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Boolean structure constants

Yibo Gao*[1] and Hai Zhu†[2]

[1] *Beijing International Center for Mathematical Research, Peking University, Beijing 100871, China*
[2] *School of Mathematical Sciences, Peking University, Beijing 100871, China*

Abstract. The Schubert problem asks for combinatorial models to compute structure constants of the cohomology ring with respect to the Schubert classes, and has been an important open problem in algebraic geometry and combinatorics that guided fruitful research in decades. In this paper, we provide an explicit formula for the (equivariant) Schubert structure constants c_{uv}^{w} across all Lie types when the elements u, v, w are boolean. In particular, in type A, all Schubert structure constants on boolean elements are either 0 or 1.

Keywords: Schubert calculus, Boolean, structure constant, Dynkin diagram

1 Introduction

Let G be a complex, connected, reductive algebraic group and B be a Borel subgroup of G with a maximal torus T. The homogeneous space G/B is called the *generalized flag variety*, which admits a *Bruhat decomposition* $\sqcup_{w \in W} X_w^\circ$ into open *Schubert cells*, whose closures are the *Schubert varieties* $\{X_w \mid w \in W\}$, indexed by the *Weyl group* $W = N_G(T)/T$. Let $\sigma_w \in H^*(G/B; \mathbb{Z})$ be the Poincaré dual of the fundamental class of X_w.

The *Schubert problem* asks for combinatorial interpretations of the structure constants $c_{uv}^{w} \in \mathbb{Z}_{\geq 0}$ of $H^*(G/B; \mathbb{Z})$ appearing in the expansion $\sigma_u \cdot \sigma_v = \sum_w c_{uv}^{w} \sigma_w$. It has been a major open problem in algebraic geometry and combinatorics for decades, guiding numerous fruitful research in recent years. We mention a few beautiful results here in the massive literature: the most classical Chevalley–Monk's formula [19], Pieri's rule [22], the separated descent case [8, 13], puzzle rules for the Grassmannian [10, 11], a survey on the equivariant Schubert calculus of the Grassmannian [21], 2 or 3 step partial flag varieties [3, 4, 12], and various others working in richer cohomology theories.

The goal of this paper is to make progress towards the Schubert problem. We describe an explicit rule (Corollary 1.3) for the Schubert structure constants c_{uv}^{w} across all Lie types when the element w is *boolean*, a previously unexplored family of the Schubert problem, with connections to the Pieri's rule [22] and hook's rule [18]. Interestingly, our formula

*gaoyibo@bicmr.pku.edu.cn
†zhuhai1686@stu.pku.edu.cn

demonstrates certain "multiplicity-freeness", especially in type A_n, where all $c_{uv}^w \in \{0,1\}$ when w is boolean. Boolean elements play an important role in the study of Schubert calculus. The Schubert variety X_w is a toric variety if and only if w is boolean [9], and the Schubert variety $X_{w_0(I)c}$ is L_I-spherical if and only if c is boolean [6, 7].

We work in the generality of the torus equivariant cohomology ring $H_T^*(G/B;\mathbb{Z})$. Let $\{\xi_w \mid w \in W\}$ be the *equivariant Schubert classes* and write $\xi_u \cdot \xi_v = \sum d_{uv}^w \xi_w$ where $d_{uv}^w \in \mathbb{Z}[\Lambda] = H_T^*(\mathrm{pt};\mathbb{Z})$ is the *equivariant Schubert structure constant*.

Remark 1.1. The Kostant–Kumar formula [14, Theorem 4.15] provides d_{uv}^w with a recursive formula [20, Theorem 4.2]. Moreover, in the boolean case, the Kostant–Kumar formula reduces to a combinatorially positive formula as follows.

Let $\mathcal{A} := \mathbb{Z}[\alpha_1, \cdots, \alpha_n]$ denote the polynomial algebra in the simple roots and define two operators $\partial_j : \mathcal{A} \to \mathcal{A}$ and $B_j : \mathcal{A} \to \mathcal{A}$ as

$$\partial_j(p) := \frac{s_j(p) - p}{\alpha_j}, \qquad B_j(p) := \begin{cases} \alpha_j \cdot s_j(p) & \text{if } s_j \in S(u) \cap S(v) \\ s_j(p) & \text{if } s_j \in S(u) \triangle S(v) \\ \partial_j(p) & \text{if } s_j \notin S(u) \cup S(v) \end{cases}.$$

Let $w \in W$ be a boolean element with support set $S(w) = \{i_1, i_2, \cdots, i_k\}$ and fix a reduced word $w = s_{i_1} s_{i_2} \cdots s_{i_k}$. Consider $u, v \leq w$ and note that u, v must be boolean as well. Furthermore, u, v are uniquely determined by their support sets $S(u), S(v) \subseteq S(w)$. According to [20, Subsection 4.4], the Kostant–Kumar formula says that

$$d_{u,v}^w = B_{i_1} \circ B_{i_2} \circ \cdots \circ B_{i_k}(1). \tag{1.1}$$

Now if $p \in \mathcal{A}$ with non-negative coefficients does not contain the variable α_j, then it can be shown using the twisted Leibniz formula [15, Theorem 11.1.7 part (h)] that $\partial_j(p)$ is a polynomial with non-negative coefficients in the simple roots. Hence Equation (1.1) is combinatorially positive in the boolean case since the root variable α_j only gets introduced when applying B_j.

However, it is not immediately from Equation (1.1) that boolean structure constants $c_{uv}^w \in \{0,1\}$ in type A which we prove in Corollary 3.14.

The following is our main theorem.

Theorem 1.2. *For boolean elements $u, v, w \in W$,*

$$d_{uv}^w = \begin{cases} \sum\limits_{u \overset{\mathsf{S}(v)}{\rightsquigarrow} w} \mathrm{mul}(u \overset{\mathsf{S}(v)}{\rightsquigarrow} w) \cdot \mathrm{wt}(u \overset{\mathsf{S}(v)}{\rightsquigarrow} w), & \textit{if there exists a boolean insertion path } v \overset{\mathsf{S}(u)}{\rightsquigarrow} w \\ 0, & \textit{otherwise} \end{cases}$$

where the summation is over all boolean insertion paths $u \overset{\mathsf{S}(v)}{\rightsquigarrow} w$.

The *boolean insertion path* consists of the *boolean insertion steps* that encode the equivariant Chevalley's rule on boolean elements. These steps are also called the *k-Bruhat order* in $H^*(\mathrm{Fl}_n;\mathbb{Z})$, and have been very useful in the Schubert problem [17, 16]. The appearance of our boolean insertion path in the formula is not surprising. What's interesting and what's unique about the boolean elements, is that these paths precisely govern the structure constants in a subtraction-free and multiplicity-free way (Proposition 3.13).

The precise definitions in Theorem 1.2 are given in Definitions 3.2 and 3.7. We remark that $\mathrm{mul}(u \overset{\mathrm{S}(v)}{\rightsquigarrow} w)\,\mathrm{wt}(u \overset{\mathrm{S}(v)}{\rightsquigarrow} w)$ can be replaced by $\mathrm{mul}(v \overset{\mathrm{S}(u)}{\rightsquigarrow} w)\,\mathrm{wt}(v \overset{\mathrm{S}(u)}{\rightsquigarrow} w)$ in Theorem 1.2, making the formula symmetric. The proof is given in Section 3.

We also have a cohomology version of Theorem 1.2.

Corollary 1.3. *For boolean elements $u, v, w \in W$,*

$$c_{uv}^{w} = \begin{cases} \sum\limits_{u \overset{\mathrm{S}(v)}{\rightsquigarrow} w} \mathrm{mul}(u \overset{\mathrm{S}(v)}{\rightsquigarrow} w), & \text{if there exists a non-equivariant boolean insertion path } v \overset{\mathrm{S}(u)}{\rightsquigarrow} w \\ 0, & \text{otherwise} \end{cases}$$

where the summation is over all non-equivariant boolean insertion paths $u \overset{\mathrm{S}(v)}{\rightsquigarrow} w$.

Furthermore, $c_{uv}^{w} \in \{0,1\}$ in type A (Corollary 3.14).

This paper is organized as follows. In Section 2, we provide the necessary background on root systems, Weyl groups, the equivariant Chevalley's formula, boolean elements and their boolean diagrams. In Section 3, we introduce the boolean insertion algorithms and prove the main Theorem (Theorem 1.2). In Section 4, we give a fast algorithm to compute c_{uv}^{w} for boolean elements.

2 Preliminaries

2.1 Root systems and Weyl groups

Let $\Phi := \Phi(\mathfrak{g}, T)$ be the *root system* of weights for the adjoint action of T on the Lie algebra $\mathfrak{g}$ of G, with a decomposition $\Phi^+ \sqcup \Phi^-$ into *positive roots* and negative roots. Let $\Delta = \{\alpha_1, \ldots, \alpha_r\} \subseteq \Phi^+$ be the corresponding set of *simple roots*, which is a basis of $\mathfrak{h}_{\mathbb{R}}^*$, the real span of all roots. Let $\langle \cdot, \cdot \rangle$ be the nondegenerate scalar product on $\mathfrak{h}_{\mathbb{R}}^*$ induced by the Killing form. For each root $\alpha \in \Phi$, denote by s_α the corresponding reflection. For simplicity of notations, write the *simple reflections* as $s_i := s_{\alpha_i}$ for $\alpha_i \in \Delta$. For each root $\alpha \in \Phi$, we have a *coroot* $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$. The *fundamental weights* $\{\omega_\alpha \mid \alpha \in \Delta\}$ are the dual basis to the simple coroots $\{\alpha^\vee \mid \alpha \in \Delta\}$. Let Λ be the weight space and we identify $\mathbb{Z}[\Lambda]$ as $\mathbb{Z}[\mathbf{t}]$, the polynomial ring in $\{t_\alpha := \omega_\alpha - s_\alpha(\omega_\alpha) \mid \alpha \in \Delta\}$.

Definition 2.1. The *Dynkin diagram* of Φ with a choice of simple root Δ is a directed graph whose vertex set is Δ with $-2\langle\alpha,\beta\rangle/\langle\beta,\beta\rangle \in \mathbb{N}$ edges going from α to β for $\alpha \neq \beta \in \Delta$.

The *Weyl group*, generated by the reflections $\{s_\beta \mid \beta \in \Phi\}$, is equipped with a *Coxeter length* function $\ell(w) := \min\{\ell \mid w = s_{i_1} \cdots s_{i_\ell}\}$. Such an expression $w = s_{i_1} \cdots s_{i_\ell}$ is called a *reduced word* of w if $\ell(w) = \ell$. The *Bruhat order* on W is generated by $w < ws_\beta$ if $\ell(w) < \ell(ws_\beta)$ for $\beta \in \Phi^+$. For $w \in W$, its *support* is

$$\mathrm{S}(w) := \{\alpha_i \mid s_i \text{ appears in any/all reduced words of } w\}.$$

The following result [1, p.351, Theorem 19.1.2] lets us do calculations in $H_T^*(G/B;\mathbb{Z})$.

Theorem 2.2 (Equivariant Chevalley's formula)**.** *For $\alpha \in \Delta$ and $v \in W$,*

$$\xi_v \cdot \xi_{s_\alpha} = (\omega_\alpha - v(\omega_\alpha))\xi_v + \sum_{\substack{w = vs_\beta \\ \ell(w) = \ell(v)+1}} \langle \omega_\alpha, \beta^\vee \rangle \, \xi_w$$

in $H_T^*(G/B;\mathbb{Z})$, where we sum over positive roots $\beta \in \Phi^+$.

2.2 Boolean elements

Definition 2.3. A Weyl group element $w \in W$ is *boolean* if its lower Bruhat interval $[\mathrm{id}, w]$ is isomorphic to a boolean lattice.

The following Lemma is straightforward by the subword property [2, Theorem 2.2.2]. See also [23, Proposition 7.3] and [5, Proposition 3, 1].

Lemma 2.4. *An element $w \in W$ is boolean if and only if w is a product of distinct simple reflections. In other words, w is boolean if and only if $\ell(w) = |\mathrm{S}(w)|$.*

We now view boolean elements visually using *boolean diagrams*.

Definition 2.5. For $w \in W$ that is boolean, its *boolean diagram* $B(w)$ is a directed graph on $\mathrm{S}(w)$ such that $\alpha_k \to \alpha_j$ if s_j appears before s_k in any/all reduced words of w.

For two boolean diagrams $B(u)$ and $B(w)$, we write $B(u) \subseteq B(w)$ if $\mathrm{S}(u) \subseteq \mathrm{S}(w)$ and if $\alpha_k \to \alpha_j$ in $B(u)$, we also have $\alpha_k \to \alpha_j$ in $B(w)$.

Example 2.6. Consider $w = s_3 s_2 s_4 s_5 s_7$ in $W(E_7)$. The Dynkin diagram of type E_7 and the boolean diagram $B(w)$ marked with solid nodes are shown in Figure 1.

Let $\mathcal{NB}$ be the ideal of $H_T^*(G/B;\mathbb{Z})$ spanned by classes ξ_w such that w is not boolean.

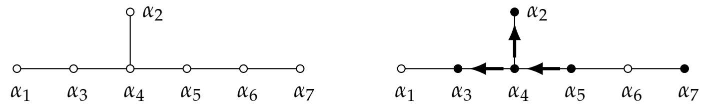

Figure 1: Left: the Dynkin diagram of type E_7. Right: $B(s_3s_2s_4s_5s_7)$.

3 Schubert structure constants for boolean elements

3.1 The boolean insertion algorithms

Now we define an operation between boolean elements $u, v \in W$ with respect to a simple root α, which is denoted by $u \overset{\alpha}{\rightsquigarrow} v$. In fact, it encodes Equivariant Chevalley's formula in Theorem 2.2 restricted to boolean elements. In particular, in type A, $u \overset{k}{\rightsquigarrow} v$ exactly means that v covers u under the k-Bruhat order. We associate each operation with a multiplicity $\mathrm{mul}(u \overset{\alpha}{\rightsquigarrow} v) \in \mathbb{N}$ and weight $\mathrm{wt}(u \overset{\alpha}{\rightsquigarrow} v) \in \mathbb{Z}[\mathbf{t}] = \mathbb{Z}[\Lambda]$ as a nonzero polynomial in $\{t_\gamma := \omega_\gamma - s_\gamma(\omega_\gamma) \mid \gamma \in \Delta\}$ with non-negative coefficients.

Definition 3.1. For boolean elements $u, v \in W$ and $\alpha \in \Delta$, we write $u \overset{\alpha}{\rightsquigarrow} v$ and call it a *boolean insertion* if one of the following mutually exclusive events happens:

1. $\alpha \in \mathrm{S}(u)$, $\ell(v) = \ell(u) + 1$, $B(u) \subseteq B(v)$ and there is a directed path in $B(v)$ from α to the unique vertex of $B(v) \setminus B(u)$. In this case, $\mathrm{wt}(u \overset{\alpha}{\rightsquigarrow} v) := 1$.

2. $\alpha \in \mathrm{S}(u)$ and $u = v$. In this case, $\mathrm{wt}(u \overset{\alpha}{\rightsquigarrow} v) := \sum_L t_\gamma$, summing over all directed paths L of the Dynkin diagram from α to some vertex $\gamma \in \mathrm{S}(u)$, which is compatible with the direction of $B(u)$. Here L is permitted to has length 0.

3. $\alpha \notin \mathrm{S}(u)$, $\ell(v) = \ell(u) + 1$ and $B(u) \subseteq B(v)$ where α is the unique vertex of $B(v) \setminus B(u)$. In this case, $\mathrm{wt}(u \overset{\alpha}{\rightsquigarrow} v) := 1$.

We say $u \overset{\alpha}{\rightsquigarrow} v$ is *non-equivariant* if (1) or (3) happens and is *equivariant* if (2) happens.

Note that a non-equivariant boolean insertion has weight 1 and changes the element, whereas an equivariant boolean insertion picks up a nontrivial weight without modifying the element. In Section 3.3, we focus only on non-equivariant insertions.

Definition 3.2. For a boolean insertion path $u^{(0)} \overset{\beta_1}{\rightsquigarrow} u^{(1)} \overset{\beta_2}{\rightsquigarrow} \cdots \overset{\beta_n}{\rightsquigarrow} u^{(n)}$, its *weight* is

$$\mathrm{wt}(u^{(0)} \overset{\beta_1}{\rightsquigarrow} u^{(1)} \overset{\beta_2}{\rightsquigarrow} \cdots \overset{\beta_n}{\rightsquigarrow} u^{(n)}) := \prod_{j=1}^{n} \mathrm{wt}(u^{(j-1)} \overset{\beta_j}{\rightsquigarrow} u^{(j)}).$$

For convenience, we also write the boolean insertion path above as $u^{(0)} \overset{B}{\rightsquigarrow} u^{(n)}$ where $B = \{\beta_1, \cdots, \beta_n\}$. Note that we need to always fix an ordering $B = \{\beta_1, \cdots, \beta_n\}$ first, before summing over boolean insertion paths $u^{(0)} \overset{B}{\rightsquigarrow} u^{(n)}$ for a fixed set B.

Remark 3.3. The ordering of B in Definition 3.2 can be arbitrary and we have freedom for practical applications. Choosing an appropriate one is especially useful in Section 4.

Example 3.4. The Dynkin diagram of type E_7 is shown in Figure 1. Let $u = s_3 s_5 s_4 s_7$ and $B(u)$ be its boolean diagram indicated by the solid vertices and directed edges shown in Figure 2. Then there are 5 boolean elements $v \in W$ satisfying $u \overset{\alpha_4}{\rightsquigarrow} v$, corresponding to

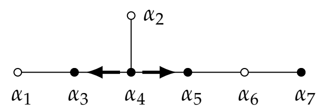

Figure 2: The boolean diagram $B(u)$ for the boolean element $u = s_3 s_5 s_4 s_7$.

the boolean terms in the expansion of $\xi_u \cdot \xi_{\alpha_4}$. One of them is u itself with the equivariant step and $\mathrm{wt}(u \overset{\alpha_4}{\rightsquigarrow} u) = t_3 + t_4 + t_5$. The other 4 elements are shown in Figure 3.

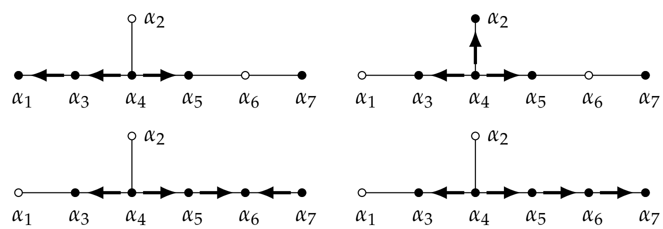

Figure 3: The boolean diagrams $B(v)$ for all the boolean elements v satisfying $s_3 s_5 s_4 s_7 = u \overset{\alpha_4}{\rightsquigarrow} v$ with a non-equivariant insertion step.

In the above example, $\alpha \in \mathrm{S}(u)$. Now choose $\alpha_6 \notin \mathrm{S}(u)$ and consider $u \overset{\alpha_6}{\rightsquigarrow} v$. Here, only non-equivariant insertion steps are possible. All of the boolean elements $v \in W$ satisfying $u \overset{\alpha_6}{\rightsquigarrow} v$ are shown in Figure 4, corresponding to the expansion of $\xi_u \cdot \xi_{\alpha_6}$.

Example 3.5. Consider a non simply-laced case. Let $u = s_2 s_3 s_4$ in $W(C_4)$ shown in Figure 5. Since there are 2 edges from α_4 to α_3 of the Dynkin diagram by Definition 2.1, there are 2 directed paths from α_4 to α_3 and 2 directed paths from α_4 to α_2 in the Dynkin diagram aligning with the direction of $B(u)$. It follows that $\mathrm{wt}(u \overset{\alpha_4}{\rightsquigarrow} u) = 2t_2 + 2t_3 + t_4$.

Definition 3.6. For $u \overset{\alpha}{\rightsquigarrow} v$, define its *multiplicity*, denoted by $\mathrm{mul}(u \overset{\alpha}{\rightsquigarrow} v)$, as follows:

1. If $u \overset{\alpha}{\rightsquigarrow} v$ is equivariant as in Definition 3.1, $\mathrm{mul}(u \overset{\alpha}{\rightsquigarrow} v) := 1$.

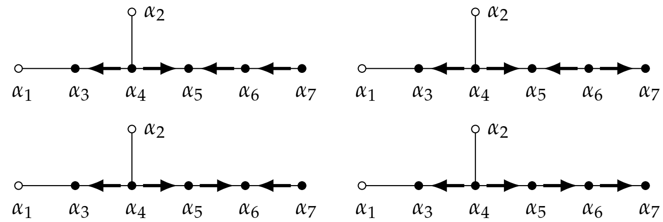

Figure 4: The boolean diagrams $B(v)$ for $s_3s_5s_4s_7 = u \overset{\alpha_6}{\rightsquigarrow} v$.

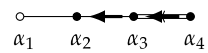

Figure 5: The boolean diagrams $B(u)$ for $u = s_2s_3s_4$

2. If $u \overset{\alpha}{\rightsquigarrow} v$ is non-equivariant as in Definition 3.1, let γ be the unique vertex of $B(v) \setminus B(u)$. Then $\mathrm{mul}(u \overset{\alpha}{\rightsquigarrow} v)$ is the number of directed paths from α to γ in the Dynkin diagram which are compatible with the direction of $B(v)$.

Note that when event (2) or (3) in Definition 3.1 occurs, $\mathrm{mul}(u \overset{\alpha}{\rightsquigarrow} v) = 1$.

Definition 3.7. For a boolean insertion path $u^{(0)} \overset{\beta_1}{\rightsquigarrow} u^{(1)} \overset{\beta_2}{\rightsquigarrow} \cdots \overset{\beta_n}{\rightsquigarrow} u^{(n)}$, its *multiplicity* is the product of the multiplicities of all its steps.

Example 3.8. Consider $u = s_2s_3s_4$ in $W(C_4)$ shown in Figure 5. The boolean insertion $u \overset{\alpha_4}{\rightsquigarrow} v$ gives $v = s_1s_2s_3s_4$ shown in Figure 6. In this case, $\mathrm{mul}(u \overset{\alpha_4}{\rightsquigarrow} v) = 2$.

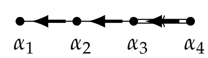

Figure 6: The boolean diagram $B(v)$ for $v = s_1s_2s_3s_4$.

Restricting Theorem 2.2 to boolean elements yields the following lemma, which is the basis of our calculations.

Lemma 3.9. *For $\alpha \in \Delta$ and a boolean element $v \in W$,*

$$\xi_v \cdot \xi_{s_\alpha} = \sum_{v \overset{\alpha}{\rightsquigarrow} w} \mathrm{mul}(v \overset{\alpha}{\rightsquigarrow} w)\, \mathrm{wt}(v \overset{\alpha}{\rightsquigarrow} w) \xi_w \mod \mathcal{NB}.$$

3.2 Multiplying Schubert classes indexed by boolean elements

Recall that once we fix an ordering $B = \{\beta_1, \cdots, \beta_n\} \subseteq \Delta$, a boolean insertion path $u = u^{(0)} \overset{\beta_1}{\rightsquigarrow} u^{(1)} \overset{\beta_2}{\rightsquigarrow} \cdots \overset{\beta_n}{\rightsquigarrow} u^{(n)} = w$ can be written as $u \overset{B}{\rightsquigarrow} w$. The following is a direct corollary of Lemma 3.9, which is obtained from applying Lemma 3.9 on $\beta_1, \ldots, \beta_n$ step by step.

Corollary 3.10. *For boolean element $u \in W$ and a set of simple roots $B \subseteq \Delta$, fix an ordering $B = \{\beta_1, \cdots, \beta_n\}$ of B, then*

$$\xi_u \prod_{\beta \in B} \xi_{s_\beta} = \sum_{u \overset{B}{\rightsquigarrow} w} \mathrm{mul}(u \overset{B}{\rightsquigarrow} w)\, \mathrm{wt}(u \overset{B}{\rightsquigarrow} w) \xi_w \mod \mathcal{NB}$$

summing over all boolean insertion paths $u = u^{(0)} \overset{\beta_1}{\rightsquigarrow} u^{(1)} \overset{\beta_2}{\rightsquigarrow} \cdots \overset{\beta_n}{\rightsquigarrow} u^{(n)} = w$.

For convenience, for $f \in H_T^*(G/B;\mathbb{Z})$, we write $[\xi_w]f$ for the coefficient of ξ_w in f expanded in the basis of the equivariant Schubert classes.

The following result is the last technical lemma for Theorem 1.2.

Lemma 3.11. *For boolean elements $u, v, w \in W$ satisfying $u \overset{\mathsf{S}(v)}{\rightsquigarrow} w$ and $v \overset{\mathsf{S}(u)}{\rightsquigarrow} w$,*

$$[\xi_w](\xi_u \cdot \xi_v) = [\xi_w]\Big(\xi_u \prod_{\beta \in \mathsf{S}(v)} \xi_{s_\beta}\Big) = [\xi_w]\Big(\Big(\prod_{\alpha \in \mathsf{S}(u)} \xi_{s_\alpha}\Big)\xi_v\Big) = [\xi_w]\Big(\Big(\prod_{\alpha \in \mathsf{S}(u)} \xi_{s_\alpha}\Big)\Big(\prod_{\beta \in \mathsf{S}(v)} \xi_{s_\beta}\Big)\Big).$$

Proof. (Sketch.) By Lemma 3.9, we have that

$$\prod_{\alpha \in \mathsf{S}(u)} \xi_{s_\alpha} = \sum_{\mathsf{S}(u') = \mathsf{S}(u)} \xi_{u'} \mod \mathcal{NB}, \qquad \prod_{\beta \in \mathsf{S}(v)} \xi_{s_\beta} = \sum_{\mathsf{S}(v') = \mathsf{S}(v)} \xi_{v'} \mod \mathcal{NB}$$

summing over boolean elements u' and v'. It follows that

$$\Big(\prod_{\alpha \in \mathsf{S}(u)} \xi_{s_\alpha}\Big)\Big(\prod_{\beta \in \mathsf{S}(v)} \xi_{s_\beta}\Big) = \sum_{u', v'} \xi_{u'} \xi_{v'} \mod \mathcal{NB} \tag{3.1}$$

summing over boolean elements $u', v' \in W$ satisfying $\mathsf{S}(u') = \mathsf{S}(u)$ and $\mathsf{S}(v') = \mathsf{S}(v)$. Note that the terms $\xi_{u'}\xi_{v'}$ on the right hand side of (3.1) are all "disjoint", after expanding in the basis of the equivariant Schubert classes and restricting to the boolean elements. Further, with Corollary 3.10, $u \overset{\mathsf{S}(v)}{\rightsquigarrow} w$ and $v \overset{\mathsf{S}(u)}{\rightsquigarrow} w$ indicate that ξ_w appears in the expansions of both $\xi_u \prod_{\beta \in \mathsf{S}(v)} \xi_{s_\beta}$ and $\Big(\prod_{\alpha \in \mathsf{S}(u)} \xi_{s_\alpha}\Big)\xi_v$, completing the proof. □

Remark 3.12. Lemma 3.11 demonstrates a very unique property of boolean elements. Let $u \in W$ be any element and let $\mathbf{u}$ be a reduced word of u. We know that $\prod_{\alpha \in \mathbf{u}} \xi_{s_\alpha}$ contains ξ_u and a lot of other terms. In general, we expect

$$[\xi_w](\xi_u \cdot \xi_v) < [\xi_w]\Big(\prod_{\alpha \in \mathbf{u}} \xi_{s_\alpha}\Big)\Big(\prod_{\beta \in \mathbf{v}} \xi_{s_\beta}\Big).$$

However, Lemma 3.11 tells us that when w is boolean, which implies that the relevant u and v are also boolean, we have an equality so that the structure constants are manageable.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 3.11, we have

$$d_{uv}^{w} = [\xi_w](\xi_u \cdot \xi_v) = [\xi_w]\Big(\xi_u \prod_{\beta \in \mathsf{S}(v)} \xi_{s_\beta}\Big).$$

We are done by applying Corollary 3.10 to the right hand side. □

3.3 Structure constants in the cohomology ring $H^*(G/B;\mathbb{Z})$

In this section, we only need to consider non-equivariant boolean insertions with weights equal to 1.

Proof of Corollary 1.3. The cohomology version can be derived from the equivariant cohomology version by setting $t_\alpha = 0$ in Theorem 1.2 for each simple root $\alpha \in \Delta$. This is equivalent to requiring each boolean insertion to be non-equivariant with weight 1. □

The following result is an interesting property of boolean insertions.

Proposition 3.13. *In the case where the Dynkin diagram is a path, fix an ordering $B = \{\beta_1, \cdots, \beta_n\}$ of a set of simple roots $B \subseteq \Delta$, then there exists at most one non-equivariant boolean insertion path $u \overset{B}{\rightsquigarrow} w$ for any boolean elements $u, w \in W$.*

In type A, all the multiplicities as in Definition 3.6 are 1. Combining Corollary 1.3 and Proposition 3.13, we arrive at the following result.

Corollary 3.14. *For boolean elements u, v, w in the Weyl group of type A, $c_{uv}^{w} = 1$ if there exist non-equivariant boolean insertion paths $u \overset{\mathsf{S}(v)}{\rightsquigarrow} w$ and $v \overset{\mathsf{S}(u)}{\rightsquigarrow} w$; $c_{uv}^{w} = 0$ otherwise.*

4 Fast algorithms for computation

In this section, we work in type A_n. We provide Algorithm 1 that determines whether there exists a non-equivariant boolean insertion path $u \overset{\mathsf{S}(v)}{\rightsquigarrow} w$ for boolean elements $u, v, w \in W$. This algorithm works by finding a good ordering of $S(v)$ for the insertion paths. The correctness and the time complexity of the algorithm is provided in Theorem 4.2. By Corollary 1.3, we can calculate the structure constants c_{uv}^{w} for boolean permutations in the symmetric group in $O(n^2)$ time as well.

Algorithm 1 Construction of a boolean insertion path

Input: Boolean elements $u, v, w \in W$.

Output: A boolean insertion path $u \overset{\mathrm{S}(v)}{\rightsquigarrow} w$ if it exists.

1: Initialize $B = B(u)$, P to be an empty list and $S = \mathrm{S}(v)$.
2: Check whether $B(u) \subseteq B(w)$. If not, return **None**.
3: For each $i \in S$ in increasing order, try the boolean insertion $B \overset{i}{\rightsquigarrow}$. If there is only one possible insertion step $B \overset{i}{\rightsquigarrow} B'$ satisfying $B' \subseteq B(w)$, remove i from S, append this step to P and replace B by B'. If no such $B' \subseteq B(w)$ exists, return **None**.
4: Repeat Step 3 until no such insertions are available.
5: Let the remaining vertices in S and $B(w) \setminus B$ be $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_m$ respectively. For $k = 1, \ldots, m$ in increasing order, do $B \overset{i_k}{\rightsquigarrow} B'$ such that the newly added vertex in B' is exactly j_k and that $B' \subseteq B(w)$. Append this sequence of insertions to P if they exist and return **None** if not.
6: Return P.

Example 4.1. Let $u = s_4s_3s_8s_{11}s_{12}$, $v = s_2s_3s_7s_6s_8s_{12}$ and $w = s_7s_6s_5s_4s_2s_3s_9s_8s_{11}s_{13}s_{12}$. In Algorithm 1, we begin with $B = B(u_0)$ where $u_0 = u$ and $S = \mathrm{S}(v) = \{2, 3, 6, 7, 8, 12\}$. The boolean diagrams of u and w and all steps in Algorithm 1 are shown in Figure 7. In the end, we obtain a boolean insertion path $u \overset{\mathrm{S}(v)}{\rightsquigarrow} w$. In fact, there is a boolean insertion path $v \overset{\mathrm{S}(u)}{\rightsquigarrow} w$ as well. Thus, by Corollary 3.14, $c_{uv}^{w} = 1$.

Theorem 4.2. *Algorithm 1 returns a boolean insertion path $u \overset{\mathrm{S}(v)}{\rightsquigarrow} w$ if it exists and otherwise returns* **None**. *The runtime of Algorithm 1 is $O(n^2)$ in type A_n.*

Remark 4.3. In arbitrary types, we can construct similar algorithms to find insertion paths and to compute structure constants c_{uv}^{w} for boolean elements with the same time complexity $O(n^2)$, where $n = \mathrm{rank}(\Phi)$. The general idea stays the same.

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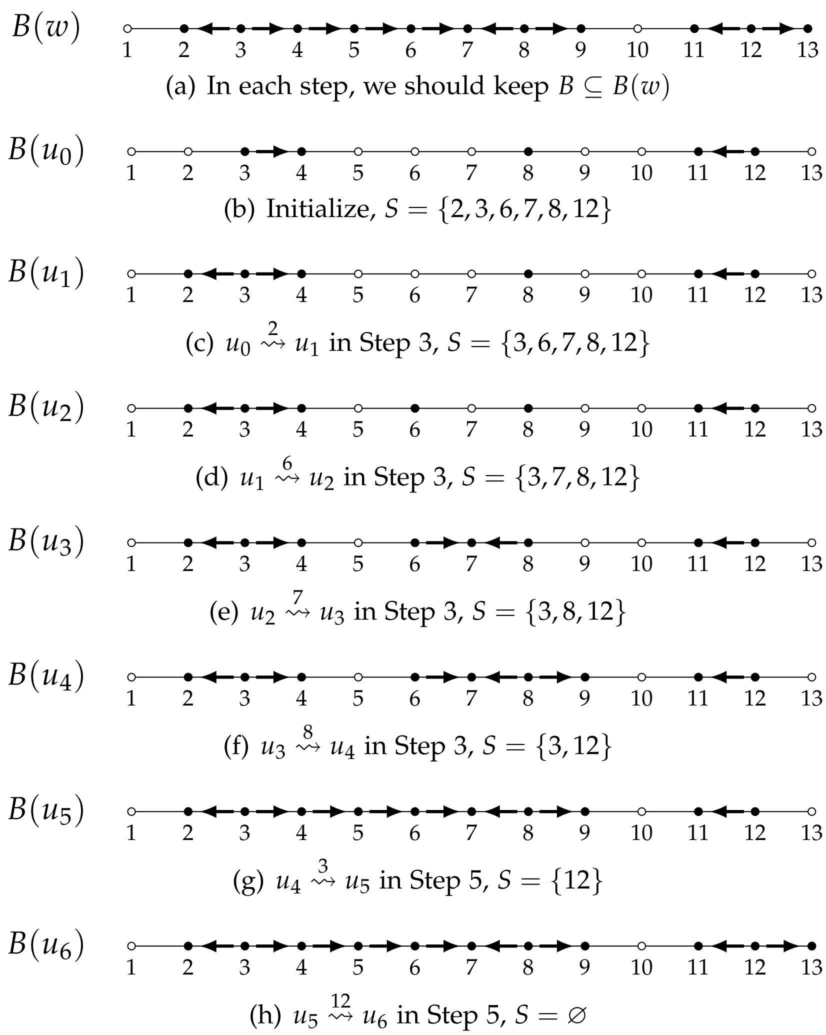

(a) In each step, we should keep $B \subseteq B(w)$

(b) Initialize, $S = \{2, 3, 6, 7, 8, 12\}$

(c) $u_0 \overset{2}{\rightsquigarrow} u_1$ in Step 3, $S = \{3, 6, 7, 8, 12\}$

(d) $u_1 \overset{6}{\rightsquigarrow} u_2$ in Step 3, $S = \{3, 7, 8, 12\}$

(e) $u_2 \overset{7}{\rightsquigarrow} u_3$ in Step 3, $S = \{3, 8, 12\}$

(f) $u_3 \overset{8}{\rightsquigarrow} u_4$ in Step 3, $S = \{3, 12\}$

(g) $u_4 \overset{3}{\rightsquigarrow} u_5$ in Step 5, $S = \{12\}$

(h) $u_5 \overset{12}{\rightsquigarrow} u_6$ in Step 5, $S = \varnothing$

Figure 7: An example for Algorithm 1, that results in a boolean insertion path $u = u_0 \overset{2}{\rightsquigarrow} u_1 \overset{6}{\rightsquigarrow} u_2 \overset{7}{\rightsquigarrow} u_3 \overset{8}{\rightsquigarrow} u_4 \overset{3}{\rightsquigarrow} u_5 \overset{12}{\rightsquigarrow} u_6 = w$.

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