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(q, t)-tau functions and path operators

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Abstract. Motivated by weighted Hurwitz theory and its connection to integrability, we introduce a (q, t)-tau function that deforms the classical case of hypergeometric tau functions using Macdonald polynomials, while simultaneously generalizing several important series that have already appeared in enumerative geometry, gauge theory, and integrability. We prove that this function is uniquely characterized by a family of differential equations and demonstrate a positive combinatorial expansion of these PDEs in terms of a new family of operators encoded by alternating paths. As a byproduct of our techniques, we establish a connection between path operators and the Delta conjecture.

Keywords: Macdonald polynomials, Hurwitz series, vertex operators, delta conjecture.

1 Motivation and main results

1.1 The (q, t)-tau functions and the first main result

Hurwitz theory is a branch of enumerative geometry that focuses on counting branched coverings of the sphere. It has been found that various generating functions in Hurwitz theory can be expressed, using monodromy and representation theory of symmetric groups, as explicit infinite series involving Schur symmetric functions. This perspective proved highly effective, revealing many beautiful connections between the old and somewhat forgotten Hurwitz theory and other modern fields such as Gromov–Witten theory, integrable probability, integrable systems, and matrix models, which gave a new life to this old topic [3]. In particular, these series are often called tau functions due to the fact that they are tau functions of certain integrable hierarchies such as the KP-hierarchy, or more generally the 2D Toda hierarchy [9].

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Schur symmetric functions have several interesting deformations, such as Jack symmetric functions (a one-parameter deformation) and more generally, Macdonald polynomials (a two-parameter deformation). These symmetric functions can furthermore be used to deform the tau functions of Hurwitz theory, by replacing occurences of Schur functions with Jack or Macdonald polynomials. This leads to a natural question: *what are the structural properties of these "deformed" tau functions?* In the case of Jack symmetric functions, it was recently proven in [4] that such a deformation can still be interpreted as a generating function for geometric objects in an extended version of Hurwitz theory that includes real (non-oriented) branched coverings. A primary novelty in these findings was the construction of an explicit PDE satisfied by the deformed tau function, which exhibited a rich combinatorial structure. It also allowed to discover further connections to the Whitakker vector of $\mathcal{W}$-algebras and topological recursion [5]. Inspired by those results, we here study a (q,t)-deformed tau function, and prove analogous PDEs in this more general setting. Let us now introduce this (q,t)-deformed tau function.

Consider two infinite sequences of variables $u_0, u_1, \dots$ and $v_0, v_1, \dots$, and define $\mathbb{K}$ as the field of formal Laurent series in these variables with coefficients in $\mathbb{Q}(q,t)$

$$\mathbb{K} := \mathbb{Q}(q,t)((u_0, u_1, \dots, v_0, v_1, \dots)). \tag{1.1}$$

We denote Λ the space of symmetric functions with coefficients in $\mathbb{K}$. If $X = x_1 + x_2 + \cdots$, $Y = y_1 + y_2 + \cdots$ are two alphabets, then Λ_X and Λ_Y will be the spaces of symmetric functions in X and Y, respectively.

Let $G_1(\hbar) = \sum_{i=0}^{\infty} u_i \hbar^i, G_2(\hbar) = \sum_{i=0}^{\infty} v_i \hbar^i \in \mathbb{Q}(q,t)[\hbar]((u_0, u_1, \dots, v_0, v_1, \dots))$, and define $G(\hbar) = \frac{G_1(\hbar)}{G_2(\hbar)}$. In particular, $G(q^j t^i) \in \mathbb{K}$ is well-defined for all $i, j \geq 0$. The G-weighted (q,t)-tau function is the series in $\Lambda_X \otimes \Lambda_Y[\![z]\!]$ defined by:

$$\tau_G(z, X, Y) := \sum_{\lambda \in \mathbb{Y}} z^{|\lambda|} \frac{\widetilde{H}_\lambda^{(q,t)}[X] \widetilde{H}_\lambda^{(q,t)}[Y]}{\left\| \widetilde{H}_\lambda^{(q,t)} \right\|_*^2} \prod_{(i,j) \in \lambda} G(q^{j-1} t^{i-1}), \tag{1.2}$$

where $\widetilde{H}_\lambda^{(q,t)}$ are modified Macdonald polynomials, see Section 2.1.

Beyond it being a natural (q,t)-deformation of the tau functions for the G-weighted b-Hurwitz numbers of [4], this function has appeared as an important generating series in enumerative geometry and mathematical physics for specific cases of the weight G. When $G(\hbar) = \prod_{i=1}^{n}(1 - \hbar u_i)$, the plethystic logarithm of $\tau_G(z, X, Y)$ was conjectured by Hausel–Lettelier–Rodriguez-Villegas [11] to be the generating series of the mixed Hodge polynomials of character varieties of the Riemann sphere. When $G(\hbar) = (1 - u\hbar)^{-1}$, the specialization $Y = 1$ corresponds to the Whittaker vector for the deformed Virasoro algebra from the $5D$ $\mathcal{N} = 1$ pure SU(2) case of the AGT conjecture [13]. Finally, the case $G(\hbar) = \hbar$ has appeared in the recent work of Bourgine and Garbali [2] as a tau function of a (q,t)-extension of the 2D Toda hierarchy[1].

Our first main theorem gives explicit PDEs that allow to obtain $\tau_G(z,X,Y)$ as their unique solution. Our family of differential operators is defined using the classical operator D_0,

$$D_0 := \sum_{k\geq 0}(-1)^k e_k[X]h_k^\perp[MX], \tag{1.3}$$

where $M=(1-q)(1-t)$ and $h_k^\perp$ is the adjoint of h_k for the Hall inner product, see Section 2.1. We then define the following operators, obtained from D_0 and[2] e_1 by commutation operations.

Throughout this paper, the series F will be $F=G_1$ or $F=G_2$, with an expansion $F(\hbar)=\sum_{i\geq 0}a_i\hbar^i$ ($a_i=u_i$ in the first case and $a_i=v_i$ in the second one). We then define

$$\mathcal{A}_F := \sum_{i\geq 0} a_i\,\mathrm{ad}^i_{-\frac{D_0}{M}}(-e_1[X]),\quad \mathcal{A}_F^{(1)} := \mathrm{ad}_{-\frac{D_0}{M}}\left(\mathcal{A}_F\right),\quad \mathcal{A}_F^{(\ell)} = \mathrm{ad}^{\ell-1}_{\mathcal{A}_F/M}(\mathcal{A}_F^{(1)}),\ \ell\geq 2,$$

where $\mathrm{ad}_A(B) := [A,B] = AB-BA$, for any operators A and B. The operators $\mathcal{A}_F^{(\ell)}$ are differential when we think of symmetric functions as functions in $(h_i)_{i\geq 1}$ for example.

Theorem 1.1. *For any $\ell\geq 1$ we have*

$$z^\ell\mathcal{A}^{(\ell)}_{G_1}(X)\cdot\tau_G(z,X,Y) = \left(\mathcal{A}^{(\ell)}_{G_2}(Y)\right)^*\cdot\tau_G(z,X,Y). \tag{1.4}$$

Moreover, $\tau_G(z,X,Y)$ is the unique function $\Phi_G\in\Lambda_X\otimes\Lambda_Y[\![z]\!]$ which satisfies these equations and such that $[z^0]\Phi_G(z,X,Y)=1$, where $\left(\mathcal{A}^{(\ell)}_{G_2}\right)^*$ is the adjoint of $\mathcal{A}^{(\ell)}_{G_2}$ with respect to the star scalar product (see Section 2.1).

1.2 Path operators and the second main theorem

Note that the operators $\mathcal{A}_F^{(\ell)}$ appearing in Theorem 1.1 are defined through multiple applications of the adjoint action, and it is not clear at all whether they possess a combinatorial interpretation. A key part of [4] that enabled a combinatorial/topological interpretation in the case of the tau function involving Jack symmetric functions was the explicit expression of analogous operators, defined by the adjoint action, as a positive sum of operators defined combinatorially using lattice paths. Our second main result

[1]In [2, 13] the authors worked with the P-version of Macdonald polynomials. We define our (q,t)-tau function in terms of the modified Macdonald polynomials, as such a function seems to have much more interesting combinatorial structure that is partially supported by the conjecture of Hausel–Lettelier–Rodriguez-Villegas.

[2]Here, we think of e_1 as the operator acting on Λ by multiplying by e_1.

demonstrates that a similar statement holds in the Macdonald case. The key idea is the following: we write the operators $\mathcal{A}_F^{(\ell)}$ by *alternating* multiplicative and derivative operators ($(-1)^k e_k$ and $h_k^\perp$). We start by defining the underlying combinatorial objects: alternating paths.

Let $N = (x_N, y_N)$ be a point of $(\mathbb{Z}_{\geq 0})^2$. We call *a path* γ from the origin $(0,0)$ to N, a sequence of points $(w_0, w_1, \ldots, w_{x_N})$ in $(\mathbb{Z}_{\geq 0})^2$ such that $w_j = (j, y_j)$, with $w_0 = (0,0)$ and $w_{x_N} = N$. Such a path is uniquely encoded by its sequence of *steps* $\gamma_j := y_j - y_{j-1} \in \mathbb{Z}$. The length of γ is x_N.

We say that a path $\gamma = (\gamma_1, \ldots, \gamma_n)$ from $(0,0)$ to N is *alternating*, if $\gamma_{2j-1} \geq 0$ and $\gamma_{2j} \leq 0$ for any $0 \leq j \leq n$. In other terms, odd steps are up steps and even steps are down steps, with the convention that a flat step is considered either an up or a down step according to parity.

We say that a point $V = (x_V, y_V)$ of γ is a *valley* of γ if x_V is even, it means that V is preceded by a down step and followed by an up step. Similarly, we say that $P = (x_P, y_P)$ is a *peak* if x_P is odd.

Definition 1.2. *Fix two integers $\ell \geq 1$ and $n \geq 0$ and let γ be an alternating path of length $2n+2$ starting at $(0,0)$ and ending at $(2n+2, \ell)$. Let $\mathbf{u} := (u_1, \ldots, u_\ell)$ be a non-decreasing sequence of integers satisfying $1 = u_1 \leq u_2 \leq \cdots \leq u_\ell \leq n+1$. For $j = 1, \ldots, n$ we define m_j as the largest integer k for which $u_k \leq j$, we also set $m_0 := 0$. We define the* height *of a valley $V = (2j, y_{2j})$ by*

$$\mathrm{ht}(V) := y_{2j} - m_j.$$

*We say that $\boldsymbol{\gamma} := (\gamma, \mathbf{u})$ is a **decorated alternating path** if all valleys of γ have non-negative heights. The integer ℓ will be called the* degree *of the path.*

Throughout the rest of the paper, we will think of these paths as alternating paths whose peaks are decorated by particles as follows: the integers $(u_j)_{1 \leq j \leq \ell}$ indicate the positions of ℓ particles labelled $1, 2, \ldots, \ell$, such that the j-th particle sits on the u_j-th peak (the peak of x-coordinate $2u_j - 1$). The integer m_j is then precisely the number of particles to the left of the valley at x-coordinate $2j$. Note that the number of particles is always equal to the y-coordinate of the end point of the path. As a consequence, the first and the last valleys of the path (which correspond to the origin and end points) have always height 0.

Example 1.3. *We give in Figure 1 an example of a decorated alternating path $\boldsymbol{\gamma} := (\gamma, \mathbf{u})$ of length 8 and of degree 4. The 4 particles are represented by the turquoise points above the peaks. The particle positions are given by $\mathbf{u} = (u_1, u_2, u_3, u_4) = (1, 2, 2, 4)$. The heights of the five valleys from left to right are equal to $0, 1, 0, 1, 0$.*

We are almost ready to define the path operators, the main object of our second main theorem. Roughly, these operators are weighted generating functions of decorated

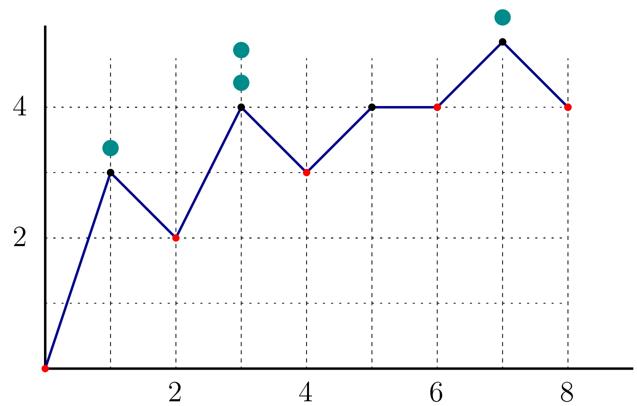

Figure 1: A decorated alternating path of length 8 and degree 4. The five valleys are represented in red.

alternating paths, where the weight associated with each path is given by the following definition.

Definition 1.4. *Let $\boldsymbol{\gamma} := (\gamma, \mathbf{u})$ be a decorated alternating path. We then associate to $\boldsymbol{\gamma}$ the valley weight*

$$\mathrm{vw}(\gamma) = \prod_{V \in \mathcal{V}(\gamma)} (qt)^{\mathrm{ht}(V)}, \tag{1.5}$$

the product being taken over the valleys of $\boldsymbol{\gamma}$. We associate to γ the operator $\mathcal{O}(\gamma)$ defined as follows: if $\gamma = (\gamma_1, \ldots, \gamma_{2n+2})$ then

$$\mathcal{O}(\boldsymbol{\gamma}) := \mathrm{vw}(\gamma)\mathcal{O}(\gamma_1)\ldots\mathcal{O}(\gamma_{2n+2}), \tag{1.6}$$

where for any integer m we define

$$\mathcal{O}(m) := \begin{cases} (-1)^m e_m[X] = h_m[-X] & \text{if } m > 0 \\ h_{-m}^{\perp}[MX] & \text{if } m < 0 \\ 1 & \text{if } m = 0. \end{cases} \tag{1.7}$$

Note that there are two ways of describing the positions of the particles in a decorated path: either by the sequence of distances between them, or by the sequence counting the number of particles on every peak. We introduce then the following definitions:

- for any vector $\alpha \in \mathbb{Z}_{\geq 0}^{\ell}$ with $|\alpha| := \sum_i \alpha_i = n$, we define $\mathbf{Q}_\alpha$ as the set of all decorated paths $\boldsymbol{\gamma} = (\gamma, \mathbf{u})$ of length $2n+2$ and degree ℓ such that $\mathbf{u} = (u_1, \ldots, u_\ell)$ satisfies $\alpha_i = u_{i+1} - u_i$, for any $1 \leq i \leq \ell - 1$ and $\alpha_\ell = n + 1 - u_\ell$. In other terms, $2\alpha_i$ is the distance between the i-th and the $i+1$-th particles, and $2\alpha_\ell$ is the distance between the last particle and the last peak (the distance here is defined as the difference between the x-coordinates).

- for any vector $\alpha \in \mathbb{Z}_{>0} \times \mathbb{Z}^n_{\geq 0}$, we define $\mathbf{Q}^\alpha$ as the set of all decorated paths γ of length $2n+2$ and degree $\ell := |\alpha|$, with α_i particles sitting on the i-th peak.

Finally, we define

$$\mathcal{Q}_\alpha := \sum_{\gamma \in \mathbf{Q}_\alpha} \mathcal{O}(\gamma), \qquad \mathcal{Q}^\alpha := \sum_{\gamma \in \mathbf{Q}^\alpha} \mathcal{O}(\gamma). \tag{1.8}$$

Example 1.5. *The operator associated to the path γ of Figure 1 is given by*

$$\mathcal{O}(\gamma) = (qt)^{1+0+1} \cdot (-1)^{3+2+1+1} e_3[X] h_1^\perp[MX] e_2[X] h_1^\perp[MX] e_1[X] e_1[X] h_1^\perp[MX].$$

Moreover $\gamma \in \mathbf{Q}_{(1,0,2,0)} = \mathbf{Q}^{(1,2,0,1)}$.

We are now ready to state the seond main result of this paper.

Theorem 1.6. *If $F(\hbar) := \sum_{i \geq 0} a_i \hbar^i \in \{G_1, G_2\}$, then*

$$\mathcal{A}_F^{(\ell)} = \sum_{\alpha \in \mathbb{Z}^\ell_{\geq 0}} a_\alpha \mathcal{Q}_\alpha, \tag{1.9}$$

where $a_\alpha := \prod_{i=1}^{\ell} a_{\alpha_i}$.

1.3 Explicit formulas for the functions $\mathcal{Q}^{\boldsymbol{\alpha}} \cdot 1$

As we will see in Equation (2.5), the (q,t)-tau function admits a simple expansion on a basis of symmetric functions built upon consecutive actions of the operators $\mathcal{A}_G^{(\ell)}$. From Theorem 1.6, these functions are in turn linear combinations of symmetric functions of the form

$$\mathcal{Q}^{\boldsymbol{\alpha}} \cdot 1 := \mathcal{Q}^{\alpha^{(1)}} \cdot \mathcal{Q}^{\alpha^{(2)}} \cdots \mathcal{Q}^{\alpha^{(r)}} \cdot 1 \tag{1.10}$$

where $\boldsymbol{\alpha} = \left(\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(r)}\right) = (\alpha_1, \ldots, \alpha_\ell)$ is a list of r vectors of respective lengths $\ell_1, \ldots, \ell_r$ of total size n. In our last main theorem, we give explicit formulas for these functions, by extracting the coefficients of some rational function with respect to a second alphabet $Z = z_1 + \cdots + z_\ell$. We start by introducing some notation.

Let $\mathcal{NI}(\boldsymbol{\alpha})$ be the set of *non-consecutive indices* of $\boldsymbol{\alpha}$ defined by:

$$\mathcal{NI}(\boldsymbol{\alpha}) := \{(i,j) | 1 \leq i < j \leq \ell, \text{ such that } i < j-1 \\ \text{or } (i = j-1 \text{ and } i \leq L_k < j \text{ for some } 1 \leq k \leq r)\}, \tag{1.11}$$

where we denote the partial sums $L_k = \sum_{1 \leq i \leq k} \ell_i$. Note that $\mathcal{NI}(\boldsymbol{\alpha})$ depends only on the sequence $(\ell_1, \ell_2, \ldots, \ell_r)$ and not α itself.

Example 1.7. *If* $\boldsymbol{\alpha} = ((2,0,0),(3))$, *then* $\mathcal{NI}(\boldsymbol{\alpha}) = \{(1,3),(1,4),(2,4),(3,4)\}$.

Theorem 1.8. *Fix* $\boldsymbol{\alpha} = (\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(r)}) = (\alpha_1, \dots, \alpha_\ell)$ *of total size* n. *For any partition* λ *of size* n *and for the alphabet* $Z = z_1 + \cdots + z_\ell$ *we have*

$$(-1)^n [m_\lambda[X]] \mathcal{Q}^{\boldsymbol{\alpha}} \cdot 1 = [z^{\boldsymbol{\alpha}}]\, e_\lambda[Z] \prod_{1 \le i < j \le \ell} \frac{1 - z_j/z_i}{(1 - qz_j/z_i)(1 - tz_j/z_i)} \prod_{(i,j) \in \mathcal{NI}(\boldsymbol{\alpha})} (1 - qtz_j/z_i),$$

$$(-1)^n [s_\lambda[X]] \mathcal{Q}^{\boldsymbol{\alpha}} \cdot 1 = [z^{\boldsymbol{\alpha}}]\, s_{\lambda'}[Z] \prod_{1 \le i < j \le \ell} \frac{1 - z_j/z_i}{(1 - qz_j/z_i)(1 - tz_j/z_i)} \prod_{(i,j) \in \mathcal{NI}(\boldsymbol{\alpha})} (1 - qtz_j/z_i),$$

$$(-1)^n [e_\lambda[X]] \mathcal{Q}^{\boldsymbol{\alpha}} \cdot 1$$
$$= \frac{1}{|\operatorname{Aut}(\lambda)|} \sum_{\sigma \in \mathfrak{S}_\ell} \left[z^{\boldsymbol{\alpha} - \sigma(\lambda)}\right] \prod_{1 \le i < j \le \ell} \frac{1 - z_j/z_i}{(1 - qz_j/z_i)(1 - tz_j/z_i)} \prod_{(i,j) \in \mathcal{NI}(\boldsymbol{\alpha})} (1 - qtz_j/z_i),$$

where λ' *is the transpose of* λ, $\sigma(\lambda_1, \dots, \lambda_\ell) = (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(\ell)})$, *and the rational functions in the RHS should be expanded as formal series in* z_j/z_i *for* $j > i$.

Interestingly, the functions of Equation (1.10) turned out to be related to some other important problems in the field, namely the Shuffle and the Delta conjectures.

1.4 Connection to the Delta conjecture

For $f \in \Lambda$, the operators Δ'_f are defined by the equations

$$\Delta'_f \widetilde{H}_\lambda^{(q,t)}[X] := f\left[\sum_{(i,j) \in \lambda} q^{j-1} t^{i-1} - 1\right] \widetilde{H}_\lambda^{(q,t)}[X]. \tag{1.12}$$

The Delta conjecture, formulated in [10] and proved by D'Adderio and Mellit in [6] states that $\Delta'_{e_k} \cdot e_n$ is positive in the monomial basis, and the coefficients count Dyck paths with weights given by the *area* and the *dinv* statistics. The Shuffle conjecture corresponds to the case $k = n - 1$. It turns out that these problems are closely related to our path operators. Indeed, we prove the following

$$\Delta'_{e_k} \cdot e_n = (-1)^n \sum_{\substack{\ell(\alpha) = k+1 \\ |\alpha| = n}} \mathcal{Q}^\alpha \cdot 1. \tag{1.13}$$

Actually, we obtain a more general formula for $\Delta_{h_l} \Delta'_{e_k} \cdot e_n$ also in terms of path operators. This last quantity is the subject of the extended delta conjecture.

As a byproduct of Theorem 1.8, we obtain explicit formulas for $\Delta_{h_l} \Delta'_{e_k} \cdot e_n$, from which we obtain [1, Theorem 4.4.1.] of Blasiak *et al.* used to prove the extended delta conjecture. While the proof of this result is based in [1] on a connection with the Schiffmann algebra, we are able to derive it here completely combinatorially using path operators. We then hope that the formulas of Theorem 1.8 might give a new approach to understand the combinatorics of the delta conjecture and other problems related to it.

2 Elements of proofs

2.1 Notation and preliminaries

We denote $\mathbb{Y}$ the set of integer partitions. For $i \geq 1$ and $\lambda \in \mathbb{Y}$, let $m_i(\lambda)$ be the number of parts of size i in λ. We then set

$$z_\lambda := \prod_{i\geq 1} m_i(\lambda)! i^{m_i(\lambda)}.$$

The *Hall scalar product* $\langle ., . \rangle$ on Λ is defined by $\langle p_\mu, p_\lambda \rangle = z_\lambda \delta_{\mu,\lambda}$, for any $\lambda, \mu \in \mathbb{Y}$, where $\delta_{\mu,\lambda}$ is the Kronecker delta, and p_μ are power-sum symmetric functions. The *star scalar product* $\langle ., . \rangle_*$ is the deformation of the Hall scalar product given by

$$\langle p_\mu, p_\lambda \rangle_* = z_\mu \delta_{\mu,\lambda} p_\mu[-M] = (-1)^{\ell(\mu)} z_\mu \delta_{\mu,\lambda} \prod_{1\leq i \leq \ell(\mu)} (1-q^{\mu_i})(1-t^{\mu_i}), \text{ for any } \mu, \lambda \in \mathbb{Y}.$$

The modified Macdonald polynomials are then orthogonal with respect to this product, $\langle \widetilde{H}^{(q,t)}_\mu, \widetilde{H}^{(q,t)}_\lambda \rangle_* = \delta_{\mu,\lambda} \left\| \widetilde{H}^{(q,t)}_\lambda \right\|^2_*$.

For any operator $\mathcal{O}$ on Λ, we denote the $\mathcal{O}^\perp$ (resp. $\mathcal{O}^*$) the adjoint of $\mathcal{O}$ with respect to the Hall scalar product (resp. the star scalar product). In particular, if $f \in \Lambda$ then $f^\perp$ is the adjoint of the multiplication by f.

We define the linear operator Π_G on Λ_X by its action on modified Macdonald polynomials $\widetilde{H}^{(q,t)}_\lambda$

$$\Pi_G \cdot \widetilde{H}^{(q,t)}_\lambda[X] := \widetilde{H}^{(q,t)}_\lambda[X] \prod_{(i,j)\in\lambda} G(q^{j-1}t^{i-1}). \tag{2.1}$$

One may notice that the G-weighted tau function is obtained by applying this operator on the Macdonald Cauchy kernel:

$$\tau_G(z, X, Y) = \Pi_G \cdot \sum_{\lambda\in\mathbb{Y}} z^{|\lambda|} \frac{\widetilde{H}^{(q,t)}_\lambda[X] \widetilde{H}^{(q,t)}_\lambda[Y]}{\left\| \widetilde{H}^{(q,t)}_\lambda \right\|^2_*}. \tag{2.2}$$

We think of the adjoint $\mathrm{ad}_A(\cdot) = [A, \cdot]$ as a linear operator on the space of operators on Λ. In particular, $\mathrm{ad}^i_A(B)$ corresponds to iterating the commutation by A i times, and if $F(\hbar) = \sum_{i\geq 0} a_i \hbar^i$, then $F(\mathrm{ad}_A) := a_0 + a_1 \,\mathrm{ad}_A + a_2 \,\mathrm{ad}^2_A + \cdots$.

2.2 Proof of the equations of 1.1: Pieri rule and the operator D_0

We will derive the equations and prove uniqueness separately. In order to prove the equations, we need to find how the operators $\mathcal{A}^{(\ell)}_F$ act on modified Macdonald polynomials, starting with $\mathrm{ad}^i_{D_0/M}(-e_1[X])$. The action of $e_1[X]$ (by multiplication) is given by the following Pieri rule, [12, Chapter VI, Equation 6.7] (see also [8, pp. I.11, I.12]).

Theorem 2.1 (Pieri Rule, [12])**.**

$$-e_1[X]\widetilde{H}_\mu^{(q,t)} = \sum_{\mu \nearrow \lambda} d^{\mu,\lambda}\widetilde{H}_\lambda^{(q,t)},$$

where the sum is taken over partitions λ obtained from μ by adding one cell, and $d^{\mu,\lambda} \in \mathbb{Q}(q,t)$.

Remarkably, no explicit expression of $d^{\mu,\lambda}$ is required in our work, only the fact that they vanish if λ is not obtained from μ by the addition of a box. Moreover, D_0 acts diagonally on Macdonald polynomials.

Theorem 2.2 ([7, Theorem 1.2])**.** *We have*

$$D_0 \cdot \widetilde{H}_\lambda^{(q,t)} = \Big(1 - M \sum_{(i,j)\in\lambda} q^{j-1}t^{i-1}\Big)\widetilde{H}_\lambda^{(q,t)}, \quad \textit{for any } \lambda \in \mathbb{Y}.$$

Lemma 2.3. *For any $\ell \geq 1$, we have*

$$\Pi_G \cdot \mathcal{A}_{G_2}^{(\ell)} \cdot \Pi_G^{-1} = \mathcal{A}_{G_1}^{(\ell)}.$$

Proof. From Theorem 2.1, Theorem 2.2 and the definition $\mathcal{A}_F = F(\mathrm{ad}_{-D_0/M}) \cdot (-e_1)$ we find

$$\mathcal{A}_F\widetilde{H}_\mu^{(q,t)} = \sum_{\mu \nearrow \lambda} d^{\mu,\lambda}F(c_{q,t}(\lambda/\mu))\widetilde{H}_\lambda^{(q,t)}, \tag{2.3}$$

where $c_{q,t}(\lambda/\mu) := q^{j-1}t^{i-1}$, with (i,j) being the only cell of λ/μ. With the definition of Π_G, this gives directly the lemma for $\ell = 1$. Then for $\ell \geq 2$, one has $\mathcal{A}_F^{(\ell)} = \frac{1}{M}[\mathcal{A}_F, \mathcal{A}_F^{(\ell-1)}]$, so the lemma is obtained by induction. □

We obtain Theorem 1.1 by combining Lemma 2.3 and the following lemma, which is a consequence of the orthogonality of Macdonald polynomials.

Lemma 2.4. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two operators on Λ which have the same homogeneous degree $\ell \geq 0$. Then*

$$z^\ell \mathcal{O}_2(X) \cdot \tau_G(z,X,Y) = \mathcal{O}_1^*(Y) \cdot \tau_G(z,X,Y) \iff \Pi_G \cdot \mathcal{O}_1 \cdot \Pi_G^{-1} = \mathcal{O}_2. \tag{2.4}$$

2.3 Proof of uniqueness in Theorem 1.1: the basis $\mathfrak{a}_{F,\lambda}$

For $F \in \{G_1, G_2\}$, and any partition λ, we define the symmetric function of degree λ

$$\mathfrak{a}_{F,\lambda} := \mathcal{A}_F^{(\lambda_1)} \cdot \mathcal{A}_F^{(\lambda_2)} \cdots \mathcal{A}_F^{(\lambda_{\ell(\lambda)})} \cdot 1.$$

Proposition 2.5. *The family $(\mathfrak{a}_{F,\lambda})_{\lambda\in\mathbb{Y}}$ is a basis of Λ.*

Idea of the proof. It is enough to prove that $\mathfrak{a}_{F,\lambda}$ is a basis when we take the specialization $q = 1$. In this case, the derivative operators ($\mathcal{O}(m)$ in Equation (1.7) for m negative) become trivial, and all the path operators are multiplicative. The proposition follows then by a triangularity argument.

Let us now explain how this proposition implies uniqueness in Theorem 1.1. For any partition λ, the differential equations Equation (1.4) allow us to write

$$\begin{aligned} z^{|\lambda|}\mathcal{A}_{G_1}^{(\lambda_1)}(X)\cdot\mathcal{A}_{G_1}^{(\lambda_2)}(X)\cdots\mathcal{A}_{G_1}^{(\lambda_{\ell(\lambda)})}(X)\cdot\tau_G(z,X,Y)& \\ = \left(\mathcal{A}_{G_2}^{(\lambda_{\ell(\lambda)})}(Y)\right)^*\cdots\left(\mathcal{A}_{G_2}^{(\lambda_2)}(Y)\right)^*\cdot\left(\mathcal{A}_{G_2}^{(\lambda_1)}(Y)\right)^*\cdot\tau_G(z,X,Y).& \end{aligned}$$

We now extract the coefficient of $z^{|\lambda|}p_{\emptyset}[Y] = z^{|\lambda|}$ (the constant term in Y). On the left-hand side we simply get $\mathfrak{a}_{G_1,\lambda}(X)$. On the right-hand side, we obtain

$$\begin{aligned} &\left[z^{|\lambda|}\right]\left(\mathcal{A}_{G_2}^{(\lambda_{\ell(\lambda)})}(Y)\right)^*\cdots\left(\mathcal{A}_{G_2}^{(\lambda_2)}(Y)\right)^*\cdot\left(\mathcal{A}_{G_2}^{(\lambda_1)}(Y)\right)^*\cdot\tau_G(z,X,Y) \\ &\quad= \left\langle p_{\emptyset}[Y], \left[z^{|\lambda|}\right]\left(\mathcal{A}_{G_2}^{(\lambda_{\ell(\lambda)})}(Y)\right)^*\cdots\left(\mathcal{A}_{G_2}^{(\lambda_2)}(Y)\right)^*\cdot\left(\mathcal{A}_{G_2}^{(\lambda_1)}(Y)\right)^*\cdot\tau_G(z,X,Y)\right\rangle_* \\ &\quad= \left\langle \mathcal{A}_{G_2}^{(\lambda_1)}(Y)\cdot\mathcal{A}_{G_2}^{(\lambda_2)}(Y)\cdots\mathcal{A}_{G_2}^{(\lambda_{\ell(\lambda)})}(Y)\cdot 1, \left[z^{|\lambda|}\right]\tau_G(z,X,Y)\right\rangle_* \\ &\quad= \left\langle \mathfrak{a}_{G_2,\lambda}(Y), \left[z^{|\lambda|}\right]\tau_G(z,X,Y)\right\rangle_*, \end{aligned}$$

where the scalar product is taken with respect to the alphabet Y. Let $\mathfrak{b}_{G_2,\lambda}$ denote the dual basis of $\mathfrak{a}_{G_2,\lambda}$ with respect to the star scalar product: $\langle \mathfrak{a}_{G_2,\lambda}, \mathfrak{b}_{G_2,\mu}\rangle_* = \delta_{\lambda,\mu}$, for any λ and μ. Then the differential equations of Theorem 1.1 imply that for any λ

$$\mathfrak{a}_{G_1,\lambda}(X) = \left[z^{|\lambda|}\mathfrak{b}_{G_2,\lambda}(Y)\right]\tau_G(z,X,Y),$$

or equivalently

$$\tau_G(z,X,Y) = \sum_{\lambda\in\mathbb{Y}} z^{|\lambda|}\mathfrak{a}_{G_1,\lambda}(X)\mathfrak{b}_{G_2,\lambda}(Y). \tag{2.5}$$

In particular, the differential equations of Theorem 1.1 characterize the function τ_G. Note that when $G_1 = G_2$, the last equation corresponds to the Macdonald–Cauchy identity.

2.4 Proof of Theorem 1.6 and Theorem 1.8

The proof of Theorem 1.6 is based on the two families of commutation relations satisfied by path operators.

Proposition 2.6 (The first commutation relation)**.** *For any $n \geq 0$,*

$$\mathcal{Q}_n = \mathrm{ad}_{-D_0/M}(\mathcal{Q}_{n-1}),$$

where $\mathcal{Q}_n$ is the operator defined in Equation (1.8) if $n \geq 0$, and $\mathcal{Q}_{-1} := -e_1[X]$.

Note that these commutation relations concern one-particle paths. They allow one to obtain the length n operators from operators associated to paths of lengths $n-1$ by applying the adjoint of D_0. The proof is based on the following identities, which are a consequence of commutation relations of vertex operators (see *e.g* [7, Theorem 1.1 and Proposition 1.2])

$$[D_0, \mathcal{O}(r)] = -M \sum_{k\geq 0} \sum_{1\leq i\leq \min(k,r)} (1+qt+\dots (qt)^{i-1})\mathcal{O}(k)\mathcal{O}(k-i)\mathcal{O}(r-i),$$
$$[D_0, \mathcal{O}(-r)] = M \sum_{k\geq 0} \sum_{1\leq i\leq \min(k,r)} (1+qt+\dots (qt)^{i-1})\mathcal{O}(-(r-i))\mathcal{O}(k-i)\mathcal{O}(-k).$$

The second commutation relation concerns paths decorated by an arbitrary number of particles. We give here a symmetrized version, which is enough to prove Theorem 1.6.

Proposition 2.7 (The second commutation relation)**.** *Let $\alpha \in \mathbb{Z}^{\ell}_{\geq 0}$ be a sequence of non-negative integers of length $\ell \geq 2$. We then have*

$$\sum_{\sigma\in\mathfrak{S}_\ell} \mathcal{Q}_{\sigma(\alpha)} = \frac{1}{M} \sum_{\sigma\in\mathfrak{S}_\ell} \left[\mathcal{Q}_{\alpha_{\sigma(\ell)}-1}, \mathcal{Q}_{\sigma(\alpha_1,\dots,\alpha_{\ell-1})}\right].$$

The proof of this proposition uses the first commutation relation and manipulations on commutators. Let us now use these propositions to prove Theorem 1.6 by induction on ℓ.

From Proposition 2.6, we get that $\mathrm{ad}^{n+1}_{-D_0/M}(-e_1) = \mathcal{Q}_n$. This implies that for $F(\hbar) = \sum_{n\geq 0} a_n \hbar^n$, we have $\mathcal{A}_F = \sum_{n\geq 0} a_n \mathcal{Q}_{n-1}$. We deduce that

$$\mathcal{A}^{(1)}_F = \sum_{n\geq 0} a_n \mathcal{Q}_n,$$

giving Theorem 1.6 for $\ell = 1$.

We now assume that Equation (1.9) holds for some $\ell \geq 1$. We then have

$$\mathcal{A}^{(\ell+1)}_F = \frac{1}{M}\left[\mathcal{A}_F, \mathcal{A}^{(\ell)}_F\right] = \frac{1}{M}\sum_{j\geq 0}\sum_{\alpha\in\mathbb{Z}^{\ell}_{\geq 0}} u_j u_\alpha \left[\mathcal{Q}_{j-1}, \mathcal{Q}_\alpha\right].$$

By applying Proposition 2.7 it comes

$$\begin{aligned}\mathcal{A}^{(\ell+1)}_F =& \frac{1}{M}\sum_{0\leq\nu_1\leq\dots\leq\nu_{\ell+1}} \frac{1}{|\operatorname{Aut}(\nu)|} u_\nu \sum_{\sigma\in\mathfrak{S}_{\ell+1}} \left[\mathcal{Q}_{\nu_{\sigma(\ell+1)}-1}, \mathcal{Q}_{\sigma(\nu_1,\dots,\nu_\ell)}\right] \\ =& \sum_{0\leq\nu_1\leq\dots\leq\nu_{\ell+1}} \frac{1}{|\operatorname{Aut}(\nu)|} u_\nu \sum_{\sigma\in\mathfrak{S}_{\ell+1}} \mathcal{Q}_{\sigma(\nu)} = \sum_{\alpha\in\mathbb{Z}^{\ell+1}_{\geq 0}} u_\alpha \mathcal{Q}_\alpha,\end{aligned}$$

where $\operatorname{Aut}(\nu) < \mathfrak{S}_{\ell+1}$ is the stabilizer of the partition ν. This finishes the proof of Theorem 1.6.

Idea of the proof of Theorem 1.8. We start by taking the normal order form in Theorem 1.6, *i.e* we move the derivative part to the right and the multiplicative part to the left. By inspecting the combinatorics of this formula, we prove that it can be written as a sum over alternating paths for which all valleys have minimal height. This gives the formulas of Theorem 1.8. □

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