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On Two Notions of Total Positivity for Generalized Partial Flag Varieties of Classical Lie Types

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Abstract. For Grassmannians, Lusztig's notion of total positivity coincides with positivity of the Plücker coordinates. This coincidence underpins the rich interaction between matroid theory, tropical geometry, and the theory of total positivity. Bloch and Karp furthermore characterized the (type A) partial flag varieties for which the two notions of positivity similarly coincide. We characterize the symplectic (type C) and odd-orthogonal (type B) partial flag varieties for which Lusztig's total positivity coincides with Plücker positivity.

Keywords: total positivity, flag varieties, symplectic group, orthogonal group

1 Introduction

Let n be a positive integer, and denote $[n] = \{1, \ldots, n\}$. The *totally positive* (resp. *nonnegative*) *part* $GL_n^{>0}$ (resp. $GL_n^{\geq 0}$) of the general linear group GL_n consists of the real invertible matrices whose minors are all positive (resp. nonnegative). The study of these spaces traces back to [15, 22]. Lusztig generalized this notion of total positivity to an arbitrary connected reductive ($\mathbb{R}$-split) algebraic group G and its partial flag varieties G/P [16, 17]. The study of total positivity has since been a nexus for fruitful interactions between algebraic geometry, representation theory, combinatorics, and physics [1, 10, 11].

Underpinning such fruitful interactions is the interplay between "parametric" and "implicit" descriptions of total positivity. The original definitions are "parametric" in nature: Lusztig defined the *Lusztig positive* (resp. *Lusztig nonnegative*) part $G^{>0}$ (resp. $G^{\geq 0}$) of G as a semigroup in G generated by certain elements (see Section 3.1). For a parabolic subgroup $P \subset G$, the Lusztig positive (resp. Lusztig nonnegative) part $(G/P)^{>0}$ (resp.

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$(G/P)^{\geq 0}$) of the partial flag variety G/P is then defined as the image of (resp. the closure of the image of) $G^{>0}$ under the projection map $G \to G/P$. Marsh and Rietsch gave a combinatorial parametrization of $(G/P)^{\geq 0}$ in terms of its Deodhar cells [18]. A challenge of studying Lusztig nonnegative flag varieties is that, from these parametric descriptions alone, it is difficult to recognize when a partial flag is Lusztig nonnegative.

To address this difficulty, one may seek an "implicit" description of Lusztig positivity for G/P in terms of positivity of suitably natural coordinates on G/P. Our main result, Theorem A, gives a simple implicit description for certain flag varieties when G is the symplectic group on $2n$ elements or the orthogonal group on $2n+1$ elements. As a motivating example, consider the *Grassmannian* $\mathrm{Gr}_{k;n}$ of k-dimensional subspaces in $\mathbb{R}^n$. Its Plücker coordinates allow one to consider the *Plücker positive* (resp. *nonnegative*) *part* of $\mathrm{Gr}_{k;n}$, defined as

$$\mathrm{Gr}_{k;n}^{\Delta>0} \text{ (resp. } \mathrm{Gr}_{k;n}^{\Delta\geq 0}) := \left\{ L \subseteq \mathbb{R}^n \;\middle|\; \begin{array}{c} L \text{ is the row-span of a real } k\times n \text{ matrix with all} \\ \text{maximal minors positive (resp. nonnegative)} \end{array} \right\}.$$

Lam [14] and, independently, Talaska and Williams [21] showed that $\mathrm{Gr}_{k;n}^{>0} = \mathrm{Gr}_{k;n}^{\Delta>0}$ and $\mathrm{Gr}_{k;n}^{\geq 0} = \mathrm{Gr}_{k;n}^{\Delta\geq 0}$. More generally, for a subset $K = \{k_1 < \cdots < k_j\} \subseteq [n-1]$, one may consider the (type A) partial flag variety

$$\mathrm{Fl}_{K;n} := \{\text{flags of subspaces } L_\bullet = (L_1 \subseteq \cdots \subseteq L_j) \text{ with } \dim L_i = k_i \text{ for all } i = 1,\dots,j\}.$$

Under the natural embedding $\mathrm{Fl}_{K;n} \hookrightarrow \prod_{i=1}^{j} \mathrm{Gr}_{k_i;n}$, its *Plücker positive* (resp. *nonnegative*) part is defined as the intersection

$$\mathrm{Fl}_{K;n}^{\Delta>0} \text{ (resp. } \mathrm{Fl}_{K;n}^{\Delta\geq 0}) := \mathrm{Fl}_{K;n} \cap \prod_{i=1}^{j} \mathrm{Gr}_{k_i;n}^{\Delta>0} \text{ (resp. } \prod_{i=1}^{j} \mathrm{Gr}_{k_i;n}^{\Delta\geq 0}).$$

Bloch and Karp [3] showed the following. The second author independently showed a similar result in the case of $K = [n-1]$ [4].

Theorem 1.1 ([3, Theorem 1.1]). The following are equivalent for a subset $K \subseteq [n-1]$:

(1) $\mathrm{Fl}_{K;n}^{>0} = \mathrm{Fl}_{K;n}^{\Delta>0}$,

(2) $\mathrm{Fl}_{K;n}^{\geq 0} = \mathrm{Fl}_{K;n}^{\Delta\geq 0}$, and

(3) K consists of consecutive integers.

In summary, these results establish the coincidence of Lusztig's positivity and Plücker positivity for partial flag varieties $\mathrm{Fl}_{K;n}$ with consecutive K. This coincidence supports the rich interaction between matroid theory, tropical geometry, and total positivity [19, 20, 12, 4]. For instance, it is used to prove that rank K positively oriented flag matroids are flag positroids and that the positive tropical flag variety equals the positive flag Dressian [6]. Here, with a view towards the theory of Coxeter matroids [7], we characterize partial flag varieties of the symplectic group Sp_{2n} (type C) and the odd-orthogonal group SO_{2n+1} (type B) for which Lusztig's positivity coincides with Plücker positivity. This yields an explicit test for membership in the Lusztig positive part of those partial flag varieties: A partial flag is Lusztig positive if and only if all of its Plücker coordinates are positive.

Let $\mathbf{e}_i$ denote the i-th standard basis vector in a coordinate space, and $\mathbf{e}_i^*$ its dual. For type C, endow $\mathbb{R}^{2n}$ with the symplectic bilinear form $\omega = \sum_{i=1}^{n}(-1)^i \mathbf{e}_i^* \wedge \mathbf{e}_{2n+1-i}^*$. For type B, endow $\mathbb{R}^{2n+1}$ with the symmetric bilinear form $Q = \sum_{i=1}^{n+1}(-1)^i \mathbf{e}_i^* \cdot \mathbf{e}_{2n+2-i}^*$. Let Sp_{2n} and SO_{2n+1} be the linear groups preserving the bilinear forms ω and Q, respectively. Recall that a subspace of a vector space with a symmetric or alternating form is *isotropic* if the restriction of the form to the subspace is trivial. The partial flag varieties of these groups have the following description in terms of isotropic subspaces: For $K \subseteq [n]$,

$$\mathrm{SpFl}_{K;2n} := \{L_\bullet \in \mathrm{Fl}_{K;2n} : \text{each } L_i \text{ in the flag } L_\bullet \text{ is isotropic with respect to } \omega\}, \quad \text{and}$$
$$\mathrm{SOFl}_{K;2n+1} := \{L_\bullet \in \mathrm{Fl}_{K;2n+1} : \text{each } L_i \text{ in the flag } L_\bullet \text{ is isotropic with respect to } Q\}.$$

We define their *Plücker positive* (resp. *nonnegative*) parts as the intersections

$$\mathrm{SpFl}^{\Delta>0}_{K;2n} \text{ (resp. } \mathrm{SpFl}^{\Delta\geq 0}_{K;2n}) := \mathrm{SpFl}_{K;2n} \cap \mathrm{Fl}^{\Delta>0}_{K;2n} \text{ (resp. } \mathrm{Fl}^{\Delta\geq 0}_{K;2n}), \quad \text{and}$$
$$\mathrm{SOFl}^{\Delta>0}_{K;2n+1} \text{ (resp. } \mathrm{SOFl}^{\Delta\geq 0}_{K;2n+1}) := \mathrm{SOFl}_{K;2n+1} \cap \mathrm{Fl}^{\Delta>0}_{K;2n+1} \text{ (resp. } \mathrm{Fl}^{\Delta\geq 0}_{K;2n+1}).$$

Our main theorem is as follows.

Theorem A. In type C, for $n \geq 2$ and a subset $K \subseteq [n]$, the following are equivalent:

(1) $\mathrm{SpFl}^{>0}_{K;2n} = \mathrm{SpFl}^{\Delta>0}_{K;2n}$,

(2) $\mathrm{SpFl}^{\geq 0}_{K;2n} = \mathrm{SpFl}^{\Delta\geq 0}_{K;2n}$, and

(3) $K = \{k, k+1, \dots, n\}$ for some $1 \leq k \leq n$.

In type B, for $n \geq 3$ and a subset $K \subseteq [n]$, the following are equivalent:

(1) $\mathrm{SOFl}^{>0}_{K;2n+1} = \mathrm{SOFl}^{\Delta>0}_{K;2n+1}$,

(2) $\mathrm{SOFl}^{\geq 0}_{K;2n+1} = \mathrm{SOFl}^{\Delta\geq 0}_{K;2n+1}$, and

(3) $K = \{k, k+1, \dots, n\}$ for some $1 \leq k \leq n$.

In type B, when $n = 2$, the statements (1) and (2) hold for all partial flag varieties $\mathrm{SOFl}_{K;5}$.

The proof of (3) $\implies$ (1) $\implies$ (2) uses the fact that, with the forms ω and Q, $\mathrm{SpFl}_{K;2n}$ and $\mathrm{SOFl}_{\mathrm{K;2n+1}}$ are compatible with type A flag varieties in a way we make precise in condition (†) of Definition 3.4. As a result, we can invoke Theorem 1.1. In fact, any flag variety satisfying condition (†) also has the appropriate analogue of (3) $\implies$ (1) $\implies$ (2). To prove (2) $\implies$ (3), we explicitly construct examples of flags where (2) fails whenever K is not of the form $\{k, k+1, \ldots, n\}$.

We briefly discuss type D. We show that the methods employed in types B and C cannot be extended to type D. However, forthcoming work of the second author [5] gives a straightforward test for positivity in the type D Grassmannian $\mathrm{OGr}(n, 2n)$.

Previous works

Karpman showed that statements (1) and (2) of Theorem A hold for Lagrangian Grassmannians, i.e. $\mathrm{SpFl}_{n;2n}$ [13]. For a general reductive ($\mathbb{R}$-split) algebraic group G of simply-laced type, Lusztig showed that Lusztig positivity for a partial flag variety G/P coincides with positivity of the coordinates from the canonical basis of a sufficiently large irreducible representation of G [17]. However, due to the "sufficiently large" condition, this does not recover any of the aforementioned results of Lam, Talaska–Williams, Bloch–Karp, or Karpman. This abstract is based on a preprint [2], which contains further details.

2 Pinnings

Let G be a connected, reductive, $\mathbb{R}$-split linear algebraic group. We often identify G with its $\mathbb{R}$-valued points. A *pinning* of G is an additional set of choices for G that is part of the input data for the definition of Lusztig positivity for G.

Fix a split maximal torus T in G, and let X be the character lattice of T. Let $\Phi \subset X$ be the set of roots of the corresponding root system. Fix a system of positive roots Φ^+, and let B_+ be the corresponding Borel subgroup of G. Let B_- be the opposite Borel subgroup such that $B_+ \cap B_- = T$. Let U_+ and U_- be the unipotent radicals of B_+ and of B_-, respectively. Let I be an indexing set for the set $\{\alpha_i : i \in I\}$ of simple roots in Φ^+. For every $i \in I$, fix a homomorphism $\phi_i : SL_2 \to G$ such that in the induced map $\mathfrak{sl}_2 \to \mathfrak{g}$ of Lie algebras, the element $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \in \mathfrak{sl}_2$ maps to a generator of the root space in $\mathfrak{g}$ of weight α_i. We then define homomorphisms $x_i : \mathbb{R} \to U_+$, $y_i : \mathbb{R} \to U_-$, and $\chi_i : \mathbb{R}^* \to T$ by

$$x_i(m) := \phi_i\left(\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}\right), \quad y_i(m) := \phi_i\left(\begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix}\right), \quad \text{and} \quad \chi_i(t) := \phi_i\left(\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}\right).$$

Definition 2.1. The data $(T, B_+, B_-, \{x_i\}_{i \in I}, \{y_i\}_{i \in I})$ is called a *pinning* for G.

When multiple groups are in play, we write superscripts of the root system name, for example T^Φ, s_i^Φ, and y_i^Φ, to distinguish between the pinnings of different groups.

A pinning of G identifies the reflection group W of the root system Φ with the Weyl group $N_G(T)/T$, as follows. For each $i \in I$, the simple reflection $s_i \in W$ is identified with $\dot{s}_i T$ where

$$\dot{s}_i := \phi_i\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right).$$

In type A_{n-1}, when $G = GL_n$, we use the *standard pinning* $(T^A, B_+^A, B_-^A, \{x_i^A\}_{i\in[n-1]}, \{y_i^A\}_{i\in[n-1]})$, defined as follows. The torus T^A consists of diagonal matrices with non-zero entries on the diagonal. The Borels B_+^A and B_-^A consist of upper and lower triangular invertible matrices, respectively. The set of simple roots is $\{\mathbf{e}_1 - \mathbf{e}_2, \ldots, \mathbf{e}_{n-1} - \mathbf{e}_n\}$. Accordingly, for each $i \in [n-1]$, the maps ϕ_i^A are given by

$$\phi_i^A\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \begin{array}{c} \\ \\ \\ i \\ i+1 \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} & & i & i+1 & & & \end{array} \\ \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & a & b & & \\ & & c & d & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \end{array},$$

where unmarked off-diagonal matrix entries are 0. The Weyl group is the permutation group $\mathfrak{S}_n$ on $[n]$ with s_i^A the transposition $(i\; i+1)$.

We now remark on notations we will use in subsequent sections. For a sequence $\mathbf{i} = (i_1, \ldots, i_\ell)$ with entries in the indexing set I of the simple roots, we denote by $\mathbf{s_i}$ the element

$$\mathbf{s_i} := s_{i_1} \cdots s_{i_\ell} \in W.$$

When clear from context, we use $\mathbf{s_i}$ to denote also the word $(s_{i_1}, \ldots, s_{i_\ell})$. Define the function $\mathbf{y_i} : \mathbb{R}^\ell \to G$ by

$$\mathbf{y_i}(a_1, \ldots, a_\ell) := y_{i_1}(a_1) \cdots y_{i_\ell}(a_\ell),$$

and similarly define $\mathbf{x_i}$, $\chi_\mathbf{i}$, and $\dot{\mathbf{s}}_\mathbf{i}$. The length ℓ of the sequence $\mathbf{i}$ is denoted $|\mathbf{i}|$.

To avoid repeated arguments, we shall often use the following general setup. The function ψ is an extra piece of combinatorial data that plays a crucial role in Definition 3.4.

Setup 2.2. Let G be a connected, reductive, $\mathbb{R}$-split linear algebraic group with a fixed pinning $(T^\Phi, B_+^\Phi, B_-^\Phi, \{x_i^\Phi\}_{i\in I}, \{y_i^\Phi\}_{i\in I})$ with simple roots indexed by I in the root system Φ. Let $\iota : G \hookrightarrow GL_N$ be an embedding, and fix a function $\psi : I \to \{\text{sequences in } [N-1]\}$. We write ψ also for the function $\{\text{sequences in } I\} \to \{\text{sequences in } [N-1]\}$ defined by

$$(i_1, \ldots, i_\ell) \mapsto \big(\text{the concatenation of } \psi(i_1), \ldots, \psi(i_\ell)\big).$$

3 Lusztig positivity and the proof of (3) $\Longrightarrow$ (1) $\Longrightarrow$ (2)

3.1 Lusztig's total positivity

Let G be a connected, reductive, $\mathbb{R}$-split linear algebraic group with a fixed pinning $(T, B_+, B_-, \{x_i\}_{i\in I}, \{y_i\}_{i\in I})$. We recall Lusztig's definition of total positivity for G.

Definition 3.1. For a sequence $\mathbf{i}$ in I such that $\mathbf{s_i}$ is a reduced expression for the longest element $w_0 \in W$, define $U_-^{>0}$ (resp. $U_-^{\geq 0}$) to be the image $\mathbf{y_i}(\mathbb{R}_{>0}^{|\mathbf{i}|})$ (resp. $\mathbf{y_i}(\mathbb{R}_{\geq 0}^{|\mathbf{i}|})$), and similarly define $U_+^{>0}$ and $U_+^{\geq 0}$ in terms of $\mathbf{x_i}$. Define $T^{>0}$ to be the subgroup of the $\mathbb{R}$-split torus T generated by the elements $\chi(t)$ for $t \in \mathbb{R}_{>0}$ and $\chi : \mathbb{R}^* \to T$ a cocharacter of T. Define the *positive* (resp. *nonnegative*) *part* of G to be

$$G^{>0} := U_-^{>0} T^{>0} U_+^{>0} \quad (\text{resp. } G^{\geq 0} := U_-^{\geq 0} T^{>0} U_+^{\geq 0}).$$

The sets $\mathbf{y_i}(\mathbb{R}_{>0}^{|\mathbf{i}|})$ and $\mathbf{y_i}(\mathbb{R}_{\geq 0}^{|\mathbf{i}|})$ depend only on the element $\mathbf{s_i} \in W$ as long as $\mathbf{s_i}$ is a reduced expression [16]. When $G = GL_n$ with the standard pinning, it is a classical result [8, 9] that $GL_n^{>0}$ (resp. $GL_n^{\geq 0}$) as defined here is the space of invertible matrices with all positive (resp. nonnegative) minors.

For a parabolic subgroup $P \subset G$ containing B_+, let $\pi : G \to G/P$ be the projection map. For $S \subseteq G/P$, we denote by $\overline{S}$ its closure in the Euclidean topology on $(G/P)(\mathbb{R})$.

Definition 3.2. Define the *positive* (resp. *nonnegative*) *part* of the partial flag G/P to be

$$(G/P)^{>0} := \pi(G^{>0}) \quad (\text{resp. } (G/P)^{\geq 0} := \overline{\pi(G^{>0})}).$$

We caution that although $G^{\geq 0}$ is the closure of $G^{>0}$ [16], the image $\pi(G^{\geq 0})$ may be strictly contained in $(G/B_+)^{\geq 0}$, since $\pi : G \to G/B_+$ may not be proper and so we may not in general conclude that $(G/P)^{\geq 0} = \pi(G^{\geq 0})$. However, note that the projection map $G/B_+ \to G/P$ is proper, and hence $(G/P)^{\geq 0}$ is the image of $(G/B_+)^{\geq 0}$. For a linear subspace $L \subset \mathbb{R}^N$, let $L^{\perp,\omega}$ denote the orthogonal complement of L with respect to the symplectic form ω, and similarly for the symmetric form Q. One can show the following:

Lemma 3.3. For a subset $K \subseteq [n]$, we have

$$\begin{aligned}
\mathrm{Sp}_{2n}/P_J^C &= \{L_\bullet \in \mathrm{Fl}_{K\cup(2n-K);2n} : L_i = L_j^{\perp,\omega} \text{ if } \dim L_i + \dim L_j = 2n\} \\
&\simeq \mathrm{SpFl}_{K;2n}, \quad \text{and} \\
\mathrm{SO}_{2n+1}/P_J^B &= \{L_\bullet \in \mathrm{Fl}_{K\cup(2n+1-K);2n+1} : L_i = L_j^{\perp,Q} \text{ if } \dim L_i + \dim L_j = 2n+1\} \\
&\simeq \mathrm{SOFl}_{K;2n+1}.
\end{aligned}$$

Consequently, we have $(\mathrm{Sp}_{2n}/P_J^C) \cap \mathrm{Fl}_{K\cup(2n-K);2n}^{\Delta>0} \simeq \mathrm{SpFl}_{K;2n}^{\Delta>0}$ and $(\mathrm{SO}_{2n+1}/P_J^B) \cap \mathrm{Fl}_{K\cup(2n+1-K);2n+1}^{\Delta>0} \simeq \mathrm{SOFl}_{K;2n+1}^{\Delta>0}$, and similarly with $\Delta \geq 0$ in place of $\Delta > 0$.

3.2 Proof of (1) $\implies$ (2)

We work now in the general setting of Setup 2.2, where $\iota : G \hookrightarrow GL_N$ is an embedding and ψ maps sequences in I to sequences in $[N-1]$. Let G have root system Φ.

Definition 3.4. We say that (ι, ψ) has *property (†)* if the following are satisfied.

(†1) For every $i \in I$, we have $\dot{s}_i^\Phi = \dot{\mathbf{s}}^A_{\psi(i)}$, and we have $y_i^\Phi(a) = \mathbf{y}^A_{\psi(i)}(f_1(a), \dots, f_{|\psi(i)|}(a))$ for some sequence $(f_1, \dots, f_{|\psi(i)|})$ of differentiable functions $f_j : \mathbb{R} \to \mathbb{R}$ such that:

- $f_j(\mathbb{R}_{>0}) \subseteq \mathbb{R}_{>0}$, and similarly for x_i^Φ and χ_i^Φ,
- $f_j(\mathbb{R}_{<0}) \subseteq \mathbb{R}_{<0}$ (so $f_j(\mathbb{R}^*) \subset \mathbb{R}^*$),
- $\lim\limits_{a \to +\infty} f_j(a) = +\infty$ for at least one $j \in \psi(i)$, and
- $\lim\limits_{a \to -\infty} f_j(a) = -\infty$ for at least one $j \in \psi(i)$.

(†2) For some sequence $\mathbf{i}$ in I such that $\mathbf{s}^\Phi_{\mathbf{i}}$ is a reduced word for the longest element $w_0 \in W^\Phi$, the word $\mathbf{s}^A_{\psi(\mathbf{i})}$ is a reduced word for the longest element of W^A.

Condition (†) asserts that (ι, ψ) is compatible with the type A flag variety in that $B_+^\Phi \subseteq B_+^A$. In particular, if (ι, ψ) satisfies (†) and P is a parabolic subgroup of GL_N containing B_+^A, then $P \cap G$ is a parabolic subgroup of G containing B_+^Φ. Note that in the definition above, y_i^A denotes a map into GL_N, where N is implicit from the context. We always take $N = 2n$ for type C and $N = 2n+1$ for type B.

Whenever we use $\iota : \mathrm{Sp}_{2n} \hookrightarrow GL_{2n}$, we fix ψ to be the function $\psi(i) = (i, 2n-i)$ for $i \in [n-1]$, and $\psi(n) = n$. Similarly, whenever we use $\iota : \mathrm{SO}_{2n+1} \hookrightarrow GL_{2n+1}$, we fix ψ to be the function $\psi(i) = (i, 2n+1-i)$ for $i \in [n-1]$, and $\psi(n) = (n, n+1, n)$.

Lemma 3.5. There exist pinnings $(T^C, B_+^C, B_-^C, \{x_i^C\}_{i \in [n]}, \{y_i^C\}_{i\in[n]})$ of Sp_{2n} and $(T^B, B_+^B, B_-^B, \{x_i^B\}_{i\in[n]}, \{y_i^B\}_{i\in[n]})$ of SO_{2n+1}, respectively, which satisfy (†).

Proof sketch. We can choose $y_i^C(m) = y_i^A(m) y_{2n-i}^A(m)$ and $y_i^B(m) = y_i^A(m) y^A_{2n+1-i}(m)$ for $i \in [n-1]$, and $y_n^C(m) = y_n^A(m)$ and $y_n^B(m) = y_n^A(\frac{m}{\sqrt{2}}) y^A_{n+1}(\sqrt{2}m) y_n^A(\frac{m}{\sqrt{2}})$. These choices satisfy condition (†1). Condition (†2) can be verified directly on any choice of reduced word for w_0^C and for w_0^B. (Since $W^C = W^B$, we actually have $w_0^C = w_0^B$. However, due to the different ψ for $\mathrm{Sp}_{2n} \hookrightarrow GL_{2n+1}$ and $\mathrm{SO}_{2n+1} \hookrightarrow GL_{2n+1}$, there are still two statements to verify here.) □

Proposition 3.6. Suppose (ι, ψ) has property (†). Then, we have the following.

(a) $G^{>0} \subseteq GL_N^{>0}$.

(b) For $J \subseteq [N-1]$, let $P_J^A \supseteq B_+^A$ be the parabolic subgroup of GL_N, so that $GL_N/P_J^A = \mathrm{Fl}_{K;N}$ where $K = [N-1] \setminus J$. Let $P = P_J^A \cap G$. Then, we have that

$$(G/P) \cap \mathrm{Fl}_{K;N}^{\Delta \geq 0} = \overline{\left((G/P) \cap \mathrm{Fl}_{K;N}^{\Delta > 0}\right)}.$$

Applying this proposition to Sp_{2n} and SO_{2n+1} yields Theorem A (1) $\Longrightarrow$ (2):

Proof of Theorem A (1) $\Longrightarrow$ (2). Since $(G/P)^{\geq 0}$ is the closure of $(G/P)^{>0}$, by Lemma 3.3, it suffices to show that the Plücker nonnegative part is the closure of the Plücker positive part. By Proposition 3.6(b), this follows from Lemma 3.5. □

3.3 Proof of (3) $\Longrightarrow$ (1)

Property (†), together with [18, Proposition 5.2, Theorem 11.3], implies the following:

Proposition 3.7. Suppose the pair (ι, ψ) satisfies the property (†). Then, one has

$$(G/B_+)^{>0} = (G/B_+) \cap \mathrm{Fl}_{[N-1];N}^{\Delta>0}.$$

We now prove Theorem A (3) $\Longrightarrow$ (1).

Proof of Theorem A (3) $\Longrightarrow$ (1). Let $n \geq 2$. As the embeddings $\iota : \mathrm{Sp}_{2n} \hookrightarrow GL_{2n}$ and $\iota : \mathrm{SO}_{2n+1} \hookrightarrow GL_{2n+1}$ satisfy (†), Definition 3.1 and Proposition 3.6(a) together imply that $\mathrm{SpFl}_{K;2n}^{>0} \subseteq \mathrm{SpFl}_{K;2n}^{\Delta>0}$ and $\mathrm{SOFl}_{K;2n+1}^{>0} \subseteq \mathrm{SOFl}_{K;2n+1}^{\Delta>0}$ for any $K \subseteq [n]$. It remains to show the reverse inclusions when $K = \{k, k+1, \ldots, n\}$ for some $k \in [n]$.

We first reduce to the case $K = [n]$ as follows. We show this reduction for the Sp_{2n} case; the case of SO_{2n+1} is similar. By Lemma 3.3, a point in $\mathrm{SpFl}_{K;2n}^{\Delta>0}$ is a point $L_\bullet$ in $(\mathrm{Sp}_{2n}/P_K^C) \cap \mathrm{Fl}_{K\cup(2n-K);2n}^{\Delta>0}$. By our assumption on K, Theorem 1.1 implies $\mathrm{Fl}_{K\cup(2n-K);2n}^{\Delta>0} = \mathrm{Fl}_{K\cup(2n-K);2n}^{>0}$. Since by definition $Fl_{K\cup(2n-K);2n}^{>0}$ is the projection of $Fl_{[2n-1];2n}^{>0}$, we may extend the flag $L_\bullet$ to a flag $\widetilde{L}_\bullet$ in $Fl_{[2n-1];2n}^{>0} = Fl_{[2n-1];2n}^{\Delta>0}$. One can show that there exists such an extension with $\widetilde{L}_\bullet$ in $(\mathrm{Sp}_{2n}/B_+) \cap \mathrm{Fl}_{[2n-1];2n}^{\Delta>0}$. The projection of $\widetilde{L}_\bullet$ to $\mathrm{Fl}_{[n];2n}$ is a point in $\mathrm{SpFl}_{[n];2n}^{\Delta>0}$. Hence, if Lusztig positivity and Plücker positivity agrees for the case of $K = [n]$, then $\widetilde{L}_\bullet \in (\mathrm{Sp}_{2n}/B)^{>0}$ so that its projection $L_\bullet$ is Lusztig positive also.

The case of $K = [n]$ follows from Proposition 3.7 and Lemma 3.3 since, by Lemma 3.5, the embeddings $\mathrm{Sp}_{2n} \hookrightarrow GL_{2n}$ and $\mathrm{SO}_{2n+1} \hookrightarrow GL_{2n+1}$ satisfy the property (†). □

4 Examples, and the proof of (2) $\Longrightarrow$ (3)

4.1 Proof of (2) $\Longrightarrow$ (3)

Let G be either Sp_{2n} or SO_{2n+1}. As we have shown (3) $\Longrightarrow$ (1) $\Longrightarrow$ (2), in particular for $K = [n]$, we have that $(G/B_+)^{\geq 0} = (G/B_+)^{\Delta \geq 0}$. We now provide examples to demonstrate the contrapositive of (2) $\Longrightarrow$ (3) for Sp_{2n} (type C). Similar examples can be constructed for SO_{2n+1} (type B). For each relevant $K \subseteq [n]$ and $J = [n] \setminus K$, we will find a Plücker nonnegative point in G/P_J that does not extend to a Plücker nonnegative point in G/B_+. Such a point cannot be in the Lusztig nonnegative part $(G/P_J)^{\geq 0}$, since $(G/P_J)^{\geq 0}$ is the projection of $(G/B_+)^{\geq 0} = (G/B_+)^{\Delta \geq 0}$. We begin from the following observation:

Lemma 4.1. For $n \geq 2$, define $[L_{1;2n}] = \begin{bmatrix} 1 & 0 & \dots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{2n}$. Then, the span of $[L_{1;2n}]$ is a point in $\mathrm{SpFl}^{\Delta \geq 0}_{1;2n}$ that does not extend to a point in $\mathrm{SpFl}^{\Delta \geq 0}_{1,2;2n}$.

We now demonstrate how to construct a point in $\mathrm{SpFl}^{\Delta \geq 0}_{K;2n} \setminus \mathrm{SpFl}^{\geq 0}_{K;2n}$ for all $n \geq 2$ and K not of the form $\{k, k+1, \dots, n\}$ for some $1 \leq k \leq n$. Fix such a subset K, and denote g to be the integer satisfying $g \notin K$ and $\{g+1, g+2, \dots, n\} \subset K$, i.e. "the first gap from the right." Denote $f = \max\{i \mid i \in K \text{ and } i < g\}$. We consider three cases.

Case (i). Suppose $g = n$. The $f \times 2n$ matrix

$$M = \begin{bmatrix} I_{f-1} & 0 & 0 \\ 0 & [L_{1;2(n-f+1)}] & 0 \end{bmatrix},$$

where I_{f-1} is the $(f-1) \times (f-1)$ identity matrix, represents a flag in $\mathrm{SpFl}^{\Delta \geq 0}_{K;2n}$. If it could be extended to a flag in $\mathrm{SpFl}_{K \cup \{f+1\};2n}$, we would have a contradiction to Lemma 4.1. Thus, M provides the desired counterexample.

Case (ii). Suppose $g = n - 1$. Let $\ell = n - f - 2$. The $n \times 2n$ matrix

$$M = \left[\begin{array}{c|c|cccccc|c} I_{f-1} & 0 & & & 0 & & & & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & (-1)^{\ell} I_{\ell} & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 1 & 0 & 0 & 0 & \end{array}\right]$$

represents a flag in $\mathrm{SpFl}^{\Delta \geq 0}_{K;2n}$. Suppose it could be extended to a flag in $\mathrm{SpFl}_{[n];2n}$. Then, restricting our flag to the coordinates in the third block of columns of M would yield a contradiction to Lemma 4.1 with $n = 3$. Thus, M provides the desired counterexample.

Case (iii). We use a construction from Bloch and Karp appearing in [3, Proof of Theorem

1.1 (ii) $\implies$ (iii)]. Suppose $g < n-1$. The $n \times 2n$ matrix

$$M = \begin{bmatrix} 0 & I_{f-1} & 0 & 0 \\ B & 0 & 0 & 0 \\ 0 & 0 & I_{n-f-3} & 0 \\ C & 0 & 0 & 0 \end{bmatrix}$$

where

$$B = (-1)^{f-1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C = (-1)^{n-f-3} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix},$$

represents a flag in $\mathrm{SpFl}^{\Delta \geq 0}_{K;2n}$. Considered as a point in $\mathrm{Fl}^{\Delta \geq 0}_{K;2n}$, it cannot be extended to a point in $\mathrm{Fl}^{\Delta \geq 0}_{[2n-1];2n}$.

5 Brief Comments on Type D

In this section, we discuss why the methods used above in type B and C do not apply in type D. We take the type D Lie group SO_{2n} to be defined as the linear subgroup of SL_{2n} that preserves a non-degenerate symmetric bilinear form Q on $\mathbb{R}^{2n}$. Specifically, we have:

$$\mathrm{SO}_{2n} := \{A \in \mathrm{SL}_{2n}(\mathbb{R}) | A^t E A = E\},$$

where E is the symmetric matrix associated with Q.

Proposition 5.1. There does not exist a choice of E (which determines the embedding $\iota : \mathrm{SO}_{2n} \hookrightarrow \mathrm{SL}_{2n}$), a pinning $(T^D, B^D_+, B^D_-, \{x^D_i\}, \{y^D_i\})$ of SO_{2n}, and a map ψ satisfying (†1) in Definition 3.4.

Despite Proposition 5.1, there are other methods for working with type D flag varieties. Upcoming work of the second author [5] describes a set $\mathcal{I}$ of Plücker coordinates such that a point in the type D Grassmannian $\mathrm{OGr}(n, 2n)$ (with a suitable form E) is positive if and only if the Plücker coordinates in $\mathcal{I}$ have a particular sign pattern; as one would expect from Proposition 5.1, some Plückers must be negative.

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