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Random Combinatorial Billiards and Stoned Exclusion Processes

Colin Defant*[1]

[1]*Department of Mathematics, Harvard University, Cambridge, MA 02139, USA*

Abstract. We introduce and study several *random combinatorial billiard trajectories*. Such a system, which depends on a fixed parameter $p \in (0,1)$, models a beam of light that travels in a Euclidean space, occasionally randomly reflecting off of a hyperplane in the Coxeter arrangement of an affine Weyl group with some probability that depends on the side of the hyperplane that it hits. In one case, we (essentially) recover Lam's reduced random walk in the limit as p tends to 0. The investigation of our random billiard trajectories relies on an analysis of new finite Markov chains that we call *stoned exclusion processes*. These processes have remarkable stationary distributions determined by well-studied polynomials such as ASEP polynomials, inhomogeneous TASEP polynomials, and open boundary ASEP polynomials; in many cases, it was previously not known how to construct Markov chains with these stationary distributions. Using multiline queues, we analyze correlations in the *stoned multispecies TASEP*, allowing us to determine limit directions for reduced random billiard trajectories and limit shapes for new random growth processes for n-core partitions.

Keywords: Combinatorial billiards, ASEP, reduced random walk, affine Weyl group, stoned exclusion process, multiline queue

1 Weyl Groups and Reduced Random Walks

Let Φ be a finite irreducible crystallographic root system spanning a Euclidean space V, and write $\Phi = \Phi^+ \sqcup \Phi^-$, where Φ^+ and $\Phi^- = -\Phi^+$ are the set of positive roots and the set of negative roots, respectively. Let W and $\widetilde{W}$ be the Weyl group and affine Weyl group of Φ, respectively. Let I be an index set so that $\{\alpha_i : i \in I\}$ is the set of simple roots, and let $\widetilde{I} = \{0\} \sqcup I$. Write $S = \{s_i : i \in I\}$ and $\widetilde{S} = \{s_i : i \in \widetilde{I}\}$ for the sets of simple reflections of W and $\widetilde{W}$, respectively. Let $\theta \in \Phi$ be the highest root of W.

Let V^* be the dual space of V. Each root $\beta \in \Phi$ has an associated coroot $\beta^\vee \in V^*$. Let $Q^\vee = \mathrm{span}_{\mathbb{Z}}\{\beta^\vee : \beta \in \Phi\} \subseteq V^*$ denote the coroot lattice of W. For $\beta \in \Phi^+$ and $k \in \mathbb{Z}$, we define the hyperplane $\mathrm{H}_\beta^k = \{\gamma \in V^* : \gamma(\beta) = k\} \subseteq V^*$. The *Coxeter arrangements* of W and $\widetilde{W}$ are $\mathcal{H}_W = \{\mathrm{H}_\beta^0 : \beta \in \Phi^+\}$ and $\mathcal{H}_{\widetilde{W}} = \{\mathrm{H}_\beta^k : \beta \in \Phi^+,\ k \in \mathbb{Z}\}$, respectively.

*colindefant@gmail.com. Supported by the National Science Foundation under Award No. 2201907 and by a Benjamin Peirce Fellowship at Harvard University.

There is a faithful right action of $\widetilde{W}$ on V^*; each simple reflection $s_i \in S$ acts via the reflection through the hyperplane $\mathrm{H}^0_{\alpha_i}$, while s_0 acts via the reflection through H^1_θ. The closures of the connected components of $V^* \setminus \bigcup_{\mathrm{H}\in\mathcal{H}_W} \mathrm{H}$ are called *chambers*, while the closures of the connected components of $V^* \setminus \bigcup_{\mathrm{H}\in\mathcal{H}_{\widetilde{W}}} \mathrm{H}$ are called *alcoves*. The *fundamental chamber* is $\mathcal{C} = \{\gamma \in V^* : \gamma(\alpha_i) \geq 0 \text{ for all } i \in I\}$, and the *fundamental alcove* is $\mathcal{A} = \{\gamma \in \mathcal{C} : \gamma(\theta) \leq 1\}$. The map $u \mapsto \mathcal{C}u$ is a bijection from W to the set of chambers. The map $u \mapsto \mathcal{A}u$ is a bijection from $\widetilde{W}$ to the set of alcoves. Two distinct alcoves are *adjacent* if they share a common facet. The alcoves adjacent to $\mathcal{A}u$ are precisely the alcoves of the form $\mathcal{A}su$ for $s \in \widetilde{S}$. Let $\mathrm{H}^{(u,s)}$ denote the unique hyperplane separating $\mathcal{A}u$ and $\mathcal{A}su$. For $\gamma \in V^* \setminus \{0\}$, let $\langle\gamma\rangle$ denote the unit vector in V^* that points in the same direction as γ. By a slight abuse of notation, we write $\langle\mathcal{A}u\rangle$ for the unit vector that points in the same direction as the center of $\mathcal{A}u$. Let $w_\circ$ denote the long element of W.

Consider the $|I|$-dimensional torus $\mathbb{T} = V^*/Q^\vee$, and let $\mathfrak{q}\colon V^* \to \mathbb{T}$ be the natural quotient map. There is a quotient map $\widetilde{W} \to W$, which we denote by $w \mapsto \overline{w}$, where $\overline{w}$ is the unique element of w such that $\mathfrak{q}(\mathcal{A}w) = \mathfrak{q}(\mathcal{A}\overline{w})$.

In [18], Lam introduced the *reduced random walk* in $\widetilde{W}$, a very natural and intriguing random walk on the set of alcoves of $\mathcal{H}_{\widetilde{W}}$ (equivalently, on $\widetilde{W}$). The reduced random walk starts at $\mathcal{A}$. Suppose that at some point in time, the walk is at an alcove $\mathcal{A}u$. A simple reflection s is chosen uniformly at random from $\widetilde{S}$. If $\mathrm{H}^{(u,s)}$ separates $\mathcal{A}u$ from $\mathcal{A}$, then the walk stays at the alcove $\mathcal{A}u$; otherwise, it transitions to $\mathcal{A}su$. Let $\widetilde{\mathbf{M}}_{\mathrm{Lam}}$ denote the reduced random walk in $\widetilde{W}$.

Let $\widehat{W} = \{w \in \widetilde{W} : \mathcal{A}w \subseteq \mathcal{C}\}$ denote the set of *affine Grassmannian* elements of $\widetilde{W}$. Lam also introduced the *affine Grassmannian reduced random walk* in $\widehat{W}$, which is the random walk $\widehat{\mathbf{M}}_{\mathrm{Lam}}$ obtained by conditioning $\widetilde{\mathbf{M}}_{\mathrm{Lam}}$ to stay within $\mathcal{C}$. By projecting $\widehat{\mathbf{M}}_{\mathrm{Lam}}$ through the natural quotient map $\widetilde{W} \to W$, Lam obtained an irreducible finite-state Markov chain $\mathbf{M}_{\mathrm{Lam}}$ on W; when W is of type A_{n-1}, it turns out that $\mathbf{M}_{\mathrm{Lam}}$ is isomorphic to the *n-species totally asymmetric simple exclusion process* (n-species TASEP) on a ring with n sites. The Markov chain $\mathbf{M}_{\mathrm{Lam}}$ can also be seen as a certain random walk on the *toric alcoves* in the torus $\mathbb{T}$. Let ζ_{Lam} denote the stationary probability distribution of $\mathbf{M}_{\mathrm{Lam}}$.

Let $\mathcal{A}u_M$ and $\mathcal{A}v_M$ denote the states at time M of the reduced random walk in $\widetilde{W}$ and the affine Grassmannian reduced random walk in $\widehat{W}$, respectively. Let

$$\psi_{\mathrm{Lam}} = \sum_{\substack{w\in W \\ w^{-1}\theta\in\Phi^+}} \zeta_{\mathrm{Lam}}(w)\theta^\vee w. \tag{1.1}$$

Lam [18] proved that with probability 1, we have

$$\lim_{M\to\infty} \langle\mathcal{A}v_M\rangle = \langle\psi_{\mathrm{Lam}}\rangle \quad \text{and} \quad \lim_{M\to\infty} \langle\mathcal{A}u_M\rangle \in \langle\psi_{\mathrm{Lam}}\rangle W.$$

Thus, the affine Grassmannian reduced random walk almost surely travels asymptotically in the direction of ψ_{Lam}, and the reduced random walk almost surely travels asymptotically in one of the finitely many directions in $\psi_{\mathrm{Lam}}W$.

2 Random Billiards

Dynamical algebraic combinatorics is a field that studies dynamical systems on objects of interest in algebraic combinatorics (see, e.g., [21, 25, 27]). *Mathematical billiards* is a subfield of dynamics concerning the trajectory of a beam of light that moves in a straight line except for occasional reflections [20, 26]. *Combinatorial billiards* is a new topic that combines these two areas, focusing on mathematical billiard systems that are in some sense rigid and discretized; these billiard systems can usually be modeled combinatorially or algebraically [1, 5, 13, 14, 15, 28]. Our first goal is to define a random billiard trajectory that resembles Lam's reduced random walk and its variants.

Fix a point z_0 in the interior of $\mathcal{A}$. For $\eta \in V^* \setminus \{0\}$, let $\mathfrak{r}_\eta$ be the ray that starts at z_0 and travels in the direction of η. Let Y_{z_0} denote the set of vectors $\eta \in Q^\vee \setminus \{0\}$ such that $\mathfrak{r}_\eta$ does not pass through the intersection of two or more hyperplanes in $\mathcal{H}_{\widetilde{W}}$.

Given $\eta \in \mathrm{Y}_{z_0}$, we can record the sequence $\mathcal{A}u_0, \mathcal{A}u_1, \ldots$ of alcoves through which $\mathfrak{r}_\eta$ passes (in particular, $\mathcal{A}u_0 = \mathcal{A}$); we then define the infinite word $\mathsf{w}_{z_0}(\eta) = \cdots s_{i_1}s_{i_0}$, where s_{i_j} is the unique simple reflection such that $u_{j+1} = s_{i_j}u_j$ (our convention is that infinite words extend infinitely to the left). The word $\mathsf{w}_{z_0}(\eta)$ is necessarily periodic (since $\eta \in Q^\vee$), and we let $N = N_\eta$ denote its period.

Fix $\eta \in \mathrm{Y}_{z_0}$ and $p \in (0,1)$. Shine a beam of light from z_0 in the direction of η. Whenever the beam of light hits a hyperplane in $\mathcal{H}_{\widetilde{W}}$ that it has not yet crossed, it passes through the hyperplane with probability p and reflects off of the hyperplane with probability $1-p$. Whenever the beam of light hits a hyperplane in $\mathcal{H}_{\widetilde{W}}$ that it has already crossed, it reflects off of the hyperplane. We call this random process a *reduced random billiard trajectory* (see Figure 1). By imposing the extra condition that the beam of light always reflects when it hits a wall of the fundamental chamber, we obtain a different random process that we call the *affine Grassmannian reduced random billiard trajectory*.

We can discretize the reduced random billiard trajectory by only keeping track of the alcove containing the beam of light and the direction that the beam of light is facing. Let u_M be the alcove containing the beam of light after it hits a hyperplane in $\mathcal{H}_{\widetilde{W}}$ for the M-th time; at this point in time, the beam of light is facing toward the facet of $\mathcal{A}u_M$ contained in the hyperplane $\mathrm{H}^{(u_M, s_{i_M})}$. In this way, we obtain a discrete-time Markov chain $\widetilde{\mathbf{M}}_\eta$ whose state at time M is the pair (u_M, M) in $\widetilde{W} \times \mathbb{Z}/N\mathbb{Z}$. We call this Markov chain a *reduced random combinatorial billiard trajectory*. In a similar manner, we can discretize the affine Grassmannian reduced random billiard trajectory to obtain the *affine Grassmannian reduced random combinatorial billiard trajectory*, which is a discrete-time Markov chain $\widehat{\mathbf{M}}_\eta$ with state space $\widehat{W} \times \mathbb{Z}/N\mathbb{Z}$. By projecting $\widehat{\mathbf{M}}_\eta$ through the natural quotient map $\widetilde{W} \times \mathbb{Z}/N\mathbb{Z} \to W \times \mathbb{Z}/N\mathbb{Z}$ given by $(w, M) \mapsto (\overline{w}, M)$, we obtain a Markov chain $\mathbf{M}_\eta$ on $W \times \mathbb{Z}/N\mathbb{Z}$, which can be seen as a random combinatorial billiard trajectory in the torus $\mathbb{T}$. Each toric hyperplane of the form $\mathfrak{q}(\mathrm{H})$ for $\mathrm{H} \in \mathcal{H}_{\widetilde{W}}$ has two sides. When the beam of light (now traveling in the torus) hits $\mathfrak{q}(\mathrm{H})$, it either passes

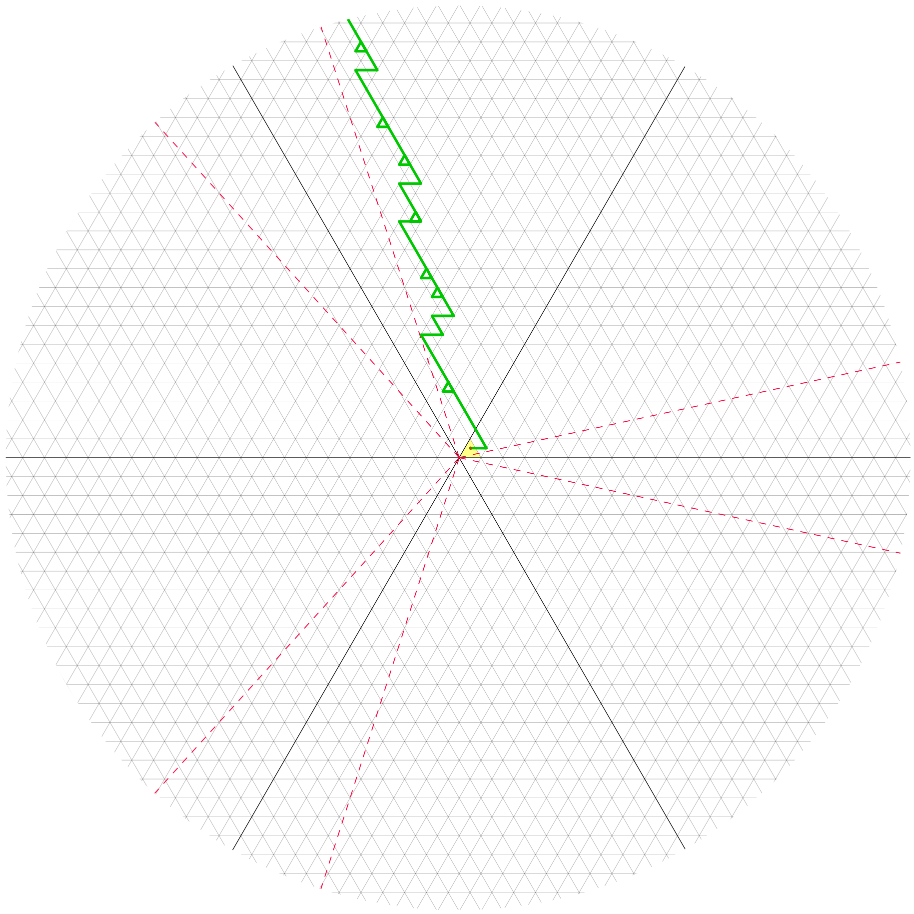

Figure 1: A reduced random billiard trajectory in the affine symmetric group $\widetilde{\mathfrak{S}}_3 = \widetilde{A}_2$ with parameter $p = 3/4$. The (green) beam of light starts in the (yellow) fundamental alcove traveling in the direction of the vector $\delta^{(3)} = (1, 1, -2)$. Occasionally, the beam of light traverses a small triangle numerous times; however, the number of times it traverses the small triangle is not discernible from the figure. The six possible asymptotic directions of the beam of light are represented by red dotted rays. The six thick black lines are the hyperplanes in the Coxeter arrangement of the finite Weyl group $\mathfrak{S}_3 = A_2$; they separate the space into six chambers.

through or reflects; the probability that it passes through is either p or 0, depending on which side of $\mathfrak{q}(\mathrm{H})$ it hits. Let ζ_η denote the stationary probability distribution of $\mathbf{M}_\eta$. We stress that ζ_η depends on the fixed parameter p.

The following theorem is an analogue of the aforementioned result due to Lam.

Theorem 2.1 ([12])**.** *Let $\eta \in \Upsilon_{z_0}$, and let $\mathsf{w}_{z_0}(\eta) = \cdots s_{i_1} s_{i_0}$. Let*

$$(u_M, M) \in \widetilde{W} \times \mathbb{Z}/N_\eta\mathbb{Z} \quad \textit{and} \quad (v_M, M) \in \widehat{W} \times \mathbb{Z}/N_\eta\mathbb{Z}$$

denote the states of $\widetilde{\mathbf{M}}_\eta$ and $\widehat{\mathbf{M}}_\eta$, respectively, at time M. Let

$$\psi_\eta = \sum_{\substack{(w,k)\in W\times\mathbb{Z}/N_\eta\mathbb{Z} \\ i_k=0 \\ w^{-1}\theta\in\Phi^+}} \zeta_\eta(w,k)\theta^\vee w.$$

With probability 1, *we have*

$$\lim_{M\to\infty}\langle \mathcal{A}v_M\rangle = \langle\psi_\eta\rangle \quad \text{and} \quad \lim_{M\to\infty}\langle \mathcal{A}u_M\rangle \in \langle\psi_\eta\rangle W. \tag{2.1}$$

Remark 2.2. There is another natural interpretation of the reduced random combinatorial billiard trajectory in terms of the *Demazure product* (see [12, 18] for the definition). Let (u_M, M) be the state of $\widetilde{\mathbf{M}}_\eta$ at time M. Let $\mathsf{w}_{z_0}(\eta) = \cdots s_{i_1}s_{i_0}$, and let x_M be the word obtained from $s_{i_{M-1}}\cdots s_{i_1}s_{i_0}$ by independently deleting each letter with probability $1-p$. Then u_M has the same distribution as the Demazure product of x_M.

Remark 2.3. One can view the aformentioned Markov chains introduced by Lam in [18] as limits of our "billiardized" Markov chains in the regime when p tends to 0.

3 Type A

Assume now that W and $\widetilde{W}$ are the symmetric group $\mathfrak{S}_n$ and the affine symmetric group $\widetilde{\mathfrak{S}}_n$. In this case, Theorem 2.1 becomes much more interesting because, for a particular choice of η, we can compute the vector ψ_η explicitly.

We can identify the index set $\widetilde{I}$ with $\mathbb{Z}/n\mathbb{Z}$ in such a way that $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ for all $i\in\mathbb{Z}/n\mathbb{Z}$. Let e_i be the i-th standard basis vector in $\mathbb{R}^n$. Then $\Phi = \Phi^+\sqcup\Phi^-$, where $\Phi^+ = \{e_i - e_j : 1\le i<j\le n\}$ and $\Phi^- = -\Phi^+$. The spaces V and V^* can each be identified with $\{(\gamma_1,\ldots,\gamma_n)\in\mathbb{R}^n : \gamma_1+\cdots+\gamma_n = 0\}$. For $i,j,k\in\mathbb{Z}$ with $1\le i<j\le n$, we have $\mathrm{H}^k_{e_i-e_j} = \{(\gamma_1,\ldots,\gamma_n)\in V^* : \gamma_i-\gamma_j = k\}$.

Let $\delta^{(n)} = -ne_n + \sum_{j\in[n]} e_j$ be the vector in V^* whose last component is $-(n-1)$ and whose other components are all equal to 1. As before, fix a point z_0 in the interior of $\mathcal{A}$. One can show that $\delta^{(n)}\in \mathrm{Y}_{z_0}$. Moreover, $N_{\delta^{(n)}} = n$, and $\mathsf{w}_{z_0}(\delta^{(n)}) = \cdots s_{i_1}s_{i_0} = \cdots\mathsf{ccc}$, where $\mathsf{c} = s_{n-1}\cdots s_1s_0$. Thus, $i_j = j\in\mathbb{Z}/n\mathbb{Z}$ for all $j\ge 0$.

Lam's Markov chain $\mathbf{M}_{\mathrm{Lam}}$ is isomorphic (in type A) to an instance of a well-studied interacting particle system known as the *multispecies TASEP*, which probabilists and statistical physicists began studying long before Lam's work [2, 16, 17, 24]. Ferrari and Martin [17] described the stationary distribution of the multispecies TASEP in terms of combinatorial objects called *multiline queues*, and Corteel, Mandelshtam, and Williams [10] interpreted this distribution in terms of specializations of certain *ASEP polynomials*. In particular, their results can be used to compute the distribution ζ_{Lam}. Ayyer and

Linusson [3] used multiline queues to prove that there is a positive scalar κ such that

$$\psi_{\mathrm{Lam}} = \kappa \sum_{1\leq i<j\leq n} (j-i)(e_i - e_j); \tag{3.1}$$

this settled a conjecture of Lam's.

Theorem 2.1 motivates us to study $\mathbf{M}_{\delta(n)}$, which we view as a "billiardization" of $\mathbf{M}_{\mathrm{Lam}}$. In Section 4, we will define a new variant of the multispecies TASEP called the *stoned multispecies TASEP*. Surprisingly, we can compute the stationary distribution of this Markov chain in terms of ASEP polynomials. In a special case, the stoned multispecies TASEP is (essentially) the same as $\mathbf{M}_{\delta(n)}$. In the full-length version of this article [12], we use multiline queues to analyze correlations in the stoned multispecies TASEP. This in turn allows us to obtain the following analogue of Ayyer and Linusson's result from (3.1).

Theorem 3.1. *The vector $\psi_{\delta(n)}$ is a positive scalar multiple of*

$$\sum_{1\leq i<j\leq n} \frac{(j-i)(2n-(i+j-1)p)}{(n-ip)(n-(i-1)p)(n-jp)(n-(j-1)p)}(e_i - e_j).$$

Note that sending p to 0 in Theorem 3.1 recovers Ayyer and Linusson's result.

Example 3.2. Let $n = 3$ so that $\delta^{(3)} = (1,1,-2)$. Using Theorem 3.1, we compute that $\langle \psi_{\delta(n)} \rangle = \langle (3-2p, p, p-3) \rangle$. Figure 1 illustrates this when $p = 3/4$; the six red dotted rays point in the directions of the vectors in $\langle (1.5, 0.75, -2.25) \rangle \mathfrak{S}_3$.

An *n-core* is an integer partition that does not have any hook lengths divisible by n. Such partitions are important due to their prominence in partition theory and representation theory. There is a natural one-to-one correspondence between n-cores and alcoves of $\mathcal{H}_{\widetilde{\mathfrak{S}}_n}$ inside the fundamental chamber $\mathcal{C}$. Using this correspondence, Lam interpreted his affine Grassmannian reduced random walk as a random growth process for n-cores. He showed that (3.1) (which Ayyer and Linusson proved later) implies an exact description of the limit shape of the (appropriately scaled) Young diagrams in this random growth process. As n tends to ∞, these limit shapes converge to the region

$$\mathbf{R}_\infty = \{(x,y) \in \mathbb{R}^2 : y \leq 0 \leq x,\ \sqrt{x} + \sqrt{-y} \leq 6^{1/4}\};$$

see [3, 18]. Note that $\mathbf{R}_\infty$ is also the limit shape that Rost derived for the *corner growth process*, a more classical random growth process for partitions [22, 23].

We can similarly interpret $\widehat{\mathbf{M}}_{\delta(n)}$ as a random growth process for n-cores. In the full-length version of this article [12], we use Theorem 3.1 to obtain an exact description of the limit shape of our random growth process. As $n \to \infty$, we find that these limit shapes converge to the region

$$\mathbf{R}_\infty^{(p)} = \{(x,y) \in \mathbb{R}^2 : y \leq 0 \leq x,\ \sqrt{(1-p)x} + \sqrt{-y} \leq (6(1-p))^{1/4}\}.$$

The remarkably simple form of the region $\mathbf{R}_\infty^{(p)}$ is ultimately due to some especially nice properties of multiline queues.

4 Stoned Exclusion Processes

Let $t \in [0,1)$. Fix a tuple $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ such that $0 \le \lambda_1 \le \cdots \le \lambda_n$. Let S_λ be the set of tuples that can be obtained by rearranging the parts of λ. Let μ_i denote the i-th part of a tuple μ. Let $\mathbf{x} = (x_1, \ldots, x_n)$. Given a tuple $\boldsymbol{y} = (y_1, \ldots, y_n)$ and a permutation $w \in \mathfrak{S}_n$, let $w\boldsymbol{y} = (y_{w^{-1}(1)}, \ldots, y_{w^{-1}}(n))$. Given integers $k, k' \in \mathbb{Z}$ and $r \in \mathbb{R}$, let

$$\mathfrak{f}_r(k,k') = \begin{cases} 1 & \text{if } k > k'; \\ r & \text{if } k < k'; \\ 0 & \text{if } k = k'. \end{cases} \tag{4.1}$$

Corteel, Mandelshtam, and Williams introduced the family $(F_\mu)_{\mu \in S_\lambda}$ of *ASEP polynomials*. These are certain homogeneous polynomials in $\mathbb{C}(t)[\mathbf{x}]$ that satisfy certain *exchange equations* (see [10, 12] for more details). It follows from the work of Cantini, de Gier, and Wheeler [9] that the polynomial $P_\lambda(\mathbf{x}) = \sum_{\mu \in S_\lambda} F_\mu(\mathbf{x})$ is a Macdonald polynomial. Corteel, Mandelshtam, and Williams [10] reproved this and gave a combinatorial formula for computing ASEP polynomials using multiline queues.

Let ASEP_λ denote the *multispecies ASEP* with state space S_λ. This is a discrete-time Markov chain in which the transition probability from a state μ to a state μ' is given by

$$\mathbb{P}(\mu \to \mu') = \begin{cases} \frac{1}{n}\mathfrak{f}_t(\mu_i, \mu_{i+1}) & \text{if } \mu' = \bar{s}_i\mu \neq \mu; \\ 1 - \sum_{\nu \in S_\lambda \setminus \{\mu\}} \mathbb{P}(\mu \to \nu) & \text{if } \mu = \mu'; \\ 0 & \text{otherwise.} \end{cases}$$

(Note that $\bar{s}_i$ is the transposition of $\mathbb{Z}/n\mathbb{Z}$ that swaps i and $i+1$; in particular, $\bar{s}_n$ swaps n and 1.) The state μ can be visualized as a configuration of particles on a ring with sites $1, \ldots, n$ (listed in clockwise cyclic order), where the particle on site i has *species* μ_i. There has been substantial attention devoted to the stationary distribution of the multispecies ASEP [2, 16, 17, 9, 10]. According to [10, 9], the stationary probability of μ in ASEP_λ is

$$\frac{F_\mu(1, \ldots, 1; t)}{P_\lambda(1, \ldots, 1; t)}.$$

When $t = 0$, the multispecies ASEP is called the *multispecies TASEP*.

Consider n *stones* $\blacktriangle_1, \ldots, \blacktriangle_n$. Let Ω denote the set of permutations $\sigma \in \mathfrak{S}_n$ such that the list $\sigma^{-1}(2), \sigma^{-1}(3), \ldots, \sigma^{-1}(n)$ is a cyclic rotation of the list $2, 3, \ldots, n$. We can view a permutation $\sigma \in \Omega$ as a certain configuration of the stones on the sites of the ring, where the stone $\blacktriangle_j$ is placed on the site $\sigma(j)$. Thus, an element of $S_\lambda \times \Omega$ is a configuration of particles and stones on the ring with sites $1, \ldots, n$. Let $\chi = (\chi_1, \ldots, \chi_n)$ be an n-tuple of nonzero real numbers such that for all $2 \leq j \leq n$, we have

$$p(j) := \frac{\chi_1 - \chi_j}{t\chi_1 - \chi_j} \in (0,1). \tag{4.2}$$

The *stoned multispecies ASEP*[1], which we denote by $\blacktriangle\mathrm{ASEP}_\lambda$, is the discrete-time Markov chain with state space $S_\lambda \times \Omega$ defined as follows. Suppose the Markov chain is in state (μ, σ). Let $i = \sigma(1)$ and $j = \sigma^{-1}(i+1)$; this means that $\blacktriangle_1$ sits on site i and $\blacktriangle_j$ sits on site $i+1$. When the Markov chain transitions, the stones $\blacktriangle_1$ and $\blacktriangle_j$ swap places, and they send a signal to the particles on sites i and $i+1$ (which have species μ_i and μ_{i+1}) telling them to swap. However, the signal only has probability $p(j)$ of actually reaching the particles. If the signal does not reach the particles, then the particles simply do not move. On the other hand, if the particles do receive the signal, then with probability $\mathfrak{f}_t(\mu_i, \mu_{i+1})$, they decide to actually follow their orders and swap places (and with probability $1 - \mathfrak{f}_t(\mu_i, \mu_{i+1})$, they stubbornly disregard the signal and do not move). Note that if $\mu_i = \mu_{i+1}$, then with probability 1, the stones $\blacktriangle_1$ and $\blacktriangle_j$ swap places and no particles move. Figure 2 illustrates the definition of $\blacktriangle\mathrm{ASEP}_\lambda$.

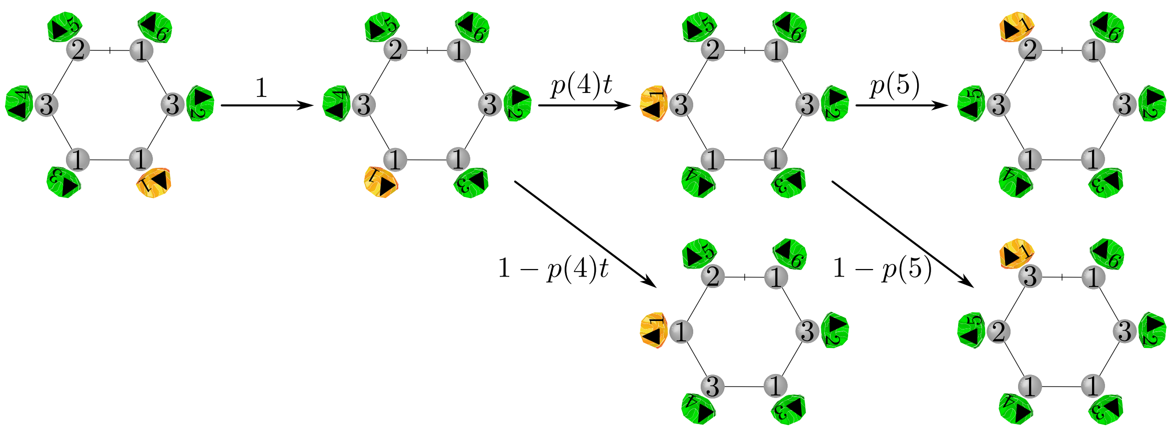

Figure 2: Some transitions in $\blacktriangle\mathrm{ASEP}_{(1,1,1,2,3,3)}$. At each step, the (gold) stone $\blacktriangle_1$ must swap places with the (green) stone immediately to its right. Then, the particles occupying the same sites as those two stones either swap places or stay put.

Theorem 4.1 ([12])**.** *The stationary probability measure π of $\blacktriangle\mathrm{ASEP}_\lambda$ is given by*

$$\pi(\mu, \sigma) = \frac{1}{Z(\lambda)} F_\mu(\sigma\chi; t),$$

[1] For the sake of brevity, we have chosen to define here a version of the stoned multispecies ASEP that is actually less general than the one defined in the full-length article [12].

where $Z(\lambda)$ is a normalization factor that only depends on λ.

Let us connect the stoned multispecies ASEP with random combinatorial billiards. To do so, we specialize to the setting where $\lambda = (1, 2, \dots, n)$. The map $\mathfrak{S}_n \to S_\lambda$ given by $w \mapsto (w^{-1}(1), \dots, w^{-1}(n))$ allows us to identify $\mathfrak{S}_n$ with S_λ.

Fix $p \in (0, 1)$. Let $\boldsymbol{\chi} = (\chi, 1, \dots, 1)$, where $\chi = \dfrac{1-p}{1-pt}$. Then (4.2) holds with $p(j) = p$ for all $2 \le j \le n$. Let $\boldsymbol{\chi}^{(k)}$ denote the n-tuple whose k-th entry is χ and whose other entries are all 1 (so $\boldsymbol{\chi} = \boldsymbol{\chi}^{(1)}$).

Now consider the following random combinatorial billiard trajectory. Start at a point in the interior of the alcove $\mathcal{A}$, and shine a beam of light in the direction of the vector $\delta^{(n)} = -ne_n + \sum_{j \in [n]} e_j$. If at some point in time the beam of light is traveling in an alcove $\mathcal{A}u$ and hits the hyperplane $\mathrm{H}^{(u, s_i)}$, then it passes through with probability $p\, \mathfrak{f}_t(\overline{u}(i), \overline{u}(i+1))$ (thereby moving into the alcove $\mathcal{A}s_i u$), and it reflects with probability $1 - p\, \mathfrak{f}_t(\overline{u}(i), \overline{u}(i+1))$. (Note that $\mathfrak{f}_t(\overline{u}(i), \overline{u}(i+1))$ only depends on the side of the hyperplane the light beam hits, not the particular alcove $\mathcal{A}u$.) Let us discretize this billiard trajectory; if the beam of light is in the alcove $\mathcal{A}u$ and it is headed toward the facet of $\mathcal{A}u$ contained in the hyperplane $\mathrm{H}^{(u, s_i)}$, then we record the pair $(u, i) \in \widetilde{\mathfrak{S}}_n \times \mathbb{Z}/n\mathbb{Z}$. Applying the quotient map $\widetilde{\mathfrak{S}}_n \times \mathbb{Z}/n\mathbb{Z} \to \mathfrak{S}_n \times \mathbb{Z}/n\mathbb{Z}$ defined by $(u, i) \mapsto (\overline{u}, i)$, we obtain a Markov chain $\mathbf{M}^{(t)}_{\delta^{(n)}}$ with state space $\mathfrak{S}_n \times \mathbb{Z}/n\mathbb{Z}$, which models a certain random combinatorial billiard trajectory in the torus $\mathbb{T}$. One can show that $\mathbf{M}^{(0)}_{\delta^{(n)}}$ is precisely the Markov chain $\mathbf{M}_\eta$ defined in Section 2 when $\eta = \delta^{(n)}$.

Given a state (w, i) of $\mathbf{M}^{(t)}_{\delta^{(n)}}$, we can encode the permutation w as usual by placing particles of species $w^{-1}(1), \dots, w^{-1}(n)$ on the sites $1, \dots, n$ (respectively) of the ring. We can also encode i by placing a gold stone on site i and placing green stones on all the other sites. (See Figure 3.) Then $\mathbf{M}^{(t)}_{\delta^{(n)}}$ is the same as $\blacktriangle\mathrm{ASEP}_{(1,2,\dots,n)}$, except we have colored the stone $\blacktriangle_1$ gold, colored the other stones green, and removed the names of all the stones. It follows from Theorem 4.1 that the stationary probability of (w, i) in $\mathbf{M}^{(t)}_{\delta^{(n)}}$ is

$$\frac{1}{Z'(n)} F_w(\boldsymbol{\chi}^{(i)}; t),$$

where $Z'(n)$ is some normalization factor only depending on n.

The entries in the tuple $\boldsymbol{\chi}$ can be chosen generically (subject to (4.2)), so Theorem 4.1 tells us that the stoned multispecies ASEP is a Markov chain whose stationary distribution is determined by ASEP polynomials evaluated at generic values. Ayyer, Martin, and Williams [4] recently studied a Markov chain called the *inhomogeneous t-PushTASEP*, which is quite different from the stoned multispecies ASEP; they found that its stationary distribution is also given by ASEP polynomials evaluated at generic values. We discovered stoned exclusion processes independently of their work while considering random

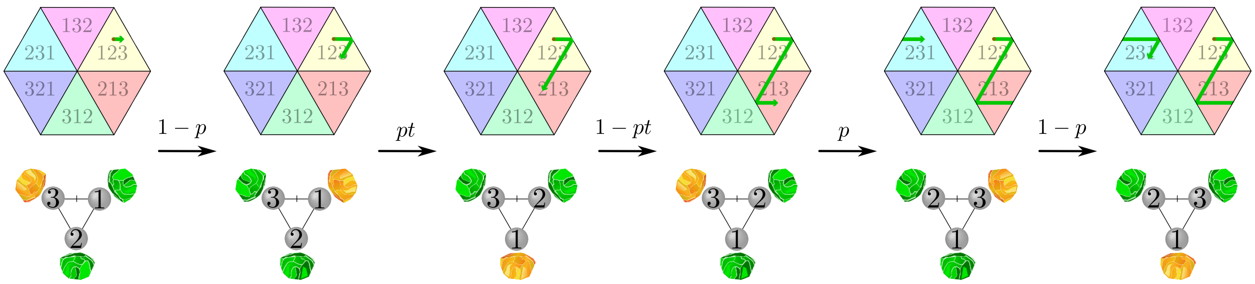

Figure 3: The 2-dimensional torus $\mathbb{T}$ is represented as a hexagon with opposite sides identified. There are 6 toric alcoves, which are drawn as triangles of different colors and labeled by the permutations in $\mathfrak{S}_3$. This figure shows a sequence of transitions in $\mathbf{M}^{(t)}_{\delta^{(3)}}$. Each state is represented both as a beam of light traveling in $\mathbb{T}$ (top) and as a configuration of particles and stones on a ring with 3 sites. Each transition is labeled with its probability.

billiard trajectories. A major advantage of our approach using stones is that it easily adapts to other settings where it is not clear how to adapt the t-PushTASEP. Indeed, in the full-length version of this article [12], we define stoned versions of other exclusion processes such as the *inhomogeneous TASEP* and the *multispecies open boundary ASEP*.

Lam and Williams [19] introduced the inhomogeneous TASEP and posed several intriguing conjectures about it, including one stating that the stationary probabilities can be expressed (up to a normalization factor) as nonnegative integral sums of Schubert polynomials. Cantini [6] introduced certain polynomials that we call *inhomogeneous TASEP polynomials* and found that very particular specializations of these polynomials determine the stationary distribution of the inhomogeneous TASEP. In [12], we introduce the stoned inhomogeneous TASEP and show that its stationary distribution is given by inhomogeneous TASEP polynomials evaluated at generic values. It was previously not known how to construct a Markov chain with this stationary distribution. In a special case, the stoned inhomogeneous TASEP can also be interpreted as a random combinatorial billiard trajectory in the torus $\mathbb{T}$ (in type A), where the probability that the light beam passes through a toric hyperplane depends on the particular toric hyperplane.

The *multispecies open boundary ASEP* is an interacting particle system on a path graph that has received an enormous amount of attention (see, e.g., [7, 8, 11]). Cantini, Garbali, de Gier, and Wheeler [8] found that the stationary distribution of the multispecies open boundary ASEP is given by particular specializations of certain polynomials called *open boundary ASEP polynomials*, which are closely related to Koornwinder polynomials. In [12], we introduce the stoned multispecies open boundary ASEP and show that its stationary distribution is given by open boundary ASEP polynomials evaluated at generic values. It was previously not known how to construct a Markov chain with this stationary distribution. In a special case, the stoned multispecies open boundary ASEP can also be interpreted as a random combinatorial billiard trajectory in the torus $\mathbb{T}$ when Φ is the root system of type C_n.

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