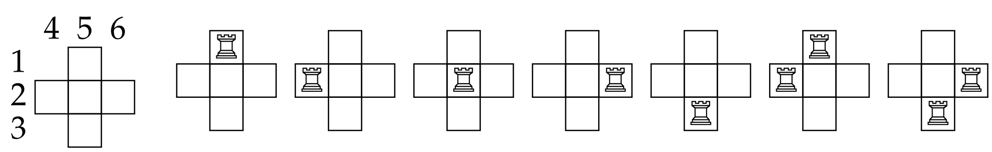


Figure 2: Board which does not admit a rook matroid structure, and the seven additional non-nesting rook placements.

Theorem 2.4. *The class of rook matroids $\mathcal{R}$ has the following properties:*

1. *$\mathcal{R}$ is closed under taking duals, direct sums but not under taking minors, free extensions, or truncations.*
2. *$\mathcal{R}$ contains the class of Schubert matroids (also known as generalized Catalan matroids).*
3. *The lattice path matroid $\mathcal{P}_{\lambda/\mu}$ is isomorphic to the non-nesting rook matroid $\mathcal{R}_{\lambda/\mu}$ if and only if λ/μ is a $332/1$-avoiding skew shape.*
4. *The Tutte polynomials of the rook matroid $\mathcal{R}_{\lambda/\mu}$ and the lattice path matroid $\mathcal{P}_{\lambda/\mu}$ are equal.*
5. *Every lattice path matroid can be obtained as the contraction of some rook matroid.*
6. *The rook matroid $\mathcal{R}_{\lambda/\mu}$ is a positroid.*

A few remarks are in order to contextualize Theorem 2.4. The failure of the class of rook matroids to be minor-closed can be partly explained by the presence of Q_6, the quaternary matroid on six elements, that is isomorphic to $\mathcal{R}_{332/1}$, and a known excluded minor of the class of lattice path matroids [7, Theorem 3.1]. Lattice path matroids form a well-studied class of transversal matroids [8, 9, 7] that are also supported on a skew shape. Property 3 yields a precise characterization for when the rook matroid is isomorphic to the lattice path matroid, that surprisingly hinges on whether Q_6 is a minor of the rook matroid. The correspondence between rook matroids and lattice path matroids is further strengthened by Property 4, which we conjecture in [2] holds more generally for all valuative invariants of matroids. Finally, and again in parallel to the lattice path matroid $\mathcal{P}_{\lambda/\mu}$ [31], the rook matroid $\mathcal{R}_{\lambda/\mu}$ can be seen to be a positroid. However, while the proof of this is straightforward for lattice path matroids due to their cyclically transversal presentation ([28, Theorem 4.6], [6, Theorem 5.2]), the rook matroid case requires the use of Oh's theorem characterizing positroids in terms of Grassmann necklaces of matroids [31]. Finally, Property 5 shows that rook matroids generalize lattice path matroids; in particular, we can obtain the positroidal structure of lattice path matroids using that of rook matroids and property (5).

2.2 Distributional properties

Let λ/μ be a skew shape with r rows and c columns, and let $r_k(\lambda/\mu)$ be the number of non-nesting rook placements on λ/μ of size k. The univariate non-nesting rook polynomial of λ/μ is defined as

$$M_{\lambda/\mu}(t) := \sum_{k=0}^{c} r_k(\lambda/\mu)t^k. \tag{2.1}$$

One motivating feature of this definition is that we can obtain the Narayana polynomials of type A and B (A001263 and A008459 respectively) as well as the Fibonacci polynomials (A011973) from $M_{\lambda/\mu}(t)$ by choosing the skew shape appropriately.

The basis-generating polynomial $r_{\lambda/\mu}(\mathbf{x},\mathbf{y})$ of $\mathcal{R}_{\lambda/\mu}$ serves as the appropriate multivariate generalization of $M_{\lambda/\mu}$, the univariate non-nesting rook polynomial in (2.1). To emphasize the role played by row and column variables separately, we use $\mathbf{x} = (x_1,\ldots,x_r)$ and $\mathbf{y} = (y_{r+1},\ldots,y_{r+c})$ for the row and column variables of $r_{\lambda/\mu}$ respectively. By Theorem 2.2, bases of rook matroids correspond to occupied row indices taken together with unoccupied column indices of rook placements; we can thus write

$$r_{\lambda/\mu}(\mathbf{x},\mathbf{y}) = \sum_{\rho\in \mathrm{NN}_{\lambda/\mu}} \prod_{i\in R(\rho)} x_i \prod_{j\in C(\rho)} y_j, \tag{2.2}$$

where as before $\mathrm{NN}_{\lambda/\mu}$ denotes the set of non-nesting rook placements on λ/μ and $R(\rho), C(\rho)$ correspond to the set of row indices occupied by ρ and the set of column indices that are not occupied by ρ respectively.

We use Lorentzian polynomials [17] to deduce the ultra-log-concavity of the non-nesting rook numbers.

Theorem 2.5. *For every skew shape λ/μ, the coefficient sequence of the non-nesting rook polynomial $M_{\lambda/\mu}(t) = \sum_{k=0}^{d} r_k(\lambda/\mu)t^k$ is ultra-log-concave with no internal zeros. That is,*

$$\left(\frac{r_k}{\binom{d}{k}}\right)^2 \geq \frac{r_{k-1}}{\binom{d}{k-1}} \cdot \frac{r_{k+1}}{\binom{d}{k+1}} \quad \textit{for all } 1 \leq k \leq d-1,$$

where d is the degree of $M_{\lambda/\mu}(t)$. Moreover, there exists a skew shape α/β such that $M_{\alpha/\beta}$ is not *real-rooted.*

Proof. The first part of the theorem follows from the fact that $r_{\lambda/\mu}$, being the basis-generating polynomial of the rook matroid, is Lorentzian [17, Theorem 3.10]. Specializing the $\mathbf{x}$ variables to t and the $\mathbf{y}$ variables to s in $r_{\lambda/\mu}(\mathbf{x}, \mathbf{y})$ yields a bivariate homogeneous Lorentzian polynomial [17, Theorem 2.10], the coefficients of which are $r_k(\lambda/\mu)$. This is equivalent to the coefficients of $M_{\lambda/\mu}(t)$ being ultra-log-concave with no internal zeros. The second part of the theorem follows from the skew shape – poset correspondence detailed in Theorem 3.4 together with Stembridge's counterexample to the Neggers–Stanley conjecture [38]. □

We note the parallel between the non-nesting rook polynomial and the classical rook polynomial: both are ultra-log-concave, but while the the full rook polynomial is real-rooted [25, 30], its non-nesting counterpart is in general *not* real-rooted. The context of this failure of real-rootedness is the Neggers–Stanley conjecture, see Section 3.1 below.

Real-rooted polynomials have a natural generalization in terms of stable polynomials. A polynomial $P(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$ is *stable* if $P(x_1, \dots, x_n) \neq 0$ whenever all x_i lie strictly in the upper half-plane. If all the coefficients of P are real we say that P is *real stable*. See [40, 16] for a survey of the theory of stable polynomials and applications in the fields of combinatorics, probability theory, statistical mechanics, and optimization. One context in which real-rootedness of a polynomial holds is if its multivariate analog is stable. Given our matroidal set-up, the multivariate analog of $M_{\lambda/\mu}$ is the basis-generating polynomial of $\mathcal{R}_{\lambda/\mu}$ in Equation (2.2). This question thus fits naturally into the theory of matroids: A matroid M is to have the *half-plane property (HPP)*, if the basis-generating polynomial of $M = (E, \mathcal{B})$ defined as

$$P_M(\mathbf{x}) = \sum_{B \in \mathcal{B}} \prod_{i \in B} x_i,$$

is stable. The class of HPP matroids was introduced and studied in [21], where a number of connections were drawn between matroids and stable polynomials, and the question was raised as to whether all transversal matroids are HPP. Although this was disproved shortly after [22], the counterexample given was not a lattice path matroid. The following result shows that even Catalan matroids [3]—lattice path matroids (or rook matroids) on the Ferrers shape $(n, n-1, \dots, 1)$—do not satisfy the half-plane property in general.

Theorem 2.6. *If M is a Catalan matroid of rank at least 10, then M is not HPP—it contains the lattice path matroid $N = \mathcal{P}_{666333}$ as a minor*[1]*, and P_N is not stable.*

Theorem 2.5 and the theorem above suggest that $M_{\lambda/\mu}$ is unlikely to be real-rooted in general; in the next section we see why this is true.

3 Neggers–Stanley conjecture

Recall that given a poset P, the *width* of P is the size of the largest antichain in P. A labeling of a poset P on n elements is a bijection $\omega : P \to [n]$. We say that ω is a *natural labeling*, or linear extension of P, if $i \prec j$ implies $\omega(i) < \omega(j)$. The *Jordan–Hölder set* $\mathcal{L}(P, \omega)$ of (P, ω) is the set of all permutations $\sigma \in \mathfrak{S}_n$ such that for every relation $i \prec j$ in P, we have that $\omega(i)$ precedes $\omega(j)$ in the one-line notation of the permutation σ.

In his PhD thesis [35], Stanley introduced the *(P, ω)-Eulerian polynomial*, also known as the *W-polynomial of P*, as the descent-generating polynomial of the Jordan–Hölder set of the labeled poset (P, ω):

$$W_{P,\omega}(t) := \sum_{\sigma \in \mathcal{L}(P,\omega)} t^{\mathrm{des}(\sigma)}.$$

When ω is natural, we simply write W_P, since the polynomial is independent of the choice of natural labeling. Observe that in the definitions of the multivariate analogs that follow, there is a dependence on the choice of natural labeling that we make explicit.

The distributional properties of $W_{P,\omega}$ were of early interest to combinatorialists working in poset theory. The following conjecture was first formulated by Neggers in 1978 for natural labelings; in 1986, Stanley extended it to arbitrary labelings. Subsequent references to this conjecture also called it the Poset conjecture [19]. For further background, see [16] or [19].

Conjecture 3.1 (Neggers–Stanley conjecture [29, 36])**.** *Let (P, ω) be a labeled poset. Then $W_{P,\omega}$ is real-rooted.*

The Neggers–Stanley conjecture was of central importance to algebraic combinatorics until its resolution in the negative in the 2000s: first by Brändén [13] who found a family of counterexamples to Stanley's formulation and then Stembridge [38] who disproved Neggers' counterpart as well. In both cases, the counterexample furnished was of a width two poset; Brändén's construction was non-naturally labeled while Stembridge's (larger) counterexample was naturally labeled. Despite this breakthrough, the question of unimodality or log-concavity of $W_{P,\omega}$ for general (P, ω) remained open. In particular, the following conjecture of Brenti has been open since 1989.

[1] If a matroid is HPP then all its minors are as well [21].

Conjecture 3.2 ([19, Conjecture 1.1])**.** *Let (P, ω) be a labeled poset. Then $W_{P,\omega}$ is log-concave with no internal zeroes.*

There have been two positive results towards this end. Reiner and Welker proved that when P is naturally labeled and graded, W_P is unimodal and symmetric [32]. Brändén then gave an elegant combinatorial proof of the same fact by demonstrating a stronger property: namely that W_P is γ-positive for the larger class of sign-graded posets [14, 15].

In the next subsection, we show a strengthening of Conjecture 3.2 for special posets P. Namely, we show that when P is naturally labeled and of width two, then W_P is ultra-log-concave. We do so by formulating an appropriate multivariate analog of W_P and recognizing it as the basis-generating polynomial of a rook matroid.

3.1 Matroidal lifts of P-Eulerian polynomials

Given a skew shape λ/μ, our goal is to obtain a suitable poset P such that the W-polynomial of P agrees with the non-nesting rook polynomial of λ/μ. We do so in Theorem 3.4. Before we state this, we introduce a multivariate analog of W_P for width two P. Our choice is inspired by, but slightly different from, the multivariate P-Eulerian polynomials considered by Brändén and Leander in [20], developed in order to place the Neggers–Stanley conjecture in a multivariate context.

For a naturally labeled poset (P, ω) of width two, we define the multivariate analog of a P-Eulerian polynomial. Let (P, ω) be a naturally labelled poset of width two. To simplify presentation, we assume throughout that P has two minimal elements and two maximal elements. Fix a decomposition of P into two chains, C_1 on r elements and C_2 on c elements. With respect to this chain decomposition[2], define $\widetilde{W}_{P,\omega} \in \mathbb{N}[x_e, y_{e'} : e \in C_1, e' \in C_2]$ as

$$\widetilde{W}_{P,\omega}(\mathbf{x}, \mathbf{y}) = \sum_{\sigma \in \mathcal{L}(P,\omega)} \prod_{i \in \mathrm{DB}(\sigma)} x_i \prod_{j \in \mathrm{CA}(\sigma)} y_j, \tag{3.1}$$

where

- $\mathrm{DB}(\sigma) := \{\sigma_i \in [r+c] : \sigma_{i-1} > \sigma_i\}$ is the set of descent bottoms of σ,
- $\mathrm{CA}(\sigma) := \{\sigma_i \in [r+c] : \sigma_i < \sigma_{i+1}, \sigma_i \in C_2\}$ is the set of column ascents, i.e. ascent bottoms of σ lying in C_2 and $\sigma_{r+c+1} := \infty$.

The latter condition implies that the element σ_{r+c} is an ascent bottom if and only if $\sigma_{r+c} = r + c$. Here CA stands for column ascent (made clear in Example 3.5). Note that $\widetilde{W}_{P,\omega}$ is multi-affine and homogeneous of degree $|C_2|$.

[2] We emphasize that when (P, ω) is naturally labeled and of width two, the multivariate polynomial $\widetilde{W}_{P,\omega}$ depends on the choice of chain decomposition of P, but the univariate polynomial $W_{P,\omega}$ does not. Irreducible width two posets have a unique decomposition into two (maximal) chains [38, Proposition 5.1].

Example 3.3. Suppose (P, ω) is the width two poset on $[5]$ with cover relations $1 \prec 3, 1 \prec 5, 2 \prec 4, 4 \prec 5, 2 \prec 3$ and chains $C_1 = \{1, 3\}$ and $C_2 = \{2, 4, 5\}$. The Jordan–Hölder set is $\mathcal{L}(P, \omega) = \{24153, 21453, 12453, 21435\ 24135, 12435, 21345, 12345\}$. The polynomial $\widetilde{W}_{P,\omega}$ is a polynomial in the variables x_1, x_3, y_2, y_4, y_5 and is equal to

$$\widetilde{W}_{P,\omega} = x_1x_3y_2 + x_1x_3y_4 + x_3y_2y_4 + x_1x_3y_5 + x_1y_2y_5 + x_3y_2y_5 + x_1y_4y_5 + y_2y_4y_5.$$

Also, recall that if $r_{\lambda/\mu}$ is the basis-generating polynomial of the rook matroid on λ/μ, then we can express it in terms of row variables $x_1, \ldots, x_r$ and column variables $y_{r+1}, \ldots, y_{r+c}$ as

$$r_{\lambda/\mu}(\mathbf{x}, \mathbf{y}) = \sum_{\rho \in \mathrm{NN}_{\lambda/\mu}} \prod_{i \in R(\rho)} x_i \prod_{j \in C(\rho)} y_j.$$

We can now state the main theorem of the section, which interprets the polynomial $\widetilde{W}_{P,\omega}$ as the basis-generating polynomial of a rook matroid.

Theorem 3.4. *There is a bijective correspondence between skew shapes λ/μ and naturally labeled posets (P, ω) of width two. Under this correspondence if $r_{\lambda/\mu}$ is the basis-generating polynomial of the rook matroid on λ/μ then $r_{\lambda/\mu}(\mathbf{x}, \mathbf{y}) \cong \widetilde{W}_{P,\omega}(\mathbf{x}, \mathbf{y})$, where $\cong$ denotes equality up to reindexing of the variables. In particular, we also have*

$$W_P(t) = M_{\lambda/\mu}(t). \tag{3.2}$$

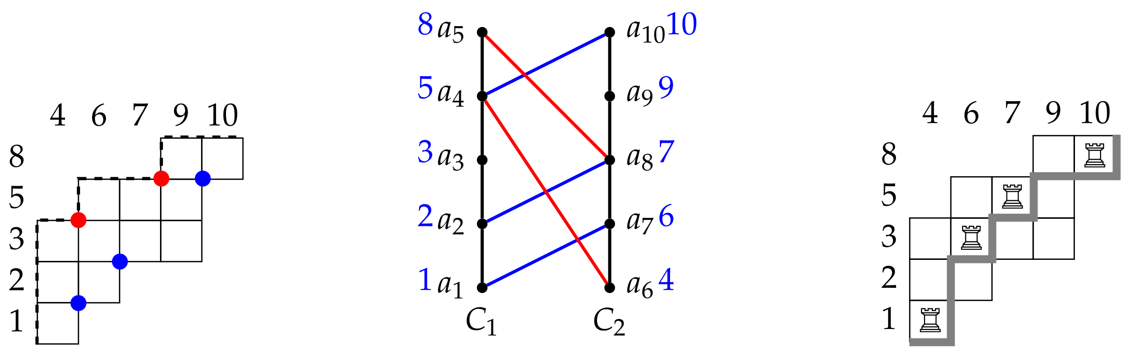
Hence[3] *the P-Eulerian polynomial W_P is ultra-log-concave for naturally labeled width two P.*

The fact that linear extensions of a poset are in bijection with lattice paths contained inside a compact polyhedral set is standard in the theory of distributive lattices [37, p. 296]. This idea underpins the above correspondence between skew shapes and posets of width two. We briefly illustrate one direction of this correspondence and the associated rooks-to-descents bijection in the following example. The reader can refer to [2] for the full proof.

Example 3.5. Consider the skew shape $\lambda/\mu = 54421/31$ shown in Figure 3a. We trace the innermost path of λ/μ with dashes and label the rows and columns according to the indices of the steps in this path. The labeled rows and columns become the chains C_1 and C_2 of the poset (P, ω) shown in Figure 3b. One can verify that ω here is indeed natural. The non-trivial cover relations of (P, ω) correspond to the inner and outer corners of the skew shape, as indicated. This correspondence between skew shapes and posets of width two is one-to-one. Non-nesting rook placements on λ/μ are in bijection with lattice paths contained inside λ/μ by specifying the valleys of the path precisely at the positions of the rook; this is shown in Figure 3c. The *path permutation* of the resulting lattice path is obtained by recording the row or column indices corresponding to the North or East steps of the path, as it is traversed from bottom to top. The path

[3] Since $\widetilde{W}_{P,\omega}$ is Lorentzian, so is its univariate specialization $W_P(t)$.

permutations are exactly the elements of $\mathcal{L}(P,\omega)$. This bijection sends the occupied row indices of rook placements to the descent bottoms of the inverses of linear extensions and unoccupied column indices to column ascents.



(a) Skew shape λ/μ with innermost path marked with dashes.

(b) Naturally labeled width two poset (P,ω) corresponding to skew shape λ/μ.

(c) Non-nesting rook placement on λ/μ with its corresponding lattice path in gray.

Figure 3: Skew shape — poset correspondence. Indices of North and East steps of the dashed path in (a) form two disjoint chains, C_1 and C_2 in (b). Blue and red cover relations correspond to outer and inner corners respectively. The lattice path in (c) has path permutation $\sigma = 41263759\,10\,8$. Here $\mathrm{DB}(\sigma) = \{1,3,5,8\}$ and $\mathrm{CA}(\sigma) = \{9\}$. The set $\mathrm{DB}(\sigma) \cup \mathrm{CA}(\sigma)$ represents the rook placement in (c).

Conjecture 3.1 follows from Theorem 3.4 for all naturally labeled posets of width two.

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