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Logarithmic concavity of polynomials arising from equivariant cohomology

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Abstract. We study the equivariant cohomology classes of torus-equivariant subvarieties of the space of matrices. For a large class of torus actions, we prove that the polynomials representing these classes (up to suitably changing signs) are covolume polynomials in the sense of Aluffi. We study the cohomology rings of complex varieties in terms of Macaulay inverse systems over $\mathbb{Z}$. As applications, we show that under certain conditions, the Macaulay dual generator is a denormalized Lorentzian polynomial in the sense of Brändén and Huh, and we give a characteristic-free extension (over $\mathbb{Z}$) of the result of Khovanskii and Pukhlikov describing the cohomology ring of toric varieties in terms of volume polynomials.

Keywords: log-concavity, equivariant cohomology, covolume polynomials, Lorentzian polynomials, Macaulay inverse systems, Schubert polynomials, toric varieties

1 Introduction

A sequence of real numbers $a_0, a_1, \ldots, a_n$ is called *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $1 \leq i \leq n-1$. Log-concave sequences naturally appear throughout algebra, combinatorics, and geometry (see the survey [31] for a thorough treatment). Recently, the theory of Lorentzian polynomials was introduced by Brändén and Huh [6] (and in [2, 3, 4] under the name of completely log-concave polynomials) and they have been instrumental in proving log-concavity results throughout mathematics [18, 19, 6].

The prototypical examples of Lorentzian polynomials are the volume polynomials of projective varieties [6]. Likewise, the covolume polynomials of Aluffi [1] are the prototypical examples of the dually Lorentzian polynomials of Ross, Süß, and Wannerer [28]. We introduce a new family of polynomials that specialize to a number of important polynomials in algebraic combinatorics, and we prove that they are covolume polynomials.

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For two permutations $u, w \in S_n$ with $w \geq u$ in Bruhat order, we define the *double Richardson polynomial* as the product of double Schubert polynomials $\mathfrak{R}_{w/u}(\mathbf{t}, \mathbf{s}) = \mathfrak{S}_u(\mathbf{t}, \mathbf{s})\mathfrak{S}_{w_0 w}(\mathbf{t}, \mathbf{s}')$, where $\mathbf{s}' = (s_n, \ldots, s_1)$ denotes the reverse of $\mathbf{s} = (s_1, \ldots, s_n)$. The double Richardson polynomial represents the torus equivariant class of matrix Richardson varieties in the cohomology ring $\mathrm{H}_T^\bullet(\mathrm{Mat}_{n,n})$ of the space of $n \times n$ matrices with the standard action of the torus $T = (\mathbb{C}^*)^n \times (\mathbb{C}^*)^n$.

Theorem 1.1 (Theorem 5.4)**.** *The (sign-changed) double Richardson polynomial $\mathfrak{R}_{w/u}(\mathbf{t}, -\mathbf{s})$ is a covolume polynomial.*

Dually Lorentzian polynomials enjoy two nice combinatorial properties: their supports are M-convex, and they are discretely log-concave. A homogenous polynomial $h = \sum_{\mathbf{n}} a_{\mathbf{n}} \mathbf{t}^{\mathbf{n}}$ of degree d with nonnegative coefficients is said to have M-convex support if $\mathrm{supp}(h)$ is the set of integer points of a generalized permutohedron in the sense of [26], and it is said to be discretely log-concave if $a_{\mathbf{n}}^2 \geq a_{\mathbf{n}+\mathbf{e}_i-\mathbf{e}_j} a_{\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j}$ for all $\mathbf{n}$ and all i, j.

Corollary 1.2 (Corollary 5.5)**.** *The following polynomials have* M*-convex support and are discretely log-concave: (sign-changed) Double Richardson polynomials $\mathfrak{R}_{w/u}(\mathbf{t}, -\mathbf{s})$, Richardson polynomials $\mathfrak{R}_{w/u}(\mathbf{t})$, (sign-changed) Double Schubert polynomials $\mathfrak{S}_u(\mathbf{t}, -\mathbf{s})$, and Schubert polynomials $\mathfrak{S}_u(\mathbf{t})$.*

We note that the M-convexity and discrete log-concavity in Corollary 1.2 recovers a result of [10] and [19]. To the best of our knowledge, the discrete log-concavity of double Schubert polynomials is new.

The proof of Theorem 1.1 follows from a general theorem that we discuss in Section 4. More precisely, we prove the following result regarding the equivariant cohomology classes of torus-equivariant subvarieties of the space of matrices.

Theorem 1.3 (Corollary 4.5)**.** *Let $\mathrm{Mat}_{m,n} = \mathbb{C}^{m \times n}$ be the space of $m \times n$ matrices with complex entries and consider the natural action of the torus $T = (\mathbb{C}^*)^m \times (\mathbb{C}^*)^n$ given by $(g, h) \cdot M = g \cdot M \cdot h^{-1}$ for all $M \in \mathrm{Mat}_{m,n}$ and $(g, h) \in T$. Let $X \subset \mathrm{Mat}_{m,n}$ be an irreducible T-variety and $C_X(t_1, \ldots, t_m, s_1, \ldots, s_n)$ be the polynomial representing the class $[X]^T$ in $\mathrm{H}_T^\bullet(\mathrm{Mat}_{m,n}) = \mathbb{Z}[t_1, \ldots, t_m, s_1, \ldots, s_n]$. Then $C_X(t_1, \ldots, t_m, -s_1, \ldots, -s_n)$ is a covolume polynomial.*

In Section 6, we also connect the theory of Macaulay inverse systems over $\mathbb{Z}$ to the theory of Lorentzian polynomials. As an application, we obtain a characteristic-free extension (over $\mathbb{Z}$) of the result of Khovanskii and Pukhlikov describing the cohomology ring of toric varieties in terms of volume polynomials. These results are presented in Theorem 6.4 and Corollary 6.6.

2 Preliminaries

In this section, we set up notation and preliminaries used throughout the paper.

2.1 Multidegrees

Here we recall the definition and basic properties of multidegrees as presented in [24, Chapter 8].

Let $R = \Bbbk[x_1, \ldots, x_n]$ be a $\mathbb{Z}^p$-graded polynomial ring over a field $\Bbbk$. Suppose M is a finitely generated $\mathbb{Z}^p$-graded R-module. Let M be a finitely generated $\mathbb{Z}^p$-graded module and $F_\bullet$ be a $\mathbb{Z}^p$-graded free R-resolution $F_\bullet : \cdots \to F_i \to F_{i-1} \to \cdots \to F_1 \to F_0$ of M. Let $t_1, \ldots, t_p$ be variables over $\mathbb{Z}$ and consider the Laurent polynomial ring $\mathbb{Z}[\mathbf{t}] = \mathbb{Z}[t_1, \ldots, t_p, t_1^{-1}, \ldots, t_p^{-1}]$, where the variable t_i corresponds with the i-th elementary vector $\mathbf{e}_i \in \mathbb{Z}^p$. If we write $F_i = \bigoplus_j R(-\mathbf{b}_{i,j})$ with $\mathbf{b}_{i,j} = (\mathbf{b}_{i,j,1}, \ldots, \mathbf{b}_{i,j,p}) \in \mathbb{Z}^p$, then we define the Laurent polynomial $[F_i]_{\mathbf{t}} := \sum_j \mathbf{t}^{\mathbf{b}_{i,j}} = \sum_j t_1^{\mathbf{b}_{i,j,1}} \cdots t_p^{\mathbf{b}_{i,j,p}}$.

Definition 2.1. *The* K-polynomial *of M is given by* $\mathcal{K}(M; \mathbf{t}) := \sum_i (-1)^i [F_i]_{\mathbf{t}}$.

We have that, even if the grading of R is non-positive and we do not have a well-defined notion of Hilbert series, the above definition of K-polynomial is an invariant of the module M and it does not depend on the chosen free R-resolution $F_\bullet$ (see [24, Theorem 8.34]).

Definition 2.2. *The* multidegree polynomial *of M is the homogeneous polynomial $\mathcal{C}(M; \mathbf{t}) \in \mathbb{Z}[\mathbf{t}]$ given as the sum of all terms in $\mathcal{K}(M; \mathbf{1} - \mathbf{t}) = \mathcal{K}(M; 1 - t_1, \ldots, 1 - t_p)$ having total degree* $\operatorname{codim}(M)$, *which is the lowest degree appearing.*

2.2 Lorentzian and covolume polynomials

A subset $J \subset \mathbb{N}^p$ is called *M-convex* if for any $\mathbf{q} = (q_1, \ldots, q_p)$ and $\mathbf{r} = (r_1, \ldots, r_p)$ in J, and any i where $q_i < r_i$, there exists j such that $q_j > r_j$ and both points $\mathbf{q} + \mathbf{e}_i - \mathbf{e}_j$ and $\mathbf{r} - \mathbf{e}_i + \mathbf{e}_j$ are also contained in J. M-convex sets are equivalent to sets of bases of *discrete polymatroids* [25] and to sets of integer points of *generalized permutohedra* [26].

Let $h(t_1, \ldots, t_p)$ be a homogeneous polynomial of degree d in $\mathbb{R}[\mathbf{t}] = \mathbb{R}[t_1, \ldots, t_p]$.

Definition 2.3 ([6]). *The homogeneous polynomial h is called* Lorentzian *if the following conditions hold:*

(i) *The coefficients of h are nonnegative.*

(ii) *The support of h is M-convex.*

(iii) $\frac{\partial}{\partial t_{i_1}} \frac{\partial}{\partial t_{i_2}} \cdots \frac{\partial}{\partial t_{i_e}} h$ *has at most one positive eigenvalue for any $i_1, \ldots, i_e \in [p]$ where $e = d - 2$.*

The normalization operator $N\left(\sum_{\mathbf{n}} a_{\mathbf{n}} \mathbf{t}^{\mathbf{n}}\right) := \sum_{\mathbf{n}} \frac{a_{\mathbf{n}}}{\mathbf{n}!} \mathbf{t}^{\mathbf{n}}$ where $\mathbf{n}! := n_1! \cdots n_p!$ preserves the Lorentzian property [6, Corollary 3.7].

In [1], Aluffi defined the notion of covolume polynomials. These polynomials arise by considering the Chow class of irreducible subvarieties of a product of projective spaces.

Definition 2.4. *Let $\mathbb{P} = \mathbb{P}^{m_1}_{\Bbbk} \times_{\Bbbk} \cdots \times_{\Bbbk} \mathbb{P}^{m_p}_{\Bbbk}$ be multiprojective space over a field $\Bbbk$. Let $X \subset \mathbb{P}$ be an irreducible subvariety of codimension c. The class of X can be written as $\sum_{|\mathbf{n}|=c} a_{\mathbf{n}} H_1^{n_1} \cdots H_p^{n_p} \in A^\bullet(\mathbb{P})$. Let $P_{[X]}(t_1, \ldots, t_p) = \sum_{|\mathbf{n}|=c} a_{\mathbf{n}} \mathbf{t}^{\mathbf{n}}$ be the polynomial associated to the class $[X]$ of $X \subset \mathbb{P}$. A polynomial $P(t_1, \ldots, t_p) \in \mathbb{R}[t_1, \ldots, t_p]$ with nonnegative real coefficients is a* covolume polynomial *if it is a limit of polynomials of the form $cP_{[X]}$ for a positive real number c and a subvariety X of $\mathbb{P}$.*

Finally, we are interested in the family of dually Lorentzian polynomials introduced by Ross, Süß, and Wannerer [28].

Definition 2.5 ([28])**.** *A polynomial $h \in \mathbb{R}[t_1, \ldots, t_p]$ is* dually Lorentzian *if the polynomial $N\left(t_1^{m_1} \cdots t_p^{m_p} h\left(t_1^{-1}, \ldots, t_p^{-1}\right)\right)$ is Lorentzian for some $\mathbf{m} = (m_1, \ldots, m_p) \in \mathbb{N}^p$.*

As shown by Aluffi [1, Proposition 2.8], covolume polynomials form a subfamily of the family of dually Lorentzian polynomials.

3 Multidegree polynomials of prime ideals in arbitrary positive gradings

In this section, working over an arbitrary positive $\mathbb{N}^p$-grading, we show that the multidegree polynomial of a prime ideal is a covolume polynomial. Our main tool is the technique of *standardization* that was used in [10, 9]. Below we discuss the case of a standard multigrading. This case is of special importance since it deals with closed subschemes of a product of projective spaces.

Remark 3.1 (Standard multigradings)**.** *Assume that R has a standard $\mathbb{N}^p$-grading and that R is the coordinate ring of $\mathbb{P} = \mathbb{P}^{m_1}_{\Bbbk} \times_{\Bbbk} \cdots \times_{\Bbbk} \mathbb{P}^{m_p}_{\Bbbk}$. Let $X \subset \mathbb{P}$ be a d-dimensional integral closed subscheme with coordinate ring R/P, where P is an R-homogeneous prime ideal. The class of X in the Chow ring of $\mathbb{P}$ is given by $[X] = \sum_{|\mathbf{n}|=d} \deg^{\mathbf{n}}_{\mathbb{P}}(X) \cdot H_1^{m_1-n_1} \cdots H_p^{m_p-n_p} \in A^*(\mathbb{P})$ where H_i represents the class of the inverse image of a hyperplane of $\mathbb{P}^{m_i}_{\Bbbk}$. We say that $\deg^{\mathbf{n}}_{\mathbb{P}}(X)$ is the multidegree of X of type $\mathbf{n}$. Then the multidegree polynomial of R/P is given by*

$$\mathcal{C}(R/P; \mathbf{t}) \;=\; \sum_{\mathbf{n} \in \mathbb{N}^p,\, |\mathbf{n}|=d} \deg^{\mathbf{n}}_{\mathbb{P}}(X) \cdot t_1^{m_1-n_1} \cdots t_p^{m_p-n_p}.$$

The volume polynomial *of X (see [6, Section 4.2]) is given by*

$$\mathrm{vol}_X(\mathbf{t}) \;=\; \int \left(H_1 t_1 + \cdots + H_p t_p\right)^d \cap [X] \;=\; \sum_{\mathbf{n} \in \mathbb{N}^p,\, |\mathbf{n}|=d} \deg^{\mathbf{n}}_{\mathbb{P}}(X) \cdot \frac{d!}{n_1! \cdots n_p!} \cdot t_1^{n_1} \cdots t_p^{n_p}.$$

By [6, Theorem 4.6], we have that $\mathrm{vol}_X(\mathbf{t})$ is a Lorentzian polynomial.

The following lemma tells us that multidegree polynomials in a standard multigrading are *dually Lorentzian*. This result first appeared in [1, Proposition 2.8] (also, see [19, Theorem 6]).

Lemma 3.2. *Keep the same notations and assumptions of Remark 3.1. Consider the polynomial*

$$F(t_1,\ldots,t_p) \;=\; t_1^{m_1}\cdots t_p^{m_p}\cdot \mathcal{C}\Big(R/P;\frac{1}{t_1},\ldots,\frac{1}{t_p}\Big).$$

Then the normalization $N(F)$ is a Lorentzian polynomial (i.e., $\mathcal{C}(R/P;\mathbf{t})$ is dually Lorentzian).

We now describe some basic properties of the process of standardization as developed in [10, 9]. For the rest of the section, the following setup is fixed.

Setup 3.3. *For $1 \le i \le n$, let $\ell_i = |\deg(x_i)|$ be the total degree of the variable x_i. Let $S = \mathbb{k}\left[y_{i,j} \mid 1 \le i \le n \text{ and } 1 \le j \le \ell_i\right]$ be a standard $\mathbb{N}^p$-graded polynomial ring such that $\deg(x_i) = \sum_{j=1}^{\ell_i}\deg(y_{i,j})$ for all $1 \le i \le n$. We define the $\mathbb{N}^p$-graded $\mathbb{k}$-algebra homomorphism*

$$\phi : R = \mathbb{k}[\mathbf{x}] \longrightarrow S = \mathbb{k}[\mathbf{y}], \quad \phi(x_i) = y_{i,1}y_{i,2}\cdots y_{i,\ell_i}.$$

For an R-homogeneous ideal $I \subset R$, we say that the extension $J = \phi(I)S$ is the standardization *of I, since $J \subset S$ is an S-homogeneous ideal in the standard $\mathbb{N}^p$-graded polynomial ring S. By a slight abuse of notation, we consider both multidegree polynomials $\mathcal{C}(R/I;\mathbf{t})$ and $\mathcal{C}(S/J;\mathbf{t})$ as elements of the same polynomial ring $\mathbb{Z}[\mathbf{t}] = \mathbb{Z}[t_1,\ldots,t_p]$.*

Finally, we are ready to present the main result of this section. It yields a large new family of dually Lorentzian polynomials.

Theorem 3.4. *Let $P \subset R$ be a prime R-homogeneous ideal. Then $\mathcal{C}(R/P;\mathbf{t})$ is a covolume polynomial.*

Proof. We sketch (due to space limitations) the idea of the proof. By [9, Theorem 7.2], the multidegree polynomals are preserved under standardization. Therefore, we have $\mathcal{C}(R/P;\mathbf{t}) = \mathcal{C}(S/J;\mathbf{t})$ where J is the standardization of P. Thus, the covolume property now follows from Lemma 3.2. □

4 Equivariant cohomology in multigraded commutative algebra

In this section, we study the equivariant cohomology of the irreducible varieties that appear in multigraded commutative algebra. Here we show that equivariant classes of multigraded varieties tend to yield covolume polynomials (up to changing the sign of negative coefficients). We follow the references [5] and [11, Chapter 5] for the basics of equivariant cohomology and equivariant K-theory.

We consider the T-equivariant cohomology ring $\mathrm{H}_T^\bullet(\mathbb{A}_\mathbb{C}^n) := \mathrm{H}^\bullet\big(\mathbb{E}T \times^T \mathbb{A}_\mathbb{C}^n\big)$ where $\mathbb{E}T$ is contractible with T acting freely (on the right). Then $\mathbb{B}T := \mathbb{E}T/T$ is a classifying space for T. Since we can take $\mathbb{E}T = (\mathbb{C}^\infty \setminus \{0\})^n$ and $\mathbb{B}T = \big(\mathbb{P}_\mathbb{C}^\infty\big)^n$, it follows that $\mathrm{H}_T^\bullet(\mathbb{A}_\mathbb{C}^n) \cong \Lambda_T := \mathrm{H}_T^\bullet(\mathrm{pt}) = \mathrm{H}^\bullet(\mathbb{B}T) = \mathbb{Z}[t_1, \ldots, t_p]$. Given a T-subvariety $X \subset \mathbb{A}_\mathbb{C}^n$, we denote by $[X]^T := [\mathbb{E}T \times^T X]$ the equivariant class of X in $\mathrm{H}_T^\bullet(\mathbb{A}_\mathbb{C}^n) \cong \mathbb{Z}[t_1, \ldots, t_p]$.

The next remark provides the connection to apply our results in Section 3 to equivariant cohomology.

Remark 4.1. *Let $X \subset \mathbb{A}_\mathbb{C}^n$ be a T-subvariety with coordinate ring R/I. Let $C_X(t_1, \ldots, t_p)$ be the polynomial representing the classes $[X]^T \in \mathrm{H}_T^\bullet(\mathbb{A}_\mathbb{C}^n)$. Then we have the equality*

$$C_X(t_1, \ldots, t_p) = \mathcal{C}(R/I; t_1, \ldots, t_p).$$

Then Theorem 3.4 poses the question of whether the class $[X]^T$ of an irreducible T-subvariety $X \subset \mathbb{A}_\mathbb{C}^n$ will always yield a covolume polynomial. This is not true for arbitrary torus actions (in particular, a simple example produces a polynomial that does not have M-convex support). For the rest of this section, we shall use the following setup that avoids the complications of that example.

Setup 4.2. *Assume that the torus T is given as $T = (\mathbb{C}^*)^q \times (\mathbb{C}^*)^{p-q}$ where "$(\mathbb{C}^*)^q$ comes with positive weights and $(\mathbb{C}^*)^{p-q}$ comes with negative weights". More precisely, we require that $\mathbf{d}_i \in \mathbb{N}^p \setminus \{\mathbf{0}\}$ for all $1 \leq i \leq q$, and $-\mathbf{d}_i \in \mathbb{N}^p \setminus \{\mathbf{0}\}$ for all $q+1 \leq i \leq p$. In this case we say that the action of T determines a* twisted positive grading *on R.*

The following lemma tells us that we can "flip" twisted positive gradings to positive gradings. A version of this lemma appeared in [10, Lemma 3.3]. Let $\widetilde{R} = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring with the same variables as R but with grading given by setting $\deg(x_i) = \mathbf{d}_i$ for $1 \leq i \leq q$ and $\deg(x_i) = -\mathbf{d}_i$ for $q+1 \leq i \leq p$. Notice that $\widetilde{R}$ has a positive $\mathbb{N}^p$-grading.

Lemma 4.3. *Assume Setup 4.2. Let $I \subset R$ be a $\mathbb{Z}^p$-graded ideal, and denote also by I the corresponding $\mathbb{N}^p$-graded ideal in $\widetilde{R}$. Then we have*

$$\mathcal{C}(\widetilde{R}/I; t_1, \ldots, t_q, t_{q+1}, \ldots, t_p) = \mathcal{C}(R/I; t_1, \ldots, t_q, -t_{q+1}, \ldots, -t_p).$$

We are now ready to present the main result of this section.

Theorem 4.4. *Assume Setup 4.2. Let $X \subset \mathbb{A}_\mathbb{C}^n$ be an irreducible T-subvariety, $C_X(t_1, \ldots, t_p)$ be the polynomial representing the class $[X]^T \in \mathrm{H}_T^\bullet(\mathbb{A}_\mathbb{C}^n)$. Then $C_X(t_1, \ldots, t_q, -t_{q+1}, \ldots, -t_p)$ is a covolume polynomial.*

Proof. The polynomial C_X representing the class $[X]^T$ is the same as the multidegree polynomial $\mathcal{C}_X$ by [23, Proposition 1.19] and the statement follows from Theorem 3.4. □

Our main application of the above theorem is the following corollary.

Corollary 4.5. *Let* $\mathrm{Mat}_{m,n} = \mathbb{C}^{m\times n}$ *be the space of* $m \times n$ *matrices with complex entries and consider the natural action of the torus* $T = (\mathbb{C}^*)^m \times (\mathbb{C}^*)^n$ *given by* $(g,h)\cdot M = g \cdot M \cdot h^{-1}$ *for all* $M \in \mathrm{Mat}_{m,n}$ *and* $(g,h) \in T$. *Let* $X \subset \mathrm{Mat}_{m,n}$ *be an irreducible* T*-variety and* $C_X(t_1,\ldots,t_m,s_1,\ldots,s_n)$ *be the polynomial representing the class* $[X]^T$ *in* $\mathrm{H}^\bullet_T(\mathrm{Mat}_{m,n}) = \mathbb{Z}[t_1,\ldots,t_m,s_1,\ldots,s_n]$. *Then we have that* $C_X(t_1,\ldots,t_m,-s_1,\ldots,-s_n)$ *is a covolume polynomial.*

5 Equivariant cohomology of matrix Richardson varieties

In this section, we study the equivariant cohomology of matrix Richardson varieties for a pair of permutations. An interesting outcome of our approach is the definition of a new family of polynomials that we call *double Richardson polynomials*. These polynomials specialize to many polynomials of interest.

Let D_w be the matrix Schubert variety of w given by the rank conditions imposed on the upper-left corner and D^w be the opposite matrix Schubert variety of w given by the rank conditions imposed on the upper-right corner. A result of fundamental importance for us is the following geometric interpretation of double Schubert polynomials.

Theorem 5.1 **([5, 22, 17]).** *The equivariant class* $[D_w]^T$ *of the matrix Schubert variety* D_w *in the cohomology ring* $\mathrm{H}^\bullet_T(\mathrm{Mat}_{n,n}) \cong \mathbb{Z}[t_1,\ldots,t_n,s_1,\ldots,s_n]$ *is given by the double Schubert polynomial* $\mathfrak{S}_w(\mathbf{t},\mathbf{s})$.

The next lemma expresses the equivariant class of the degeneracy locus D^w.

Lemma 5.2. *The equivariant class* $[D^w]^T \in \mathrm{H}^\bullet_T(\mathrm{Mat}_{n,n})$ *of* D^w *is given by the double Schubert polynomial*

$$\mathfrak{S}_{w_0w}(t_1,\ldots,t_n,s_n,\ldots,s_1)$$

of w_0w *with a reverse ordering of the variables* $s_1,\ldots,s_n$.

We are now ready to introduce the two objects that interest us in this section.

Definition 5.3. *For a pair of permutations* (w,u) *in* S_n*, we have*

(i) $D_u^w := D_u \cap D^w$ *is the* matrix Richardson variety.

(ii) $\mathfrak{R}_{w/u}(\mathbf{t},\mathbf{s}) := \mathfrak{S}_u(\mathbf{t},\mathbf{s})\mathfrak{S}_{w_0w}(\mathbf{t},\mathbf{s}')$ *is the* double Richardson polynomial, *where* $\mathbf{s}' = (s_n,\ldots,s_1)$ *denotes the reverse of* $\mathbf{s} = (s_1,\ldots,s_n)$.

We point out that the matrix Richardson variety D_u^w is a reduced and irreducible T-subvariety of $\mathrm{Mat}_{n\times n}$. We have that D_u^w is nonempty if and only if $w \geq u$ in the Bruhat order, and when it is nonempty, it has dimension $\dim(D_u^w) = \ell(w) - \ell(u)$. For details on Richardson varieties, the reader is referred to [29, 27, 7].

Our main result in this section is the following theorem.

Theorem 5.4. *For two permutations $u, w \in S_n$ with $w \geq u$ in Bruhat order, the following statements hold:*

(i) *The double Richardson polynomial $\mathfrak{R}_{w/u}(\mathbf{t}, \mathbf{s})$ presents the equivariant class of the matrix Richardson variety D_u^w in $\mathrm{H}_T^\bullet(\mathrm{Mat}_{n\times n})$.*

(ii) *The (sign-changed) double Richardson polynomial $\mathfrak{R}_{w/u}(\mathbf{t}, -\mathbf{s})$ is a covolume polynomial.*

We have specializations to many polynomials of interest:

(i) $\mathfrak{R}_{w_0/u}(\mathbf{t}, \mathbf{s}) = \mathfrak{S}_u(\mathbf{t}, \mathbf{s})$ is the double Schubert polynomial.

(ii) $\mathfrak{R}_{w_0/u}(\mathbf{t}, \mathbf{0}) = \mathfrak{S}_u(\mathbf{t}, \mathbf{0}) = \mathfrak{S}_u(\mathbf{t})$ is the ordinary Schubert polynomial.

(iii) We say that $\mathfrak{R}_{w/u}(\mathbf{t}) = \mathfrak{R}_{w/u}(\mathbf{t}, \mathbf{0})$ is the (ordinary) *Richardson polynomial*.

We can now obtain some consequences for certain polynomials of interest in algebraic combinatorics.

Corollary 5.5. *The polynomials mentioned above have* M*-convex support and are discretely log-concave.*

6 Macaulay dual generators of cohomology rings

In this section, we study the Macaulay dual generators of cohomology rings of smooth complex varieties. We prove that under suitable positivity assumptions the Macaulay dual generators are Lorentzian polynomials. Before presenting our results on cohomology rings, which is our main interest, we develop basic ideas regarding the notion of Gorenstein algebras over a base ring (these developments are probably known to the experts but we could not find a suitable reference for us). For details, the reader referred to [16, Chapter 21], [8, Chapter 3]. Throughout this section we use the following setup.

Setup 6.1. *Let A be a Noetherian ring and R be a positively graded finitely generated algebra over $R_0 = A$. Choose a positively graded polynomial ring $S = A[x_1, \ldots, x_n]$ with a graded surjection $S \twoheadrightarrow R$ and write $R \cong S/I$ for some homogeneous ideal $I \subset S$. Let $\mathfrak{m} := (x_1, \ldots, x_n) \subset S$ be the graded irrelevant ideal. Let $\delta_i := \deg(x_i) > 0$ and $\delta := \delta_1 + \cdots + \delta_n$. For any $\mathfrak{p} \in \mathrm{Spec}(A)$, we denote by $\kappa(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ the residue field at $\mathfrak{p}$ and we set $S(\mathfrak{p}) := S \otimes_A \kappa(\mathfrak{p}) \cong \kappa(\mathfrak{p})[x_1, \ldots, x_n]$ and $R(\mathfrak{p}) := R \otimes_A \kappa(\mathfrak{p})$.*

We are interested in the following relative notion of Gorenstein.

Definition 6.2. *We say that R is a* Gorenstein algebra over *A if the natural morphism $f : \mathrm{Spec}(R) \to \mathrm{Spec}(A)$ is a Gorenstein morphism (see [30, Tag 0C02]). This means that R is A-flat and the fiber $R(\mathfrak{p}) = R \otimes_A \kappa(\mathfrak{p})$ is a Gorenstein ring for all $\mathfrak{p} \in \mathrm{Spec}(A)$. We also say that R is* Artinian Gorenstein over *A if $f : \mathrm{Spec}(R) \to \mathrm{Spec}(A)$ is a finite Gorenstein morphism.*

For a graded S-module M, we denote the *B-relative graded Matlis dual* by

$$M^{\vee_A} \;=\; {}^*\mathrm{Hom}_A(M, A) \;:=\; \bigoplus_{\nu \in \mathbb{Z}} \mathrm{Hom}_A\left([M]_{-\nu}, A\right).$$

The next proposition is similar to [12, Lemma 2.10].

Proposition 6.3. *Assume that $f : \mathrm{Spec}(R) \to \mathrm{Spec}(A)$ is flat. Let e be the common dimension of the fibers of f. Then R is Gorenstein over A if and only if the following two conditions are satisfied:*

(i) $\mathrm{Ext}_S^i(R, S) = 0$ *for all* $0 \le i \le n$ *such that* $i \neq n - e$.

(ii) $\mathrm{Ext}_S^{n-e}(R, S)$ *is A-flat and it is generated locally by one element as an R-module.*

Motivated by the above proposition, when R is Gorenstein over A and e is the common dimension of the fibers, we say that

$$\omega_{R/A} \;:=\; \mathrm{Ext}_S^{n-e}\big(R, S(-\delta)\big)$$

is a relative canonical module of R over A.

Assume that R is a finitely generated A-module. Let $d > 0$ be a positive integer and $\rho : R \to A(-d)$ be a graded A-linear map (following tradition we call this map an *orientation*). We say that R *satisfies Poincaré duality with respect to the orientation* ρ if

$$R \otimes_A R \;\to\; A(-d), \quad r_1 \otimes r_2 \mapsto \rho(r_1 r_2)$$

is a perfect pairing. This means that the pairing induces a graded isomorphism $R \cong {}^*\mathrm{Hom}_A(R, A)(-d)$ of R-modules. In particular, we have perfect pairings $R_i \otimes_A R_{d-i} \to A$ for all i. We consider the inverse polynomial ring $T = A[y_1, \ldots, y_n]$ where y_i is identified with x_i^{-1}. The S-module structure of T is given by setting that $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \cdot y_1^{\beta_1} \cdots y_n^{\beta_n}$ is equal to $y_1^{\beta_1 - \alpha_1} \cdots y_n^{\beta_n - \alpha_n}$ if $\beta_i \ge \alpha_i$ for all $1 \le i \le n$ and to 0 otherwise. Then T is a negatively graded polynomial ring with $\deg(y_i) = -\delta_i$. We have the natural isomorphisms $T \cong {}^*\mathrm{Hom}_A(S, A) \cong \mathrm{H}_{\mathfrak{m}}^n(S)(-\delta)$.

The following theorem extends well-known results about Artinian Gorenstein algebras (over a field) to our current relative setting over a Noetherian base ring. In particular, we focus on cohomology rings over $\mathbb{Z}$.

Theorem 6.4. *Let X be a d-dimensional smooth complex algebraic variety. Suppose that the cohomology ring $R = \bigoplus_{i=0}^{d} \mathrm{H}^{2i}(X, \mathbb{Z})$ is a flat $\mathbb{Z}$-algebra (i.e., it is $\mathbb{Z}$-torsion-free). Let $\rho : R_d = \mathrm{H}^{2d}(X, \mathbb{Z}) \to \mathbb{Z}$ be the natural degree map. Choose a graded presentation $R \cong S/I$ where $S = \mathbb{Z}[x_1, \ldots, x_n]$, $\delta_i = \deg(x_i) > 0$, and I is homogeneous ideal. Let $\delta = \delta_1 + \cdots + \delta_n$. Then the following statements hold:*

(i) *R is Gorenstein over $\mathbb{Z}$.*

(ii) *We have the isomorphisms $\omega_{R/\mathbb{Z}} = \mathrm{Ext}^d_S(R, S(-\delta)) \cong {}^*\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Z}) \cong R(d)$.*

(iii) *Consider the inverse polynomial ring $T = \mathbb{Z}[y_1, \ldots, y_n]$ with the identification $y_i = x_i^{-1}$. The ideal $I \subset S$ is given as the annihilator $I = \{g \in S \mid g \cdot G_R = 0\}$ of the inverse polynomial $G_R(y_1, \ldots, y_n) = \sum_{i_1\delta_1+\cdots+i_n\delta_n=d} \rho\big(x_1^{i_1} \cdots x_n^{i_n}\big) y_1^{i_1} \cdots y_n^{i_n} \in T$.*

(iv) *Assume that X is complete and that each x_i is equal to the first Chern class $c_1(L_i)$ of a nef line bundle L_i on X. Then the normalization $N(G_R) \in \mathbb{R}[y_1, \ldots, y_n]$ of G_R is a Lorentzian polynomial.*

Remark 6.5. *Following standard notation, we say that the polynomial $G_R \in T = A[y_1, \ldots, y_n]$ presented in Theorem 6.4 is the* Macaulay dual generator of R over A.

We close this subsection with one application in toric geometry. Let $P_1, \ldots, P_n$ be lattice polytopes in $\mathbb{Z}^d$ and $\mathrm{vol}(P_i)$ denote the Euclidean volume of P_i where the unit hypercube has volume 1. By [13, Proposition 7.4.9], the volume $\mathrm{vol}(y_1P_1 + \cdots + y_nP_n)$ of the Minkowski sum of polytopes is a homogeneous polynomial of degree d, and we write

$$\mathrm{vol}(y_1P_1 + \cdots + y_nP_n) = \sum_{\alpha} \frac{1}{\alpha!} \mathrm{MV}_\alpha(P_1, \ldots, P_n)\, y_1^{\alpha_1} \cdots y_n^{\alpha_n}$$

where $\mathrm{MV}_\alpha(P_1, \ldots, P_n)$ is called the *mixed volume* of $(P_1, \ldots, P_n)$ of type α.

Now, we provide a characteristic-free (over $\mathbb{Z}$) extension of the celebrated result of Khovanskii and Pukhlikov [21] showing that the cohomology ring (over $\mathbb{Q}$) of certain toric varieties can be expressed in terms of differential operators that annihilate the volume polynomial. We follow the notations in [14]. Let X_Σ be a smooth complete toric variety and $\rho_1, \ldots, \rho_n$ be all one-dimensional rays in Σ. We have the following isomorphism $\mathrm{H}^\bullet(X_\Sigma, \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n]/(\mathcal{I} + \mathcal{J})$ with $D_{\rho_i} \mapsto x_i$, where $\mathcal{I} = \big(x_{i_1}, \ldots, x_{i_s} \mid i_j$ are distinct and $\rho_{i_1} + \cdots + \rho_{i_s}$ is not a cone of $\Sigma\big)$ and $\mathcal{J}$ is the ideal generated by linear forms $\sum_{i=1}^n \langle m, u_i \rangle x_i$ for all $m \in M$. This interpretation is proved in [20] and [15]. Write $d = \dim X_\Sigma$ and $n = |\Sigma(1)|$. The volume polynomial of a toric variety is defined as $V(y_1, \ldots, y_n) := \int_{X_\Sigma} (\sum_{i=1}^n y_iD_i)^d$ where $D_i = D_{\rho_i}$.

Corollary 6.6. *There is an isomorphism of $\mathbb{Z}$-algebras $\mathrm{H}^\bullet(X_\Sigma, \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n]/I$, where $I = \{g \in \mathbb{Z}[x_1, \ldots, x_n] \mid g \cdot N^{-1}(V(y_1, \ldots, y_n)) = 0\} = \mathcal{I} + \mathcal{J}$ with the identification $x_i = y_i^{-1}$ as described in Theorem 6.4. If each D_i is nef and P_i is the corresponding polytope, then*

$$G_R(y_1, \ldots, y_n) \;=\; \sum_{\alpha_1+\cdots+\alpha_n=d} \mathrm{MV}_\alpha(P_1, \ldots, P_n)\, y_1^{\alpha_1} \cdots y_n^{\alpha_n}.$$

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