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Geometric Realizations of ν-Associahedra via Brick Polyhedra (Extended Abstract)

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Abstract. Brick polytopes constitute a remarkable family of polytopes associated to the spherical subword complexes of Knutson and Miller. They were introduced for finite Coxeter groups by Pilaud and Stump, who used them to produce geometric realizations of generalized associahedra arising from the theory of cluster algebras of finite types. In this paper, we present an application of the vast generalization of brick polyhedra for general subword complexes (not necessarily spherical) recently introduced by Jahn and Stump.

More precisely, we show that the ν-associahedron, a polytopal complex whose edge graph is the Hasse diagram of the ν-Tamari lattice introduced by Préville-Ratelle and Viennot, can be geometrically realized as the complex of bounded faces of the brick polyhedron of a well chosen subword complex. We also present a suitable projection to the appropriate dimension, which leads to an elegant vertex-coordinate description.

Keywords: Coxeter groups, subword complexes, brick polyhedra, associahedra, Tamari lattices.

1 Introduction

The purpose of this work is to present an application of brick polyhedra of general subword complexes to produce geometric realizations of ν-associahedra.

There are several known connections between brick polytopes and generalizations of the associahedron. A main core for such connections is Knutson and Miller's theory of subword complexes. Subword complexes are certain simplicial complexes motivated by the study of Gröbner geometry of Schubert varieties [11, 10]. One of the first connections between subword complexes and associahedra was discovered by Pilaud and Pocchiola in [12] using a slightly different terminology (of sorting networks), which was rediscovered using the subword complex terminology in [16]. A generalization for arbitrary finite Coxeter groups is due to Ceballos, Labbé and Stump in [3], who showed that c-cluster complexes arising in the theory of cluster algebras of finite type [8] can

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be obtained as well chosen subword complexes. The dual graph of the cluster complex, also known as the mutation graph for cluster algebras, is the edge graph of a well known polytope called the generalized associahedron [7].

This last connection motivated the introduction of brick polytopes for spherical subword complexes by Pilaud and Stump [14], who generalized the notion of brick polytopes in type A by Pilaud and Santos in [13]. One of the main results in [14] provides a geometric realization of the generalized associahedron as the brick polytope of a spherical subword complex. Later on, Jahn and Stump presented a generalization of brick polyhedra for arbitrary subword complexes (not necessarily spherical) of finite type [9], who nicely connected them to the combinatorics and geometry of Bruhat intervals and Bruhat cones in Coxeter groups. Our work presents the first application of brick polyhedra to produce geometric realizations of ν-associahedra.

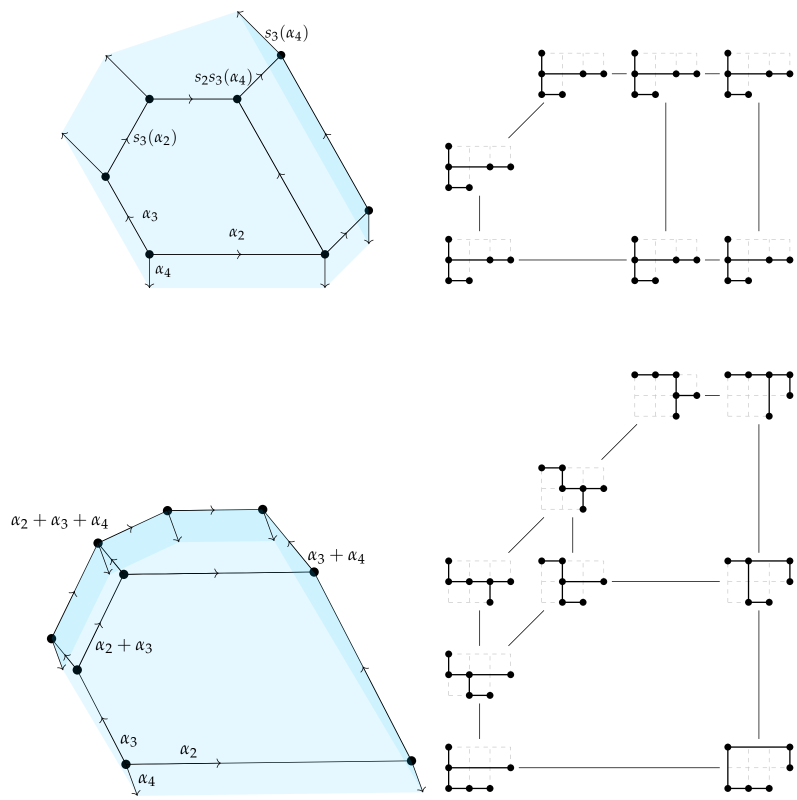

Figure 1: Comparison of the ν-brick polyhedron and ν-associahedron for $\nu = ENEEN$ (top) and for $\nu = EENEN$ (bottom).

Given a lattice path ν, consisting of finitely many north steps N and east steps E, the ν-associahedron [4] is a polytopal complex whose edge graph is the Hasse diagram of the ν-Tamari lattice introduced by Préville-Ratelle and Viennot in [15], and whose face poset is the poset of interior faces of the corresponding ν-Tamari complex [4] ordered by reverse inclusion. The case $\nu = (NE)^n$ recovers the classical associahedron, whose edge graph is the Hasse diagram of the classical Tamari lattice, which can be defined as the rotation poset of rooted binary trees and plays a fundamental role in many areas in mathematics, computer science and physics. The case $\nu = (NE^m)^n$ recovers the m-Tamari lattices of Bergeron [1], whose interval enumeration has beautiful conjectural connections to the theory of trivariate diagonal harmonics in representation theory. Using techniques from tropical geometry, Ceballos, Sarmiento and Padrol produced the first geometric realizations of ν-associahedra [4], solving an open problem of Bergeron in this more general set up.

In this paper, we present a second geometric realization of the ν-associahedron as the complex of bounded faces of the brick polyhedron of a well chosen (non-spherical) subword complex. We also provide a suitable projection, in the special case where ν has non consecutive north steps, which provides a realization of the appropriate dimension, with a beautiful and elegant vertex-coordinate description. The brick polyhedron and the projection are illustrated for two examples in Figure 1. A 3-dimensional example (resulting dimension after projecting) is illustrated in Figure 8.

2 Brick polyhedra

Throughout this work, we restrict our study to finite Coxeter groups, subword complexes and brick polyhedra of type A.

A Coxeter system (W, S) of type A_n consists of the Coxeter group $W := \mathcal{S}_{n+1}$ of permutations of $[n+1]$, which acts on the space $\{x \in \mathbb{R}^{n+1} \mid x_1 + \cdots + x_{n+1} = 0\}$ by permuting coordinates. It is finitely generated by simple transpositions $S := \{s_p \mid p \in [n]\}$ with $s_p = (p, p+1)$. The root system is defined by $\Phi = \{e_p - e_q \mid p \neq q \in [n+1]\}$, and can be partitioned into positive roots $\Phi^+ = \{e_i - e_j \mid 1 \leq i < j \leq n+1\}$ and negative roots $\Phi^- = \{e_j - e_i \mid 1 \leq i < j \leq n+1\}$. The simple roots are $\Delta = \{\alpha_p := e_p - e_{p+1} \mid p \in [n]\}$ and the fundamental weights are $\nabla = \{\omega_p := \sum_{q \leq p} e_q \mid p \in [n]\}$.

Definition 2.1 (Subword complex [10])**.** For a Coxeter system (W, S), let $Q = (q_1, ..., q_m)$ be a word in the generators S of W and let $w \in W$. The *subword complex* $\mathcal{SC}(Q, w)$ is the simplicial complex whose facets are subsets $I \subseteq [m]$ such that $Q_{[m]\setminus I}$ is a reduced expression for w. Here Q_J denotes the subword of Q with positions at J.

We can now define two important functions associated with brick polyhedra.

Definition 2.2 (Root and Weight Function [3, 14])**.** Given a facet I of $\mathcal{SC}(Q, w)$, the *root function* is the map $r(I, \cdot) : [m] \to \Phi$ defined by $r(I, k) := \prod Q_{\{1,...,k-1\}\setminus I}(\alpha_{q_k})$. We call $R(I) := \{\{r(I, i) \mid i \in I\}\}$ the *root configuration* of I. The other function is the *weight function* $\omega(I, \cdot) : [m] \to \Phi$, defined by $\omega(I, k) := \prod Q_{\{1,...,k-1\}\setminus I}(\omega_{q_k})$.

Definition 2.3 (Brick polyhedron [9]). The *brick vector* of a facet $I \in \mathcal{SC}(Q, w)$ is

$$b(I) := -\sum_{k=1}^{m} \omega(I, k).$$

The *Bruhat cone* of a non-empty subword complex $\mathcal{SC}(Q, w)$ is defined by

$$\mathcal{C}^+(w, \mathrm{Dem}(Q)) := \mathrm{cone}\{\beta \in \Phi^+ \mid w \prec_B s_\beta w \leq_B \mathrm{Dem}(Q)\},$$

where $\mathrm{Dem}(Q) = \max_{\leq_B}\{\prod Q_X \mid X \subseteq \{1, ..., m\}\}$ denotes the Demazure product of Q [10, Lemma 3.4 (1)], and $\leq_B, \prec_B$ denote the Bruhat order and its cover relation. The *brick polyhedron* $\mathcal{B}(Q, w)$ is the Minkowski sum of the convex hull of all brick vectors and the Bruhat cone:

$$\mathcal{B}(Q, w) := \mathrm{conv}\{b(I) \mid \text{I facet of } \mathcal{SC}(Q, w)\} + \mathcal{C}^+(w, \mathrm{Dem}(Q)).$$

At first glance, brick polyhedra do not seem natural, but they turn out to have very nice properties related to the combinatorics and geometry of the corresponding subword complex [9]. We aim to relate this to the combinatorics and geometry of ν-associahedra.

3 The ν-Tamari lattice and the ν-associahedron

We start by introducing the concept of ν-Tamari lattices using the conventions in [5]. We denote by ν a lattice path with finitely many east and north steps. Let F_ν be the Ferrers diagram weakly above ν, inside the smallest rectangle containing ν. We denote by A_ν the set of lattice points weakly above ν, which are inside F_ν. For a lattice point $p \in A_\nu$, we denote by $d(p)$ the lattice distance from p to the top-left corner of F_ν.

Definition 3.1 (ν-tree [5]). For $p, q \in A_\nu$, we say that p and q are ν-incompatible, denoted $p \not\sim q$, if and only if p is southwest (SW) of q or p is northeast (NE) of q, and the smallest rectangle containing p and q lies completely inside F_ν. A *ν-tree* is a maximal collection of pairwise ν-compatible elements in A_ν. Its elements are called *nodes* and the top left corner is called *root*. We associate a rooted binary tree to each ν-tree T by connecting every $p \in T$ other than the root to the next in north or west direction, see Figure 3 (Left).

Definition 3.2 (ν-Tamari lattice [5]). Two ν-trees T, T' are related by a right rotation if T' can be obtained from T by exchanging $q \in T$ with $q' \in T'$ in as in Figure 2 with $p, r \in T, T'$. The *ν-Tamari lattice* is the rotation poset of ν-trees. An example of the Hasse diagram of the ν-Tamari lattice for $\nu = ENEEN$ is the edge graph of Figure 4.

Definition 3.3 (ν-Tamari Complex [5]). The *ν-Tamari complex* is the simplicial complex $\mathcal{TC}(\nu)$ of pairwise ν-compatible sets in A_ν. The dimension of a face I is $\dim(I) = |I| - 1$. The facets are the ν-trees.

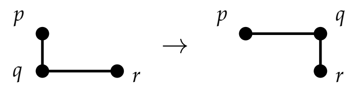

Figure 2: Right rotation.

Definition 3.4 (ν-subword complex $\mathcal{SC}(Q_\nu, w_\nu)$ [5])**.** Given a lattice path ν we label each lattice point $p \in A_\nu$ by the transposition $s_{d(p)+1}$, see Figure 3 (Middle) for an example. Furthermore, define Q_ν as the word obtained by reading the associated transpositions from bottom to top, and the columns from left to right. The element w_ν is the product of transpositions in the complement of a ν-tree, see Figure 3 (Right). The complements of ν-trees are reduced expressions of w_ν in Q_ν and the effect of a rotation keeps w_ν constant.

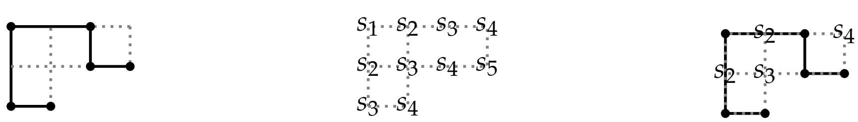

Figure 3: Left: a ν-tree for $\nu = ENEEN$. Middle: lattice points A_ν labeled by transpositions; the corresponding word is $Q_\nu = (s_3, s_2, s_1, s_4, s_3, s_2, s_4, s_3, s_5, s_4)$. Right: complement of ν-tree and its corresponding element $w_\nu = s_2 s_3 s_2 s_4$.

The following result from [5] provides a nice description of the ν-Tamari complex as a well chosen subword complex.

Theorem 3.5 ([5])**.** *The ν-subword complex $\mathcal{SC}(Q_\nu, w_\nu)$ is isomorphic to the ν-Tamari complex.*

The interior faces of the ν-Tamari complex can be characterized as follows.

Definition 3.6 ([6])**.** A node q in a ν-tree T is called an ascent if there exists a node in T to the north and another to the east of q. Equivalently, ascents of T are the nodes of T on which we can apply a right rotation.

Lemma 3.7 ([6])**.** *The interior faces I of the ν-Tamari complex are in bijective correspondence with pairs (T, A), where T is a ν-tree and A is a subset of its ascents, via the map $I = T \setminus A$.*

As we can observe from Figure 4, the ν-Tamari lattice has a very rich underlying geometric structure. Its Hasse diagram can be geometrically realized as the edge graph of a polytopal complex called the ν-associahedron [4]. The construction in [4] uses techniques from tropical geometry. The goal of this work is to give new realizations in terms of brick polyhedra. The following is a purely combinatorial definition.

Definition 3.8 (ν-Associahedron [4]). The *ν-associahedron* is a polytopal complex induced by an arrangement of tropical hyperplanes, whose poset of faces (ordered by containment) is anti-isomorphic to the poset of interior faces of the ν-Tamari complex.

Corollary 3.9 ([6]). *The faces of the ν-associahedron are in correspondence with pairs (T, A), where T is a ν-tree and A is a subset of its ascents. The dimension of the face corresponding to (T, A) is the cardinality $|A|$.*

Example 3.10 (ν-Associahedron for $\nu = ENEEN$). Consider the ν-subword complex for $\nu = ENEEN$, the ν-associahedron is shown in Figure 4, and its edge graph is the Hasse diagram of the ν-Tamari lattice. The interior face I_1 illustrated in Figure 4 corresponds to the orange line segment, while the interior face I_2 corresponds to the red pentagon. Note that $I_2 \subseteq I_1$, but the face corresponding to I_1 is contained in the face corresponding to I_2. The containment poset of interior faces is reversed.

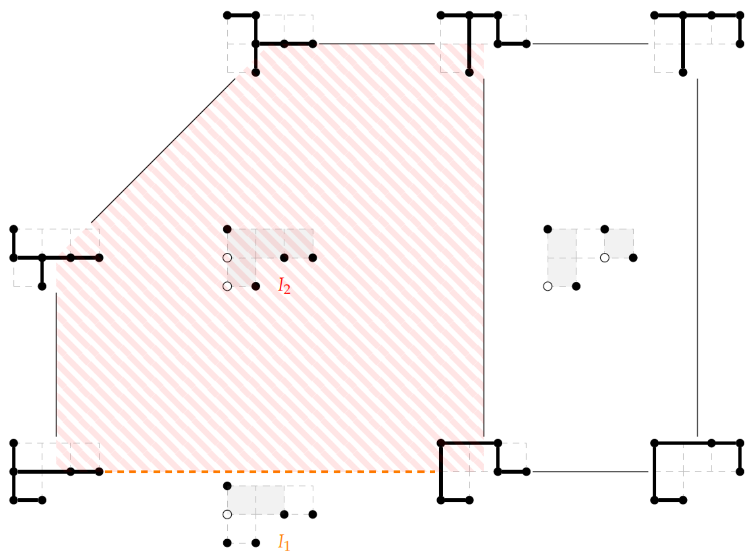

Figure 4: ν-Associahedron for $\nu = ENEEN$.

Example 3.11 (ν-Brick Polyhedron for $\nu = ENEEN$). The word Q_ν and element w_ν are $Q_\nu = (s_3, s_2, s_1, s_4, s_3, s_2, s_4, s_3, s_5, s_4)$, and $w_\nu = s_3 s_2 s_3 s_4$. The brick polyhedron $\mathcal{B}(Q_\nu, w_\nu)$ is of dimension 4 in a 5 dimensional space, so we can not really draw it. However, to get a feeling about how it looks like, we can remove the letters s_1 and s_5 from Q_ν. They are contained in every facet (are non-flippable) and give rays in the brick polyhedron. If we call $\widetilde{Q_\nu}$ the resulting word $\widetilde{Q_\nu} = (s_3, s_2, s_4, s_3, s_2, s_4, s_3, s_4)$ then the brick polyhedron $\mathcal{B}(\widetilde{Q_\nu}, w_\nu)$ is of dimension 3 and is illustrated in Figure 5 (Left). The Bruhat cone is given by $\mathcal{C}^+(w_\nu, \mathrm{Dem}(Q_\nu)) = \mathrm{cone}\{\alpha_4, s_3(\alpha_4)\}$.

4 A geometric realization via brick polyhedra

The goal of this work is to show that the ν-associahedron can be geometrically realized as the complex of bounded faces of a brick polyhedron. For this, we consider the ν-subword complex from previous section. The *ν-brick polyhedron* $\mathcal{B}(Q_\nu, w_\nu)$ is defined as the brick polyhedron of the ν-subword complex $\mathcal{SC}(Q_\nu, w_\nu)$. The following is our main result, which can be observed from Examples 3.10 and 3.11, illustrated in Figure 5.

Theorem 4.1. *The ν-associahedron is geometrically realized as the polytopal complex of bounded faces of the ν-brick polyhedron $\mathcal{B}(Q_\nu, w_\nu)$. In other words, the poset of bounded faces of $\mathcal{B}(Q_\nu, w_\nu)$ is anti-isomorphic to the poset of interior faces of the ν-subword complex ($\cong$ ν-Tamari complex).*

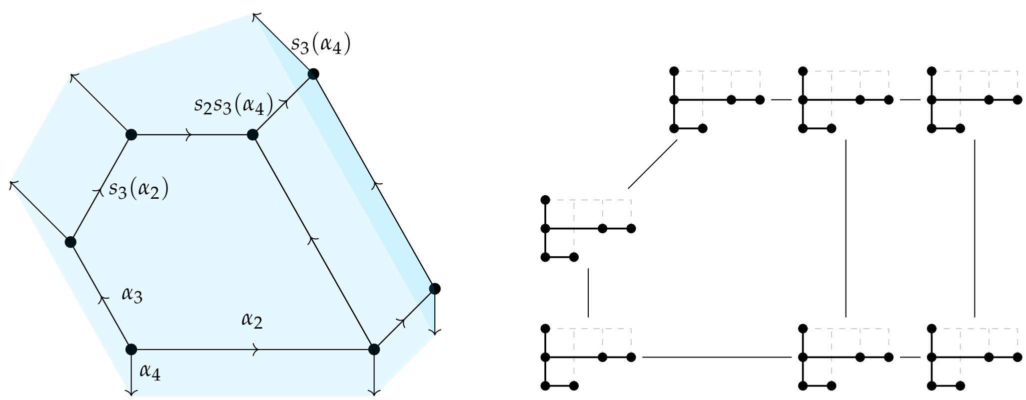

Figure 5: Comparison of the ν-brick polyhedron and ν-associahedron for $\nu = ENEEN$.

4.1 Faces of brick polyhedra

In order to prove Theorem 4.1, it is useful to have a better understanding of the faces of brick polyhedra in general. For this purpose, we use a notion of modified brick polyhedra that we define now.

Definition 4.2 (Modified Bruhat Cone $\mathcal{C}^{I,+}$)**.** We denote by $\mathcal{SC}^I(Q, w)$ the set of all faces in the subword complex $\mathcal{SC}(Q, w)$ that contain a given face $I \in \mathcal{SC}(Q, w)$:

$$\mathcal{SC}^I(Q, w) := \{J \in \mathcal{SC}(Q, w) : I \subseteq J\}.$$

For $J \in \mathcal{SC}^I(Q, w)$, the modified root configuration $R^I(J)$ is given by

$$R^I(J) := \{r(J, j) \mid j \in J \setminus I\}.$$

We define the modified Bruhat cone $\mathcal{C}^{I,+}$ as

$$\mathcal{C}^{I,+} := \bigcap_{J \in \mathcal{SC}^I(Q,w)} \text{cone } R^I(J).$$

Definition 4.3 (Modified Brick Polyhedron $\mathcal{B}^I(Q,w)$)**.** For a face $I \in \mathcal{SC}(Q,w)$ of a non-empty subword complex, the *modified brick polyhedron* $\mathcal{B}^I(Q,w)$ is the polyhedron

$$\mathcal{B}^I(Q,w) := \text{conv}\{b(J) \mid \text{J facet of } \mathcal{SC}(Q,w) \text{ and } I \subseteq J\} + \mathcal{C}^{I,+}.$$

Proposition 4.4. *Every face F of a brick polyhedron $\mathcal{B}(Q,w)$ is of the form $\mathcal{B}^I(Q,w)$ for some face $I \in \mathcal{SC}(Q,w)$.*

This proposition states that every face of a brick polyhedron $\mathcal{B}(Q,w)$ is a modified brick polyhedron $\mathcal{B}^I(Q,w)$; however, we remark that not every $\mathcal{B}^I(Q,w)$ is a face of $\mathcal{B}(Q,w)$. The Proposition was essentially proved in [9] using the description of faces via linear functionals, see [9, Remark 4.11] and [9, Corollary 3.24 and Proposition 4.6]. Our approach is a bit different and is based on the following lemma, cf. [9, Proposition 4.6 and Remark 4.11]. For a face F of $\mathcal{B}(Q,w)$, we denote by V_F the vector space spanned by F.

Lemma 4.5. *Consider a face F of the brick polyhedron $\mathcal{B}(Q,w)$, let $J \in \mathcal{SC}(Q,w)$ be a facet such that the brick vector $b(J) \in F$ and $J^F := \{j \in J : r(J,j) \in V_F\}$. The following hold:*

1. *for $I = J \setminus J^F$, we have $F = \mathcal{B}^I(Q,w)$ and*
2. *for $J' \in \mathcal{SC}(Q,w)$ a facet, we have $I \subseteq J'$ if and only if $b(J') \in F$.*

4.2 Bounded faces of ν-brick polyhedra

The following result gives a complete characterization of the bounded faces of the ν-brick polyhedron.

Theorem 4.6. *The bounded faces of the ν-brick polyhedron $\mathcal{B}(Q_\nu, w_\nu)$ are exactly the $\mathcal{B}^I(Q_\nu, w_\nu)$, for $I = T \setminus A$, where T is a ν-tree and $A \subseteq T$ is a subset of ascents. Moreover,*

$$\mathcal{B}^I(Q_\nu, w_\nu) = conv\{b(J) \mid I \subseteq J \text{ a facet of } \mathcal{SC}(Q_\nu, w_\nu)\},$$

and its face poset is the reverse containment poset on the set $\{J \in \mathcal{SC}(Q_\nu, w_\nu) : I \subseteq J\}$.

Our proof of this theorem is based on the following two key lemmas.

Lemma 4.7. *Let $I = T \setminus A$, where T is a ν-tree and $A \subseteq T$ a subset of ascents, and define $\beta_t := r(T,t)$, $t \in T$. There exists a linear functional f such that*

$$f(\beta_a) = 0 \text{ for } a \in A \text{ and } f(\beta_t) > 0 \text{ for } t \in T \setminus A.$$

We denote by $Q_{\nu,I}$ the word obtained from Q_ν by deleting the letters with positions in I, and consider the corresponding subword complex $\mathcal{SC}(Q_{\nu,I}, w_\nu)$.

Lemma 4.8. *Let $I = T \setminus A$, where T is a ν-tree and $A \subseteq T$ a subset of ascents, then $\mathcal{SC}(Q_{\nu,I}, w_\nu)$ is a spherical and root independent subword complex.*

Our main result, Theorem 4.1, follows from Theorem 4.6.

5 A projection

Since the dimension of the ν-brick polyhedron is usually much higher than the dimension of the ν-associahedron, it is interesting to study suitable projections to obtain figures in the appropriate dimension. In this section, we provide an elegant projection in the case where ν has no two consecutive north steps.

For convenience, we consider paths of the form $\nu = (NE^{k_n}) \cdots (NE^{k_1})$, where $k_i \geq 1$. Notice that we are adding a north step N at the beginning and some east steps at the end of the path, but this does not affect the combinatorics of the bounded components of the brick polyhedron. One can double check that the ν-brick polyhedron $\mathcal{B}(Q_\nu, w_\nu) \subseteq \mathbb{R}^{n+2+\Sigma(k_i-1)}$. Since the first and last coordinates of the brick vectors $b(T)$ are constant for every ν-tree T, we omit them, and write $\widetilde{b}(T)$ for the resulting vectors. Moreover, we denote by $\widetilde{\mathcal{B}}(Q_\nu, w_\nu)$ the result of omitting the first and last coordinates of the intersection of the brick polyhedron $\mathcal{B}(Q_\nu, w_\nu)$ with the affine subspace defined by the first and the last coordinates being those constant numbers. In particular, the bounded components of $\mathcal{B}(Q_\nu, w_\nu)$ are in correspondence with the bounded components of $\widetilde{\mathcal{B}}(Q_\nu, w_\nu)$. After removing the first and the last coordinates, we have

$$\widetilde{\mathcal{B}}(Q_\nu, w_\nu) \subseteq \mathbb{R}^{n+\Sigma(k_i-1)} = \mathbb{R}^N.$$

We denote by $x_I := \sum_{i \in I} x_i$. Our projection uses the sets $\widetilde{M}_1, \ldots, \widetilde{M}_n$, which are defined recursively by setting $\widetilde{M}_j$ to be the last $(k_i - 1)$ elements of $[N] \setminus \bigcup_{i=1}^{j-1} \widetilde{M}_i$, and let $M_j = \widetilde{M}_j \cup \{j\}$ for $j \in [n]$.

Definition 5.1 (Projection)**.** We define the maps

$$\pi_1 : \mathbb{R}^N \longrightarrow \mathbb{R}^n, (x_1, ..., x_N) \mapsto (x_{M_1}, ..., x_{M_n}) \in \mathbb{R}^n,$$

$$\pi_2 : \mathbb{R}^n \to \mathbb{R}^{n-1}, (x_{M_1}, ..., x_{M_n}) \mapsto (x_{M_1}, x_{M_1} + x_{M_2}, ..., x_{M_1} + ... + x_{M_{n-1}}).$$

Finally, the projection $\pi : \mathbb{R}^N \to \mathbb{R}^{n-1}$ is defined by the composition $\pi = \pi_2 \circ \pi_1$.

Theorem 5.2. *Let $\nu = (NE^{k_n}) \cdots (NE^{k_1})$ with $k_i \geq 1$ (no consecutive north steps). Then the projection $\pi : \mathbb{R}^N \to \mathbb{R}^{n-1}$ of the bounded components of $\widetilde{\mathcal{B}}(Q_\nu, w_\nu)$ is a realization of the ν-associahedron of the desired dimension.*

Moreover, the coordinates of the projected vertices can be simply described as follows. Let T_0 be the minimal ν-tree of the ν-Tamari lattice. We denote by

$$y(T) = (y_1, \dots, y_{n-1}) := \pi(\widetilde{b}(T)) - \pi(\widetilde{b}(T_0)).$$

These y-coordinates are the translation of the projection of the bounded components of the ν-brick polyhedron by the vector $-\pi(\widetilde{b}(T_0))$. In particular, $y(T_0) = (0, \dots, 0)$.

It turns out that these new coordinates can be described in a very simple and elegant combinatorial way. First, we label the horizontal lines of the Ferrers diagram determined by the path ν from 1 to $n-1$, from top to bottom, omitting the top most row. We denote by $P_i = P_i(T)$ the unique shortest path connecting the root to the left most node of T on the i-th horizontal line, and let $\mathrm{area}(P_i)$ be the number of boxes left to P_i. Two examples are illustrated in Figure 6. Magically, $y_i = \mathrm{area}(P_i)$.

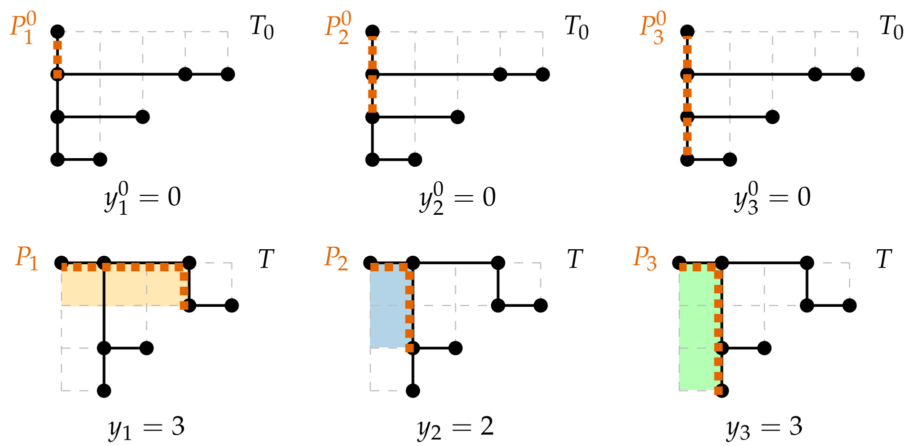

Figure 6: The canonical coordinates $y(T_0) = (y_1^0, y_2^0, y_3^0) = (0,0,0)$, and $y(T) = (y_1, y_2, y_3) = (3,2,3)$ for the two ν-trees T_0 and T. The entry $y_i(T)$ is the area (i.e. number of boxes to the left) of the path $P_i(T)$ connecting the root to the leftmost node of T at level i (increasing from top to bottom).

Theorem 5.3. *Let $\nu = (NE^{k_n}) \cdots (NE^{k_1})$ with $k_i \geq 1$ (no consecutive north steps). For a ν-tree T we have $y(T) = (y_1, ..., y_{n-1})$, where $y_i = \mathrm{area}(P_i(T))$. These coordinates determine a realization of the ν-associahedron.*

Remark 5.4. As a consequence, the projection $\pi : \mathbb{R}^N \to \mathbb{R}^{n-1}$ of the bounded components of $\widetilde{\mathcal{B}}(Q_\nu, w_\nu)$ is a translation of the canonical realization of the ν-associahedron described in [2]. This may be regarded as a Loday-like realization of the ν-associahedron, because in the classical case $\nu = (NE)^n$ both realizations are affinely equivalent.

Example 5.5. We continue the Examples 3.10 and 3.11, $\nu = NENEENE$. The projected points coincide with the canonical realization by [2], up to a translation, see Figure 7.

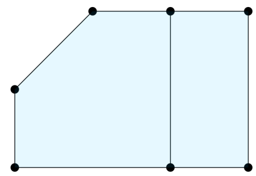

Figure 7: Projection of the bounded components of $\widetilde{\mathcal{B}}(Q_\nu, w_\nu)$ for $\nu = NENEENE$.

Example 5.6. For $\nu = NENENEENE = (NE^1)(NE^1)(NE^2)(NE^1)$, we obtain $N = n + \sum(k_i - 1) = 4 + (0+1+0+0) = 5$. Furthermore, $M_1 = \{1\}$, $M_2 = \{2,5\}$, $M_3 = \{3\}$, and $M_4 = \{4\}$. So we group coordinates 2 and 5 of $\widetilde{b}(T)$ together. In order to illustrate how the projection works, let us consider the two ν-trees T and T_0 in Figure 6. We obtain:

$$\begin{aligned} b(T) &= -(17,13,13,9,13,2,0) & b(T_0) &= -(17,16,12,10,10,2,0) \\ \widetilde{b}(T) &= -(13,13,9,13,2) & \widetilde{b}(T_0) &= -(16,12,10,10,2) \\ \pi_1(\widetilde{b}(T)) &= -(13,15,9,13) & \pi_1(\widetilde{b}(T_0)) &= -(16,14,10,10) \\ \pi(\widetilde{b}(T)) &= -(13,28,37) & \pi(\widetilde{b}(T_0)) &= -(16,30,40) \end{aligned}$$

The difference between the corresponding projected brick vectors is

$$y(T) = (y_1, y_2, y_3) = \pi(\widetilde{b}(T)) - \pi(\widetilde{b}(T_0)) = (3,2,3).$$

As we can see from Figure 6, the entry $y_i = \text{area}(P_i)$ counts the number of boxes left to the path P_i connecting the root to the left most node of T at level i.

Although the ν-brick polyhedron $\mathcal{B}(Q_\nu, w_\nu) \subseteq \mathbb{R}^7$, the projection of its bounded components lies in $\mathbb{R}^3$ and is illustrated in Figure 8.

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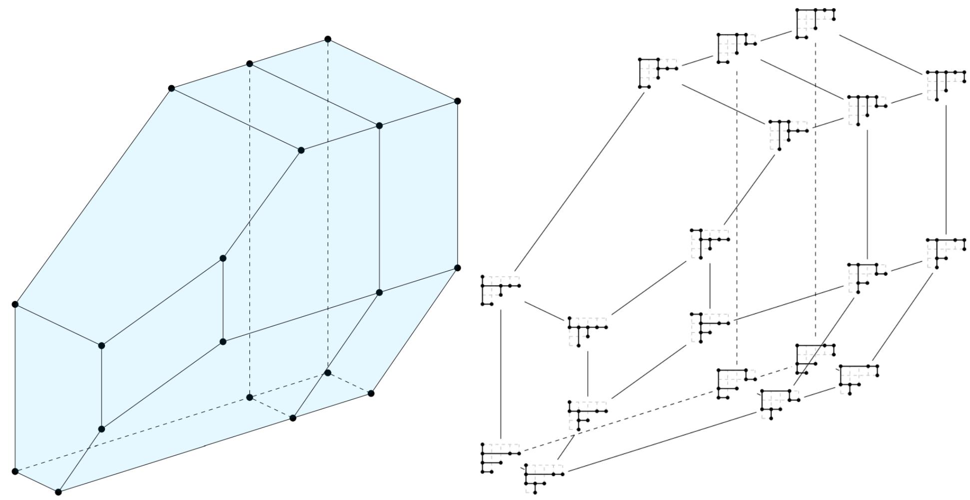

Figure 8: Left: Projection of the bounded components of $\widetilde{\mathcal{B}}(Q_\nu, w_\nu)$, Right: ν-associahedron.

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