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Differential transcendence and walks on self-similar graphs

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Abstract. Symmetrically self-similar graphs are an important type of fractal graph. Their Green functions satisfy order one iterative functional equations. We show when the branching number of a generating cell is two, either the graph is a star consisting of finitely many one-sided lines meeting at an origin vertex, in which case the Green function is algebraic, or the Green function is differentially transcendental over $\mathbb{C}(z)$. The proof strategy relies on a recent work of Di Vizio, Fernandes and Mishna. The result adds evidence to a conjecture of Krön and Teufl about the spectrum of this family of graphs. A long version of this abstract with complete proofs is available [4].

Keywords: Green functions, fractals, random walks, differential transcendence

1 Introduction

The discretization of fractals as graphs produces a rich object of study. Fractal graphs possess a regularity which facilitates combinatorial and graph theoretic analysis. A uniform random walk on a graph $X = (V, E)$ starts at some vertex, and proceeds along the edges of the graph. We assume the degree of every vertex is bounded and that every edge incident to a given vertex is equally likely to be chosen. A random walk is a Markov chain $(v_i)_{i=0}^n$ encoded by a pair (X, P), governed by the probability transition matrix P. The $[x, y]$ entry of P is the transition function $p(x, y) := \delta_{\{x,y\}\in E}\frac{1}{\deg(x)}$. The probability of a path $x_0, x_1, \ldots, x_n$, given that we start from x_0, is

$$\mathbb{P}_{x_0}[v_0 = x_0, v_1 = x_1, \ldots, v_n = x_n] = p(x_0, x_1)p(x_1, x_2)\cdots p(x_{n-1}, x_n).$$

The probability of a walk starting at x ending at y after n steps is denoted $p^{(n)}(x, y)$ and is equal to $\mathbb{P}_x[v_n = y]$. A *Green function* of (X, P), denoted $G(x, y|z)$, is a generating function of probabilities:

$$G(x, y|z) := \sum_{n\geq 0} p^{(n)}(x, y)z^n.$$

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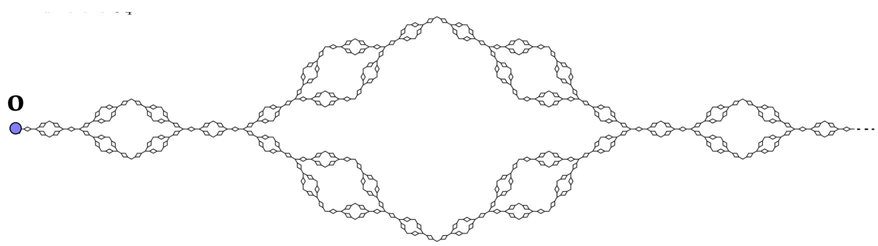


Figure 1: Close up on a symmetrically self-similar graph with branching number 2 near its origin **o**. The generating cell is pictured in Figure 2(a).

Here we focus on walks that start and end at some specified origin vertex labeled **o**, and hence refer to the Green function for a graph, and use the shorthand $G(z)$. Precisely,

$$G(z) := G(\mathbf{o}, \mathbf{o}|z) = \sum_{n\geq 0} p^{(n)}(\mathbf{o}, \mathbf{o})z^n.$$

This Green function, and the functional equations it satisfies, can tell you a surprising amount about X, and possibly even objects the graph encodes (like groups in the case of a Cayley graph). We focus here on different notions of transcendency. Recall, a series in the ring of formal power series $\mathbb{C}[[z]]$ is said to be *algebraic* if it satisfies a non-trivial polynomial equation, and is *D-finite* over $\mathbb{C}(z)$ if its derivatives span a finite dimensional vector space over $\mathbb{C}(z)$. A function is *differentiably algebraic* if it satisfies a non-trivial polynomial differential equation, and is *differentially transcendental* if is not differentiably algebraic. These categories are useful in the case of Cayley graphs: algebraicity and D-finiteness of a Green function are each correlated with structural properties of the group [2].

1.1 Symmetrically self-similar graphs

Here, we consider only symmetrically self-similar graphs with bounded geometry. They are discretizations of fractals and appear in a variety of different domains such as quantum information protocol design, the study of Brownian motion on fractals, and group theory [9, 10]. They are generated in a blow up construction from a finite cell graph. We recall a precise definition in a moment, but the Sierpiński graph is a classic example. The Green functions of these graphs are well studied [6, 8], including, the asymptotic behaviour of the excursion probabilities. Krön and Teufl [8] show that asymptotically, as n tends towards infinity, $p^{(n)}(\mathbf{o}, \mathbf{o})$ tends to $n^{\beta}w(n)$ where β is related to the geometry of the graph, and $w(n)$ is some oscillating function.

In the case of the Sierpiński graph, $\beta = \log 5/ \log 3$. Because β is irrational, we can deduce by the classic Structure Theorem for singularities of solutions to differential equations [13, Theorem 19.1 p. 111] that the Green function is not D-finite. Indeed, in almost all examples that have been studied, non-D-finiteness (and hence transcendence) of the Green function can be deduced immediately from the singular expansion. The graphs with a line as the cell graph (See Example 2.1 below for details) are the only known examples with algebraic Green functions.

Recently, Di Vizio *et al.* [3] showed that the Green function of the Sierpiński graph is not just transcendental, but actually differentially transcendental[1]. There are no similarly applicable structure theorems for differentially algebraic functions. Instead, the proof of differential transcendence uses the key result of Grabner and Woess [6] that the Green function of the Sierpiński graph satisfies an iterative equation of the form

$$G(z) = f(z)G(d(z)), \tag{1.1}$$

where f and d are explicit rational functions and are themselves Green functions of a finite graph related to the generating cell. By Di Vizio *et al.*, series solutions to such equations are either algebraic, or are differentially transcendental. In the present work we generalize this approach to the full class of the symmetrically self-similar graphs.

1.2 Transcendence of the Green functions

We are more generally motivated to understand connections between combinatorial structure, and generating function behaviour. We conjecture the following, which gives strong insight into transcendence from a combinatorial perspective.

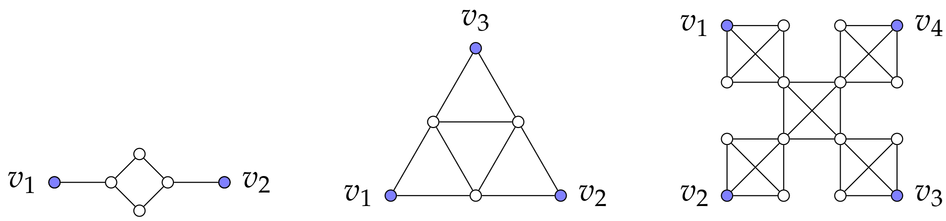
Conjecture 1.1. *The Green function of a symmetrically self-similar graph with bounded geometry is algebraic if, and only if, the graph is a star consisting of finitely many one-sided lines with exactly one origin vertex in common. Otherwise, it is differentially transcendental.*

We prove this conjecture for a sub-case in the main theorem of this paper, of which Figure 1 is an example. The hypotheses depend on the branching number θ of X, which is defined below. Here we state our Main Theorem which appears in Section 3 as Theorem 3.1.

Theorem 1.2 (Main result)**.** *Let X be a symmetrically self-similar graph with finite geometry, origin o, and branching number $\theta = 2$. Either the graph is a star consisting of finitely many one-sided lines coinciding at o or the Green function $G(\mathbf{o}, \mathbf{o}|z)$ of X, is differentially transcendental over $\mathbb{C}(z)$.*

Malozemov and Teplyaev [12] showed that the spectrum of self-similar graphs consists of the Julia set of a rational function and a (possibly empty) set of isolated eigenvalues that accumulate to the Julia set. Krön and Teufl conjectured this Julia set is an interval if, and only if, the graph is a star consisting of finitely many one-sided lines with exactly

[1] Also called hypertranscendental in some literature



(a) $\theta = 2$, $\mu = 6$ (b) $\theta = 3$, $\mu = 3$ (c) $\theta = 4$, $\mu = 5$

Figure 2: Three cell graphs, with their extremal vertices in blue.

one origin vertex in common, otherwise it is a Cantor set. Our main result resolves the conjecture of Krön and Teufl in a narrow case. Furthermore, our result demonstrates an important connection between the spectrum of infinite self-similar graphs, and the transcendent nature of Green functions.

There are still some open questions:

1. The proof of the main theorem applies whenever we can prove G has no singularity less than -1. A better understanding of the spectrum could facilitate a combinatorial characterization of this case.

2. Some infinite Cayley graphs are also fractal, and their Green functions satisfy functional equations of the type in Equation (1.1). However, they are easily proved to be algebraic [1]. Can we characterize a wider class of fractal graphs with algebraic Green functions?

2 Symmetrically self-similar graphs

The class of symmetrically self-similar graphs can be defined in two different ways: either as a fixed point to a particular type of graph morphism or as the limit of a blow up process. Different authors have conisdered it, including Malozemov and Teplyaev [11], Krön [7], and Krön and Teufl [8]. In this abstract we can, at best, sketch out one description, but finer details of the construction are well defined in these references.

We consider graphs $X = (V, E)$ with vertex set V and edge set E. Here we restrict our attention to undirected, simple graphs with no loops nor multiple edges. Given a set of vertices $C \subset V$, we define the boundary, denoted θC, to be the set of all vertices in $X \setminus C$ which are adjacent to some vertex in C. We write $\hat{C}$ for the subgraph of X spanned by C and its boundary.

A symmetrically self-similar graph X is generated from a finite cell graph C that it ultimately contains. There are restrictions on its boundary θC. The closure of the cell, $\hat{C}$, satisfies a symmetry condition. The automorphism group $\mathrm{Aut}(\hat{C})$ of $\hat{C}$ acts

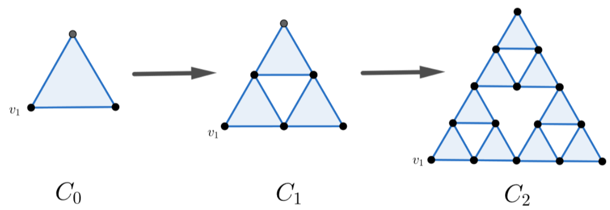


Figure 3: $C_0 = K_3$, $C_1 = \hat{C}$ and C_2. From C_i to C_{i+1} the shaded area are replaced by a copy $\hat{C}$. The limit graph in this case is the Sierpiński graph.

doubly transitively on θC, which means that it acts transitively on the set of ordered pairs $\{(x, y) \mid x, y \in \theta C,\ x \neq y\}$, where $g((x, y))$ is defined as $(g(x), g(y))$ for any $g \in \mathrm{Aut}(\hat{C})$. There are some restrictions which roughly translate into $\hat{C}$ being built from μ cliques of the same order, θ. These are two essential parameters for a graph.

Krön and Teufl [8, Theorem 1] describe an iterative process to construct a self-similar graph starting from $\hat{C}$. Roughly, the μ copies of K_θ in $\hat{C}$ are "blown up", and are replaced by a copy of the $\hat{C}$, as pictured in Figure 3. The sequence of substitutions converges to a unique limit graph C_∞.

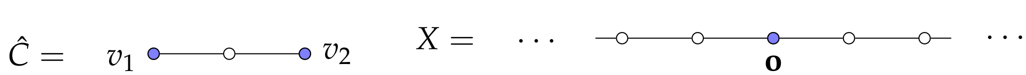
Krön showed [7, Theorem 1] that in this process, there is either a unique fixed origin cell, or a unique origin vertex, that is fixed in this blow up process. In Figure 3 it is v_1, and in the limit we denote it here $\mathbf{o}$. In this work, we consider graphs with a unique origin vertex.

The graph C_∞ is a symmetrically self-similar graphs, but so are finitely many copies of C_∞ coinciding at the origin vertex. Remark, the Green function for either will be the same.

The boundary vertices θC are called the *extremal vertices*. There must be θ of them to do the substitution so this is the *branching number* of X. The closure cell $\hat{C}$ is *the* cell associated to the graph. The edges of $\hat{C}$ can be partitioned into μ complete graphs on θ vertices. Thus, $\mu = \frac{2|E|}{\theta(\theta-1)}$ and it corresponds to the usual mass scaling factor of self-similar sets.

Figure 2 has three examples of cell graphs, their extremal vertices, and the values of θ and μ. Remark, if a self-similar graph is bipartite, its cell graph is bipartite.

Example 2.1 (A simple star graph)**.** Let $\hat{C}$ be a path on three vertices. There are two extremal vertices, so $\theta = 2$. The blow up process roughly divides each edge in two at every step, and so C_∞ a one-sided infinite line with origin at v_1. We could join two lines at the origin to make the symmetrically self-similar graph that is a bi-finite line:



The Green function here is a standard combinatorial problem: $p^{(2n)}(\mathbf{o},\mathbf{o}) = \binom{2n}{n}2^{-2n}$, from which we deduce $G(z) = \frac{1}{\sqrt{1-z^2}}$. This is an example of a star graph, and we see immediately that the generating function is algebraic. The central claim of this work is that a path is the only cell that leads to an algebraic Green function.

2.1 Green functions

One of the main results of Krön and Teufl in [8] is a functional equation satisfied by $G(z)$ that exploits the iterative nature of the blow up construction. Define d to be the *transition function*: the generating function of the probabilities that the simple random walk on $\hat{C}$ starting at o in θC hits a vertex in $\theta C \setminus \{\mathbf{o}\}$ for the first time after exactly n steps. Furthermore f is the *return function*: the generating function of f_n, the probability that the random walk on $\hat{C}$ starting at $\mathbf{o}$ returns to $\mathbf{o}$ after n stages without hitting a vertex in $\theta C \setminus \{\mathbf{o}\}$. Since the start is considered the first visit, $f_0 = 1$, and thus $f(0) = 1$. We also use $r(z)$, the probability function of first return which satisfies $f(z) = \frac{1}{1-r(z)}$. The series $d(z)$ is the generating function of d_n, the probability of walks on $\hat{C}$ that start at v_1 and end the first v_i ($i \neq 1$) they visit. Because $\hat{C}$ is finite, f and d are expressed in terms of Green functions of a finite graph. They are well understood: they are rational functions that can be computed as the determinant of a specific matrix. We first give the functional equation, and then consider singular expansions of G, d and f.

Lemma 2.2 (Krön and Teufl [8, Lemma 3])**.** *Let $G(z)$ be the Green function for a symmetrically self-similar graph with origin $\mathbf{o}$. Then, the following equation holds for all z in the open unit disc:*

$$G(z) = f(z)G(d(z)). \tag{2.1}$$

It was first proved for the Sierpiński graph by Grabner and Woess [6] before being generalized for all self-similar graphs by Krön [7]. We inspect the components of this equation more closely. The Green function $G(u, v|z)$ of a finite graph with transition probability matrix P is the $[u, v]$ entry of $(I - zP)^{-1}$. Analytically, we know a fair amount about G in this case. The structure of the Green function and its expansion near its dominant singularity is captured in the comprehensive Theorem V.7 and Lemma V.1 in [5]. We do not copy them here, but they are very important to our analysis. Remark the result does not apply to bipartite graphs, as P fails the condition of aperiodicity, but there are workarounds. The singular expansion of f (and d) is of the form $\kappa(1 - z/\rho)^{-1} + O(1)$, as $z \to \rho$ for a real, positive κ. Furthermore, as $\hat{C}$ is finite, d and f are rational.

In the case of symmetrically self-similar graphs with bounded geometries Krön and Teufl exploit their functional equation to determine the singular expansion of $G(z)$ near $z = 1$. Here $\tau = d'(1)$ and $\alpha = f(1)$.

Theorem 2.3 (Krön and Teufl [8, Theorem 5])**.** *Let X be a symmetrically self-similar graph with bounded geometry and origin vertex o. Then there exists a 1-periodic, holomorphic function ω on some horizontal strip around the real axis such that the Green function G has the local singular expansion*

$$\forall |z| < 1, \quad G(z) = (z-1)^{\eta}\left(\omega\left(\frac{1-z}{\tau}\right) + o(z-1)\right), \quad \eta = \frac{\log \mu}{\log \tau} - 1$$

Recall that μ is the number of cliques that form $\hat{C}$. Moreover, even though it is not directly stated, it follows from a quick analysis of the asymptotics of G at 1 that $\eta = \frac{\log \alpha}{\log \tau}$. In the case of the line graph, as $G(z) = \frac{1}{\sqrt{1-z^2}}$, we see that ω is constant and $\eta = -1/2$. Krön and Teufl conjecture that the converse is also true: if ω is constant, then $\hat{C}$ is a line.

The domain of analyticity of G is governed by the dynamic behaviour of f and d, and is well studied by Krön [7] and Krön and Teufl [8]. The asymptotic behaviour of the coefficients of an algebraic series is well understood, and explains why if ω is not constant, then G is not algebraic. More is true, thanks to a recent result of Di Vizio *et al.*

Theorem 2.4 (Di Vizio, Fernandes, Mishna [3, Special case of Theorem C])**.** *Suppose $y(z) \in \mathbb{C}[[z]]$ is a series that satisfies the following iterative equation, with $a(z), b(z) \in \mathbb{C}(z)$:*

$$y(z) = a(z)y(b(z)).$$

If additionally $b(z)$ satisfies the following conditions : $b(0) = 0$; $b'(0) \in \{0, \text{roots of unity}\}$; and no iteration of $b(z)$ (i.e $b \circ b \circ \cdots \circ b(z)$)is equal to the identity, then either there is some $N \in \mathbb{N}^*$, such that $y(z)^N$ is rational, or y is differentially transcendental over $\mathbb{C}(z)$.

Putting these results together we have our workhorse result.

Corollary 2.5. *Let $G(z)$ be the Green function of a symmetrically self-similar graph with origin* $\mathbf{o}$*. Then, $G(z)$ is either algebraic or differentially transcendental over $\mathbb{C}(z)$. If G is algebraic, then there exists a minimal N such that $G^N = P/Q$ with $P, Q \in \mathbb{C}[z]$, P and Q co-prime and Q monic.*

Proof. We verify that the hypotheses of Theorem 2.4 hold. We set b to d, f to a and y to G. By definition, two vertices on the boundary of the cell have distance at least two away, so $0 = d_0 = d(0)$ and $0 = d_1 = d'(0)$. No iteration of $d(z)$ is the identity, since each iteration increases the number of initial terms equal to 0. The rest follows from Theorem 2.4. □

We note additionally that there is a unique analytic continuation of any Green function $G(u, v, |z)$ to some domain beyond the radius of convergence. If G is algebraic, using the polynomial equation it satisfies we can verify that Equation (1.1) holds in that domain. Under these results, we know that G^N is rational, and it will be convenient to work with this function.

2.2 Further analysis on $f(z)$, $d(z)$ and $r(z)$

Recall that d_n, f_n and r_n are the coefficients of respective series d, f and r expanded at 0. Since they come from walks on finite graphs, we can write d and f explicitly as quotients of determinants, and deduce that the radius of convergence ρ_d (respectively ρ_f and ρ_r) of d, (f and r), given by the dominant singularity is bounded above by 1 in each case. As these are Green functions, we can deduce key facts about these elements that we use in subsequent proofs. The proofs are a mix of basic facts on Green functions, and series analysis. The proofs are in the full version of this abstract.

Lemma 2.6. *For ρ_d, ρ_f and ρ_r as defined above the following are true:*

- *ρ_d and ρ_f are each poles of order 1;*
- *$\rho_d = \rho_f < \rho_r$;*
- *$\forall z \in (1, \rho_d)$, $d'(z) > 1$ and therefore $d(z) > z$; for $z \in [0, \rho_r]$ if $f(z) = \infty$, then $z = \rho_f^{-1}$.*

Example 2.7. We can continue the example of the line to help visualize these results. $G(z)^2 = \frac{1}{1-z^2}$ is a rational function. We can similarly directly compute the transition and return probabilities by direct argument: $d = \frac{2}{2-z^2}$, $f(z) = d(z) + 1$, $r(z) = z^2/2$ hence $\rho_d = \rho_f = \sqrt{2}$. Using this we can verify that G satisfies Equation (1.1). Remark, $G(z)^2$ has simple poles at 1 and -1.

3 Proof the Main Theorem

Recall the statement of the Main Theorem:

Theorem 3.1. *Let X be a symmetrically self-similar graph with bounded geometry, origin $\mathbf{o}$ and branching number $\theta = 2$. Either the graph is a star consisting of finitely many one-sided lines coinciding at $\mathbf{o}$ or the Green function $G(\mathbf{o}, \mathbf{o}|z)$ is differentially transcendental over $\mathbb{C}(z)$.*

The proof is split into two key lemmata that we prove in Section 3.2 and Section 3.3 respectively.

Sketch of proof of Theorem 3.1. First by [7, Lemma 2] a star centered at an origin vertex if, and only $\hat{C}$ is a line. By Corollary 2.5, G is either differentially transcendental or algebraic. If G is algebraic and X is bipartite, then by Lemma 3.4, $\hat{C}$ is a line graph. Otherwise, if X is not bipartite, by Lemma 3.8, G is differentially transcendental. □

Throughout this section, we assume X is a symmetrically self-similar graph with bounded geometry, and origin $\mathbf{o}$. When we assume that G is algebraic, we write $G^N = P/Q$, as in Corollary 2.5.

3.1 The structure of an algebraic $G(z)$

In this section, assume that G is algebraic.

Lemma 3.2. *The only possible real pole of G^N greater than 0 is 1*

Proof. If G is algebraic, then ω in Theorem 2.3 is constant and by Krön [7, Theorem 7] the singularities of $G(z)$ are real, and contained in the set $(-\infty,-1] \cup [1,\infty)$. Furthermore, they are poles of G^N. Towards a contradiction, assume there is some real z_0, with $z_0 > 1$ that is either a pole or a zero. Since G^N has only a finite number of zeroes and poles, we can assume that z_0 is the smallest one strictly greater than 1. Since ρ_d is the first pole of $d(z)$ and d has non-negative coefficient, $d((1,\rho_d)) = (1,\infty)$ and d is increasing on $(1,\rho_d)$, and so there exists a $z' \in (1, \min(z_0,\rho_d))$ such that $d(z') = z_0$. Similarly, f analytic and defined on the interval $[1,\rho_d)$. Recall $f_0 = 1$ and f is increasing on this interval so z' is neither a zero nor a pole of f. Next consider Equation (1.1) and take Nth powers: $\frac{G(z)^N}{f(z)^N} = G(d(z))^N$. Since z_0 is a pole of G^N,

$$\lim_{z\to z'} G(d(z))^N = \lim_{z\to z_0} G(z_0)^N = \infty,$$

but $\lim_{z\to z'} \frac{G(z)^N}{f(z)^N} = (G(z')/f(z'))^N$, which is finite since z' is not a pole of G and z' is not a zero of f. This contradiction establishes the result. $\square$

Lemma 3.3. *Under the hypotheses above,* $\deg Q \geq N$.

Proof. We apply the central functional equation to $G^N = P/Q$ and consider expansions of G, f, and d around $\rho_f = \rho_d$. Now, $G(z)$ is not singular at ρ_f, since $\rho_f > 1$, nor is G zero at ρ_f, by Lemma 3.2. Here κ_f and κ_d are positive real constants which correspond to f and d given by [5, Theorem 7.]. We compute the following limits in $\mathbb{R}$:

$$\begin{aligned}
G(\rho_f)^N &= \lim_{z\to\rho_f} G(z)^N \\
&= \lim_{z\to\rho_f} f(z)^N P(d(z))/Q(d(z)). \\
&= \lim_{z\to\rho_f} \left(\frac{\kappa_f^N}{(1-z/\rho_f)^N} + O(z-\rho_f)^{-N+1} \right) \frac{P(\kappa_d(1-z/\rho_f)^{-1} + O(1))}{Q(\kappa_d(1-z/\rho_f)^{-1} + O(1))} \\
&= \lim_{u\to\infty} (\kappa_f^N u^N + O(u)^{N-1}) P(\kappa_d u + O(1))/Q(\kappa_d u + O(1)) \\
&= \lim_{u\to\infty} \kappa_f^N u^N P(\kappa_d u)/Q(\kappa_d u).
\end{aligned}$$

We have applied the variable change $u = (1-z/\rho_f)^{-1}$, and as z approaches ρ_f, u is arbitrarily large. The degree of Q as a polynomial in z must be larger than N since the value of the last limit is non-zero. $\square$

Analyzing the last limit, we see that $\lim_{u\to\infty} P(\kappa_d u)/Q(\kappa_d u) = 0$ hence

$$0 = \lim_{u\to\infty} G(u)^N = \lim_{u\to\infty} G(u). \tag{3.1}$$

We prove the main result in two stages depending on whether or not X is bipartite.

3.2 Case 1: X is bipartite

First we consider the case of bipartite graphs. In this case it must be that $\theta = 2$, since any complete graph on three or more vertices is not bipartite. The main result of this subsection is the following:

Lemma 3.4. *Let X be a symmetrically self-similar graph with bounded geometry, origin* **o***. If X is bipartite, and G is algebraic, then $\hat{C}$ is a path.*

We proceed by examining the form of Q.

Lemma 3.5. *Let X be a symmetrically self-similar graph with bounded geometry, origin* **o***. If X is bipartite, and G is algebraic, then $Q(z) = a_Q(1-z)^{-\eta N}(1+z)^{-\eta N}$.*

With a_Q a constant such that $Q(z)$ is a monic polynomial. Recall that $\eta = \frac{\log\mu}{\log\tau} - 1$.

Proof. By Lemma 3.2 the only positive singularity of G is at 1, and by Theorem 2.3, the exponent of the singular expansion is ν. As G is an even function with real singularities, the only other singularity is -1. As taking positive integer powers does not introduce singularities, the singularities of G are all roots of Q. From the singular expansion of G near 1, we can deduce the order of the root 1 in Q, and since the function is even, it is the same as the order of the root -1, and the result follows. $\square$

Lemma 3.6. *If X is bipartite, then $\alpha \leq \mu$ with equality if and only if $\hat{C}$ is a line graph.*

Recall that $\alpha = f(1)$.

Proof intuition. That there is equality in the case of a line graph follows from [8, Example 1]. As $\theta = 2$, μ is the number of edges in C and $\alpha = f(1)$, and α is the average number of returns for walks enumerated by the return function.

The proof is technical, but relies on a straightforward intuition that among all finite cell graphs $\hat{C}$ where v_1 and v_2 are a fixed distance apart, the average number of times a random walk starting at v_1 hits v_1 before v_2 (i.e. $f(1)$) is minimized when the graph is a line. $\square$

Corollary 3.7. *Under the hypotheses of Lemma 3.6, $\eta \geq -\dfrac{1}{2}$ with equality if, and only if, $\hat{C}$ is a line graph.*

Proof. Putting together the above equalities, we have the following:

$$\eta = -\frac{\log \alpha}{\log \tau} = \frac{\log \mu}{\log \tau} - 1 \geq \frac{\log \alpha}{\log \tau} - 1 = -\eta - 1.$$

Solving for η gives $\eta \geq -\frac{1}{2}$. □

Proof of Lemma 3.4. Recall $G^N = P/Q$ with $\deg Q \geq N$ from Lemma 3.3 and $\deg Q = -2\eta N$ by Lemma 3.5. Thus, $-2\eta N \geq N$, hence $\eta \leq -1/2$. This, coupled with Corollary 3.7 suggests that $\eta = -1/2$ precisely. By Corollary 3.7 this equality holds only if $\hat{C}$ is a line graph. □

3.3 Case 2: X is non-bipartite

In this section we assume additionally that X is not bipartite and that $\theta = 2$. We will show that if G is algebraic, and if G^N only has a pole at $z = 1$, then we can deduce a contradiction.

Lemma 3.8. *Let X be a non-bipartite symmetrically self-similar graph with origin* $\mathbf{o}$*, bounded geometry and branching number $\theta = 2$. Then the Green function $G(z)$ is differentially transcendental over $\mathbb{C}(z)$.*

Recall that if G is algebraic, then ω in Theorem 2.3 is constant, and thus the singularities of G are contained in the set $(-\infty, -1] \cup [1, \infty)$. We also know that 1 is the only singularity on the unit disc [8, p. 13], and that there are no singularities in $(1, \infty)$ by Lemma 3.2. We next show that there are also no singularities in $(-\infty, -1)$.

Lemma 3.9. *If X is not bipartite, $\theta = 2$, and G is algebraic, then G has no singularity or zero of smaller than -1.*

Proof. Since ρ_d is a pole of order 1 of d, we can deduce that $\lim_{z \to \rho_d^+} d(z) = -\infty$. We can show f has neither zero nor pole in the interval (ρ_f, ρ_r), and that d also has no pole in the interval (ρ_f, ρ_r). Therefore, by intermediate value theorem, given $z_0 \in (-\infty, -1)$, there is some pre-image $z' \in (\rho_f, \rho_r)$ such that $d(z') = z_0$. Since $z' > \rho_f$, z' is a positive real number greater than 1, therefore, z' is neither a pole nor a zero of G. We substitute this into the main functional equation to determine that $G(z_0)$ is finite and nonzero. □

Corollary 3.10. *Under these conditions, the only zero of Q is at 1. It is a zero of order $-N\eta$.*

Proof of Lemma 3.8. By Lemma 3.5, $\deg Q = -N\eta$. However, by Lemma 3.3, $\deg Q \geq N$. This is a contradiction, as $-\eta < 1$. Thus, there are no non-bipartite symmetrically self-similar graphs with both $\theta = 2$ and G algebraic. □

Anytime that we can establish that G is algebraic, and that Q only has a zero at 1, this argument will apply.

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