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Canon Permutation Posets

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Abstract. A permutation of the multiset $\{1^m, 2^m, \ldots, n^m\}$ is a *canon permutation* if the subsequence formed by the jth copy of each element of $[n] := \{1, 2, \ldots, n\}$ is identical for all $j \in [m]$. Canon permutations were introduced by Elizalde and are motivated by pattern-avoiding concepts, such as (quasi-)Stirling permutations. He proved that the descent polynomial of canon permutations exhibits a surprising product structure; as a further consequence, it is palindromic. Our goal is to understand canon permutations from the viewpoint of Stanley's (P, ω)-partitions, along the way generalizing Elizalde's definition and results. We start with a labeled poset P and extend it in a natural way to canon labelings of the product poset $P \times [n]$. The resulting descent polynomial has a product structure which arises naturally from the theory of (P, ω)-partitions. When P is graded, this theory also implies palindromicity. We include results on weak descent polynomials, an amphibian construction between canon permutations and multiset permutations, as well as γ-positivity and interpretations of descent polynomials of canon permutations.

Résumé. Una permutación del multiconjunto $\{1^m, 2^m, \ldots, n^m\}$ es una *permutación canon* si la subsecuencia formada por la j-ésima copia de cada elemento de $[n] := \{1, 2, \ldots, n\}$ es idéntica para todo $j \in [m]$. Las permutaciones canon fueron introducidas por Elizalde y están motivadas por conceptos que evitan patrones, como por ejemplo permutaciones (cuasi-)Stirling. Elizalde demostró que el polinomio de descensos de cualquier permutación canon exhibe una sorprendente estructura de producto; por lo tanto, es palindrómico. Nuestro objetivo es entender las permutaciones canon desde el punto de vista de las (P, ω)-particiones de Stanley, generalizando, a medida que avanzamos, la definición y los resultados de Elizalde. Partiendo de un poset etiquetado P, lo extendemos de manera natural a *etiquetamientos canon* del poset producto $P \times [n]$. El polinomio de descensos resultante tiene una estructura de producto que surge naturalmente de la teoría de (P, ω)-particiones. Cuando P es graduado, esta teoría también implica palindromicidad. Incluimos resultados sobre polinomios de descensos débiles,

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una construcción que generaliza permutaciones canon y permutaciones de multiconjuntos, así como la γ-positividad e interpretaciones de polinomios de descensos de permutaciones canon.

Keywords: labeled poset, canon permutation, Eulerian polynomial, Narayana number, gamma positivity.

1 Introduction

A permutation π of the multiset $\{1^m, 2^m, \ldots, n^m\}$ is a *canon permutation* if the subsequence formed by the jth copy of each element of $[n] := \{1, 2, \ldots, n\}$ is identical for all $j \in [m]$. For instance, 223143213144 is a canon permutation of $\{1^3, 2^3, 3^3, 4^3\}$, with the sequence formed by the jth copy being 2314. Canon permutations were introduced by Elizalde [7] and are motivated by pattern-avoiding permutations of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$, such as Stirling [9] and quasi-Stirling [1] permutations. For $m = 2$, canon permutations are sometimes called *nonnesting permutations*; they are precisely the permutations of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ avoiding the patterns 1221 and 2112. Following the notation in [7], we denote by $\mathcal{C}_n^{m,\sigma}$ the set of canon permutations whose jth copy of each entry forms the permutation $\sigma \in S_n$ and by $\mathcal{C}_n^m$ the set of all canon permutations for given m, n. Here, S_n denotes the set of permutations of $[n]$.

As usual, we call j a *descent* of π if $\pi(j+1) < \pi(j)$ and denote by $\operatorname{des}(\pi)$ the number of descents of π. We further denote by $\operatorname{wdes}(\pi)$ the number of *weak descents* of π, i.e., the number of positions j such that $\pi(j+1) \leqslant \pi(j)$. Elizalde's work centered on understanding the distribution of descents of canon permutations by studying the *descent polynomial*

$$C_n^m(x) := \sum_{\pi \in \mathcal{C}_n^m} x^{\operatorname{des}(\pi)}.$$

Let $A_n(x)$ be an Eulerian polynomial (i.e., the descent polynomial of permutations on $[n]$) and $N_n(x)$ a Narayana polynomial, enumerating high peaks in Dyck paths (we give a precise definition later). Both of these polynomials are *palindromic*, that is, their coefficient sequences are symmetric. Elizalde found that the descent polynomial $C_n^2(x)$ has the following surprisingly simple structure [7, Theorem 2.1].

Theorem 1.1 (Elizalde)**.** *For $n \geqslant 1$, $C_n^2(x) = A_n(x)\, N_n(x)$. In particular, $C_n^2(x)$ is palindromic.*

As Elizalde noted, the palindromicity of $C_n^2(x)$ is a priori unexpected. Elizalde proved a bivariate generalization of Theorem 1.1, involving also the number of the so-called *plateaus* of permutations. Another consequence of the bivariate extension of Theorem 1.1 is that the number of canon permutations with r weak descents is the same as that for $2n - r$ descents. (We will see an explanation of this palindromicity in Proposition 5.2.) In a follow-up paper [8], Elizalde extended Theorem 1.1 from $m = 2$ to the general case.

Here the Narayana polynomial is generalized so that the role of Dyck paths is replaced by standard Young tableaux of rectangular shape. Elizalde's proofs are bijective, and the resulting descent polynomial $C_n^m(x)$ is, again surprisingly, palindromic, due to the (older) fact that the generalized Narayana polynomials are palindromic [12].

Our goal is to understand Theorem 1.1 and its extensions from the viewpoint of Stanley's (P, ω)-partitions [10], along the way generalizing Elizalde's definition and results. We describe our *ansatz* next. Consider a poset P with m elements and a labeling ω (i.e., a bijection $\omega : P \to [m]$). We extend this labeling, via a given $\sigma \in S_n$, to the *canon labeling* $\omega \times \sigma$ of the poset $P \times [n]$ (here we think of $[n]$ as a poset—a chain) defined by

$$(\omega \times \sigma)(p, j) := \omega(p) + (\sigma(j) - 1)m\,.$$

A *linear extension* of $P \times [n]$ is defined, as usual, as an order-preserving bijection $\pi : [mn] \to P \times [n]$; writing π as a word in terms of the canon labeling $\omega \times \sigma$ of $P \times [n]$, it is a short step to view π as a permutation of the multiset $\{1^m, 2^m, \dots, n^m\}$. Let $\mathcal{C}_n^{P,\omega\times\sigma}$ consist of all such multiset permutations, and define

$$\mathcal{C}_n^{P,\omega} := \left\{ \mathcal{C}_n^{P,\omega\times\sigma} : \sigma \in S_n \right\}$$

which we call the set of all *canon permutations* of the labeled poset (P, ω). The classical canon permutations $\mathcal{C}_n^m$ stem from the case that $P = [m]$ is a chain and ω is the identity (i.e., $[m]$ is *naturally labeled*). For instance, starting with the 2-element chain poset, $P = [2]$, and the identity permutation of $[4]$, id $\in S_4$, the canon labeling $\omega \times \sigma$ of the product poset $[2] \times [4]$ is shown in Figure 1 (left). In this example, $\mathcal{C}_4^{P,\omega}$ is identified with the set $\mathcal{C}_4^2$ of canon permutations of $\{1, 1, 2, 2, 3, 3, 4, 4\}$.

Finally, we define the *canon polynomial*

$$C_n^{P,\omega}(x) := \sum_{\pi \in \mathcal{C}_n^{P,\omega}} x^{\operatorname{des}(\pi)}.$$

Again, we note the special case $C_n^m(x) = C_n^{[m],\mathrm{id}}(x)$. Indeed, in all but one of our applications, we will use a natural labeling for ω. The one exception (at least in this current work) is captured by the *weak descent polynomial* $C_n^{P,v}(x)$, where the labeling v is reverse natural. In the case $P = [m]$, this corresponds to the polynomial enumerating weak descents in canon permutations of $\{1^m, 2^m, \dots, n^m\}$. We will see that it is a translate of $C_n^{P,\omega}(x)$ for the posets that we study (Proposition 5.2 below). Our generalization of Theorem 1.1 and its extensions is as follows.

Theorem 1.2. *Let P be a poset with a natural labeling ω. Then $C_n^{P,\omega}(x) = A_n(x)\, h^*_{P\times[n]}(x)$. Furthermore, if P is graded then $C_n^{P,\omega}(x)$ is palindromic.*

Here P being *graded* means that all maximal chains have the same length, and $h^*_{P\times[n]}(x)$ is the numerator of the rational generating function of the order polynomial of $P \times [n]$, which is in a sense the descent polynomial of $P \times [n]$; we will give detailed definitions in Section 3. In fact, Theorem 1.2 is a special case of a general result (Theorem 3.5 below).

In Section 2, we give a poset model whose descent polynomial is the Narayana polynomial. The underlying posets turn out to be structured such that all labeled versions of them, in the sense of (P, ω)-partitions, have similar descent polynomials, from which Theorem 1.1 follows in a few short steps, as we outline in Section 3. The philosophy of our *ansatz* is that we consider the poset from Section 2, whose descent polynomial is the Narayana polynomial, and then sum (P, ω)-descent polynomials over certain labelings ω of this fixed poset, which gives rise to the product structure exhibited in Theorem 1.1. Our proof generalizes immediately to the canon polynomial $C_n^m(x)$ for general m. The palindromicity of canon polynomials follows organically from the structure of the involved posets; indeed, we give three direct explanations for this palindromicity (Theorem 3.5, Proposition 3.6, and Theorem 4.2 below).

In Section 4, we extend our results to subposets of $P \times [n]$ with some relations missing between different copies of P, i.e., those of the form $(p, j) \prec (p, j+1)$. These subposets give intermediary descent polynomials situated between those for canon permutations and multiset permutations. We show that palindromicity extends to this class.

A distributional property that is stronger than palindromicity (and unimodality [5]) is γ-positivity: a palindromic polynomial of degree d is *γ-positive* if its coefficients are non-negative when expressed in the *γ-basis* $\{x^i(x+1)^{d-2i} : 0 \leqslant i \leqslant \lfloor d/2 \rfloor\}$ (see, for instance, [2]). In Section 5, we show that our viewpoint implies that the descent polynomials of canon permutations are γ-positive and give a combinatorial interpretation for their coefficients, based on a result of Brändén. We conjecture the intermediate class of polynomials constructed in Section 4 to have γ-positive descent polynomials and ask for a γ-coefficient interpretation, which would recover γ-positivity exhibited by both canon permutations and multiset permutations.

2 Narayana Polynomials as Descent Polynomials

We start, as a warm-up of sorts, by realizing the Narayana polynomials as descent polynomials of the posets $[2] \times [n]$. As usual, the set D_n of *Dyck paths* consists of all lattice paths from $(0,0)$ to (n,n) with steps $e := (1,0)$ and $n := (0,1)$ that do not go above the diagonal $y = x$. A *peak* in a Dyck path is an occurrence of two adjacent steps *en*. A peak is called a *high peak* if these steps do not touch the diagonal. We denote the number of high peaks of $D \in D_n$ by $\mathrm{hpea}(D)$. For example, the Dyck path from $(0,0)$ to $(4,4)$ appearing in Figure 1 has three peaks out of which one is a high peak. We refer to the polynomial $N_n(x) := \sum_{D \in D_n} x^{\text{hpea}(D)}$ as a *Narayana polynomial*. (Our convention is slightly nontraditional, as the *Narayana numbers* are the coefficients of $x\, N_n(x)$.)

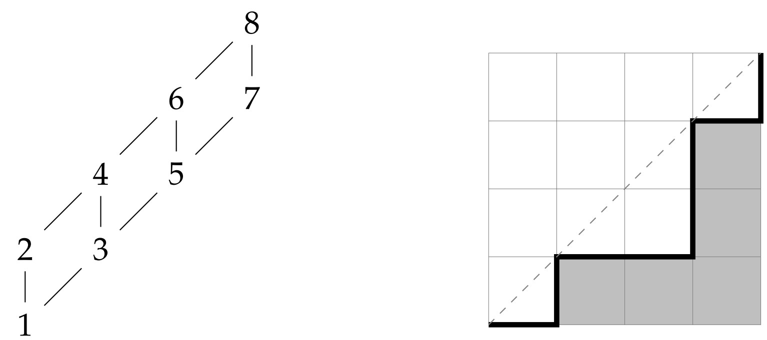

Figure 1: The poset $[2] \times [4]$ with a natural labeling (left). The Dyck path corresponding to the linear extension 12354678 of $[2] \times [4]$ (right).

The following can be proved bijectively.

Theorem 2.1. *The linear extensions of the poset* $[2] \times [n]$ *are in bijection with the Dyck paths in* D_n*. Furthermore, the descents of a linear extension of* $[2] \times [n]$*, labeled via* id $\times$ id*, are in bijection with the high peaks of the corresponding Dyck path.*

For example, the linear extension $12354678 \in \mathcal{L}([2] \times [4], \text{id} \times \text{id})$ corresponds to the Dyck path shown in Figure 1.

Corollary 2.2. *Let* $\mathcal{L}([2] \times [n])$ *denote the set of linear extensions of* $[2] \times [n]$*. Then*

$$\sum_{\pi \in \mathcal{L}([2] \times [n])} x^{\text{des}(\pi)} = N_n(x)\,.$$

The analogy between descents of linear extensions and high peaks of Dyck paths extends to weak descents and (ordinary) peaks, and the bijection discussed extends similarly.[1]

Remark 2.3. Viewing the descent polynomial of $[2] \times [n]$ as the Ehrhart h^*-polynomial of the order polytope of $[2] \times [n]$, it is a curious fact that the positive root polytope of type A_{n-1} has the same h^*-polynomial [6, Example 6].

3 Order Polynomials of Labeled Posets

We now recall some fundamental definitions and results on labelled posets and their order polynomials and generating functions [10].

[1] We note that Corollary 2.2 also follows from work of Sulanke [12].

Definition 3.1. *Let P be a poset of cardinality m with a given labeling $\omega : P \to [m]$. A (P,ω)-*partition *is a map $\sigma : P \to \mathbb{Z}_{\geqslant 0}$ satisfying the following conditions:*

- *If $s \leqslant t$ in P, then $\sigma(s) \leqslant \sigma(t)$; in other words, σ is order-preserving.*[2]
- *If $s < t$ and $\omega(s) > \omega(t)$, then $\sigma(s) < \sigma(t)$.*

Let $\Omega_{P,\omega}(j)$ be the number of (P,ω)-partitions $\sigma : P \to [j]_0 := \{0,1,\ldots,j\}$.

The function $\Omega_{P,\omega}(j)$ turns out to be a polynomial of degree m, called the *order polynomial* of (P,ω).[3] Subsequently, we may define the *h^*-polynomial* of a labeled poset (P,ω), $h^*_{P,\omega}(x)$, via

$$\sum_{j\geqslant 0} \Omega_{P,\omega}(j)\, x^j \;=\; \frac{h^*_{P,\omega}(x)}{(1-x)^{m+1}}\,.$$

When ω is a natural labeling, we denote $h^*_{P,\omega}(x)$ by $h^*_P(x)$. For example, we can rephrase Corollary 2.2 as $h^*_{[2]\times[n]}(x) = N_n(x)$.

Parallel to the classical case, we may think of a linear extension σ of P as a permutation of ω; we denote the set of all such linear extensions by $\mathcal{L}(P,\omega)$. The fundamental property of order polynomials is the following [10, Proposition 13.3].

Theorem 3.2 (Stanley). $h^*_{P,\omega}(x) = \sum_{\sigma\in\mathcal{L}(P,\omega)} x^{\operatorname{des}(\sigma)}$.

Given a poset P with a labeling ω and a chain C of P, a *descent* of C is any occurrence of a cover relation $a < b$ in C with $\omega(a) > \omega(b)$. We define $\operatorname{Des}(C,\omega)$ to be the set of descents in C and let $\operatorname{des}(C,\omega)$ denote the cardinality of $\operatorname{Des}(C,\omega)$. The following theorem, which was stated in [11, Theorem 4.1] using the notion of shift equivalence, provides a sufficient condition for the h^*-polynomials corresponding to two different labelings of a given poset to be the same up to a shift.

Theorem 3.3 (Stembridge). *Let P be a poset with two labelings ω and ω' such that for each $j \in P$ there exists t_j with the following conditions:*

- *if j is minimal then $t_j = 0$;*
- *if j covers i then for any maximal chain C containing i and j*

$$t_j - t_i = \begin{cases} 1 & \text{if } i \in \operatorname{Des}(C,\omega)\setminus\operatorname{Des}(C,\omega')\,,\\ -1 & \text{if } i \in \operatorname{Des}(C,\omega')\setminus\operatorname{Des}(C,\omega)\,,\\ 0 & \text{otherwise;}\end{cases}$$

[2]Stanley defines (P,ω)-partitions in an order-*reversing* fashion; the present definition mirrors that of the usual order polynomial.

[3]This definition is found in the literature most frequently for the case where ω is a natural (i.e., order-preserving) labeling.

- *there exists k such that $t_j = k$ for any maximal j.*

Then $h^*_{P,\omega}(x) = x^k\, h^*_{P,\omega'}(x)$.

We note that our conditions in Theorem 3.3 imply that every maximal chain C in the order ideal $\langle j \rangle$ satisfies $t_j = \mathrm{des}(C,\omega) - \mathrm{des}(C,\omega')$. By choosing ω' to be natural (and so there are no descents) and $t_j = \mathrm{des}(C,\omega)$ for any choice of maximal chain C in $\langle j \rangle$ for a given $j \in P$, we obtain the following result which appeared in [3, Corollary 2.4], phrased there in the language of sign-graded posets.[4]

Corollary 3.4. *Consider a poset P with a labeling ω such that all maximal chains have the same number k of descents. Then* $h^*_{P,\omega}(x) = x^k\, h^*_P(x)$.

Our results in the remainder of this section concern posets with maximal chains with the same number of descents for the sake of simplicity, but most arguments extend to the situation described in Theorem 3.3. The next theorem follows from Corollary 3.4 and the fact that the h^*-polynomial of a naturally labeled poset is palindromic if and only if the poset is graded [10, Proposition 19.3].

Theorem 3.5. *Consider a poset P with a labeling ω such that all maximal chains have the same number k of descents. Then* $C_n^{P,\omega}(x) = x^k\, A_n(x)\, h^*_{P\times[n]}(x)$. *Moreover, if P is graded then* $C_n^{P,\omega}(x)$ *is palindromic.*

Starting with $P \times [n]$, we now construct a new poset $P \check{\times} [n]$ of cardinality $(m+1)n$ by adding n elements with no relation among them but which cover all maximal elements of $P \times [n]$. The poset in Figure 2 gives an example where $P = [2]$ and $n = 3$. We extend a given labeling ω of P, first to a labeling $\omega \times \mathrm{id}$ of $P \times [n]$, and then, to a labeling $\omega \check{\times} \mathrm{id}$ of $P \check{\times} [n]$, by giving the new elements any labels that are larger than those in $\omega \times \mathrm{id}$.

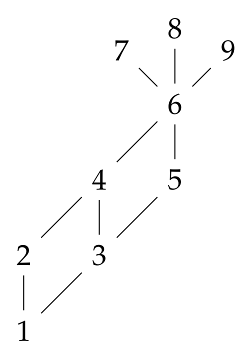

Figure 2: A labeled poset $(P \check{\times} [n], \omega \check{\times} \mathrm{id})$ such that $h^*_{P \check{\times} [n], \omega \check{\times} \mathrm{id}}(x) = C_3^2(x)$.

[4]The notion that all maximal chains of (P,ω) have the same number of descents is slightly different than (P,ω) being sign-graded; when P is graded, the two notions coincide.

Proposition 3.6. $C_n^{P,\omega}(x) = h^*_{P \check{\times} [n], \omega \check{\times} \mathrm{id}}(x)$.

This gives a direct proof for the palindromicity of $C_n^{P,\omega}$ for $\omega = \mathrm{id}$ as $P \check{\times} [n]$ is graded.

4 Amphibians

In this section, we will study a broader family of linear extensions/multiset permutations whose descent polynomials are also palindromic.

Throughout this section, we denote by Q a subposet of $P \times [n]$ with some relations missing between different copies of P, i.e., those of the form $(p, j) \prec (p, j+1)$. The motivation for studying these subposets comes from the following observation. When $P = [m]$, the linear extensions of Q can be interpreted (in the same way as before) as a collection of multiset permutations of $\{1^m, 2^m, \ldots, n^m\}$, where conditions weaker than those for canon permutations are imposed. When $P = [m]$ and we remove all relations of the form $(p, j) \prec (p, j+1)$ for all $j \in [m] \backslash \{1\}$ (where 1 can be replaced with any other element of $[m]$), we recover the set of multiset permutations of $\{1^m, 2^m, \ldots, n^m\}$ whose subsequence formed by the first copy of each element in $[n]$ is fixed. Similarly to canon permutations, summing over the corresponding descent polynomials for all canon labelings $\mathrm{id} \times \sigma$ of Q for $\sigma \in S_n$ gives the descent polynomial of multiset permutations of $\{1^m, 2^m, \ldots, n^m\}$. Since the descent polynomials corresponding to $\{1^m, 2^m, \ldots, n^m\}$ and $\mathcal{C}_n^m$ are both palindromic, it is natural to ask if palindromicity extends to more subposets Q of $P \times [n]$. Theorem 4.2 below confirms this. We note that the notion of *palindromicity* here might involve polynomials with zero constant terms (or even more zero coefficients), and so we state, in each case, the relevant functional equation.

Let Q be a subposet of $P \times [n]$ with some of the cover relations of the form $(p, j) \prec (p, j+1)$ removed. We define the *dissonant canon polynomial*

$$C^{Q,\omega}(x) := \sum_{\sigma \in S_n} h^*_{Q, \omega \times \sigma}(x).$$

The name is inspired from the case when $P = [m]$ and Q is a subposet of $[m] \times [n]$ with some of the edges of the form $(p, j) \prec (p, j+1)$ removed, where $p \in P \backslash \{q\}$ for some fixed $q \in P$. This last condition fixes one of the subsequences and therefore ensures that $C^{Q,\omega}$ is counting multiset permutations of $\{1^m, \ldots, n^m\}$ without doublecounting.

If all maximal chains in (P, ω) have the same number k of descents, then any maximal chain in $(Q, \omega \times \sigma)$ will contain between k and $k + \mathrm{des}(\sigma)$ descents. In particular, from Theorem 3.3 we can deduce the following.

Corollary 4.1. *Consider a poset P with a labeling ω such that all maximal chains in (P, w) have the same number k of descents. Then, for any Q as above*

$$h^*_{Q, \omega \times \sigma}(x) = x^k \, h^*_{Q, \mathrm{id} \times \sigma}(x).$$

Corollary 4.1 is used to prove the following theorem.

Theorem 4.2. *Consider a poset P with a labeling ω such that all maximal chains in (P, w) have the same number k of descents. Then, for any Q as above, the dissonant canon polynomial $C^{Q,\omega}(x)$ is palindromic, in the sense that*

$$x^{m(n-1)+2k}\, C^{Q,\omega}(\tfrac{1}{x}) = C^{Q,\omega}(x)\,.$$

A key point in the proof of Theorem 4.2 is the decomposition of the mentioned polynomial into (a total of $\frac{n!}{2}$) palindromic polynomials. A natural question is to identify the corresponding bijections between the permutations captured by the coefficients, which would result in a bijective proof of Theorem 4.2. This would also yield bijections in the subcase of canon permutations, addressing questions raised by Elizalde [7, Problem 4.1]. The palindromicity of the dissonant canon polynomial $C^{Q,\mathrm{id}}(x)$ in Theorem 4.2 also gives rise to the following problem.

Question 4.3. For which subposets Q of $P\times[n]$ is $C^{Q,\mathrm{id}}(x)$ γ-positive? Can we describe the γ-coefficients in a unified way?

In the next section (Corollary 5.1), we will see that the answer to Question 4.3 is positive for canon permutations, which correspond to the poset $Q = [m] \times [n]$. The answer is also positive for the poset corresponding to all multiset permutations, i.e., the poset where all conditions of the form $(p, j) \prec (p, j+1)$ are removed. This follows from [3] by realizing the set of multiset permutations of $\{1^m, 2^m, \ldots, n^m\}$ as linear extensions of the poset consisting of n m-element chains, labeled with the regular canon labeling. A combinatorial interpretation for those γ-coefficients is given in [4]. The two combinatorial interpretations (corresponding to the set of multiset permutations and to canon permutations) rely on partitioning permutations into classes, but the classes are different in each case. Question 4.3 asks for a unified interpretation of the γ-coefficients for both corresponding polynomials, as well as other γ-positive polynomials that arise for different subposets Q. We conjecture that this class will contain all such Q, at least for $P = [m]$.

Conjecture 4.4. *The dissonant canon polynomial $C^{Q,\mathrm{id}}(x)$ is γ-positive whenever $P = [m]$.*

5 γ-positivity of Canon Permutations

Theorem 1.2 shows that certain distributional properties shared among the polynomials $A_n(x)$ and $h^*_{P\times[n]}(x)$, such as palindromicity and γ-positivity, transfer to the canon polynomial $C_n^{P,\omega}(x)$. A direct proof of the γ-positivity of $C_n^m(x)$ can be derived from Brändén's work in [3] using the poset $[m]\check{\times}[n]$ constructed in Proposition 3.6 (here

$P = [m]$). Below, we discuss a combinatorial interpretation of the γ-coefficients of $C_n^m(x)$, a consequence of a group action on permutations due to Foata and Strehl extended to posets by Brändén [4].

Following the notation in [4], we consider the map $\rho : [m]\check{\times}[n] \to \mathbb{Z}_2$ with values $\rho(q) = 0$ if the maximal chains in the poset ideal $\langle q \rangle$ have even length and $\rho(q) = 1$ if the length is odd. For a permutation $\pi = \pi_1 \cdots \pi_{(m+1)n} \in \mathcal{L}([m]\check{\times}[n])$, let us call $j \in [(m+1)n-1]$ a *ρ-descent* of π if $\pi_{j+1} < \pi_j$ or $\rho(\pi_{j+1}) < \rho(\pi_j)$. We say that a ρ-descent $j \in [(m+1)n-1]$ of π is a *double ρ-descent* if $j-1$ and j are both ρ-descents in π, or $j = 1$. The following result can now be derived from [4, Section 6].

Corollary 5.1. *Writing*

$$C_n^m(x) = h^*_{[m]\check{\times}[n]}(x) = \sum_{i=0}^{\lfloor \frac{m(n-1)}{2} \rfloor} \gamma_i \, x^i (1+x)^{m(n-1)-2i},$$

the coefficient γ_i equals the number of linear extensions $\pi = \pi_1 \cdots \pi_{(m+1)n} \in \mathcal{L}([m]\check{\times}[n])$ such that, for $d = \lfloor \frac{m+n-1}{2} \rfloor$,

- *π has exactly $i + d$ ρ-descents,*
- *π has no double ρ-descents,*
- *$\pi_{(m+1)n-1} < \pi_{(m+1)n}$ if $\rho(\pi_{(m+1)n-1}) = \rho(\pi_{(m+1)n}) = 1$.*

The weak-descent polynomial of canon permutations is palindromic and γ-positive with a combinatorial interpretation specified by Corollary 5.1, as discussed in the following proposition. (The palindromicity of the weak-descent polynomial of canon permutations for the multiset $\{1,1,2,2,\ldots,n,n\}$ was first observed in [7, Corollary 2.3].)

Proposition 5.2. $\displaystyle\sum_{\sigma \in \mathcal{C}_n^m} x^{\mathrm{wdes}(\sigma)} = x^{m-1} C_n^m(x) = x^{m-1} A_n(x)\, h^*_{[m]\times[n]}\,.$

Let Q be a subposet of $[m] \times [n]$ with some of the edges of the form $(p, j) \prec (p, j+1)$ removed, where $p \in P \backslash \{q\}$ for some fixed $q \in P$. It follows by similar arguments as in the proof of Proposition 5.2 that the dissonant canon polynomial $C^{Q,\mathrm{id}}(x)$ (resp. $C^{Q,v}(x)$) is the descent (resp. weak-descent) polynomial of the set of permutations of $\{1^m, 2^m, \ldots, n^m\}$ that satisfy the constraints imposed by the edges of Q between copies of $[m]$.

Corollary 5.3. *Let Q and v be as above. The dissonant canon polynomial $C^{Q,\mathrm{id}}(x)$ is palindromic in the sense that*

$$x^{m(n-1)}\, C^{Q,\mathrm{id}}(\tfrac{1}{x}) = C^{Q,\mathrm{id}}(x)\,.$$

The dissonant canon polynomial $C^{Q,v}(x)$ corresponding to weak descents of multiset permutations is palindromic, in the sense that

$$x^{m(n+1)-2}\, C^{Q,\omega}(\tfrac{1}{x}) = C^{Q,\omega}(x)\,.$$

Corollary 5.3 provides a class of sets of permutations of $\{1^m, 2^m, \dots, n^m\}$ with palindromic (weak) descent polynomial, which simultaneously generalizes canon permutations and the collection of all permutations of $\{1^m, 2^m, \dots, n^m\}$.

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