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Twists, r-dimer covers, and web duality

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Abstract. We study a twisted version of Fraser–Lam–Le's higher boundary measurement map using face weights instead of edge weights, thereby providing Laurent monomial expansions, in Plücker coordinates, for twisted web immanants for Grassmannians. Combined with our computation that web immanants for $\mathrm{Gr}(3, 12)$ and $\mathrm{Gr}(4, 12)$ correspond to webs indexed by transposed standard Young tableaux, we recover and extend formulas of Elkin–Musiker–Wright for twists of certain cluster variables.

Keywords: cluster algebras, Grassmannians, webs, dimer covers, tableaux, twist map

1 Introduction

Our work will compare the coordinate ring of the Grassmannian, denoted $\mathbb{C}[\widehat{\mathrm{Gr}}(k, n)]$, and the space of *tensor invariants*, denoted $\mathcal{W}(\mathbb{C}^r)$. Here, $\mathrm{Gr}(k, n)$ is the space of k-dimensional linear subspaces of $\mathbb{R}^n$, and $\widehat{\mathrm{Gr}}(k, n)$ denotes the affine cone over $\mathrm{Gr}(k, n)$ using the Plücker embedding. $\mathbb{C}[\widehat{\mathrm{Gr}}(k, n)]$ is generated by the *Plücker coordinates*, defined as $k \times k$ minors of a full rank $k \times n$ matrix, up to *Plücker relations*. We denote by Δ_J the Plücker coordinate corresponding to the determinant of the k columns indexed by subset $J \subset [n]$. This description gives a natural $\mathbb{N}^n$-grading on $\mathbb{C}[\widehat{\mathrm{Gr}}(k, n)]$, where the piece associated to $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$ is generated by products of Plücker coordinates with column i represented λ_i times.

Let $\mathcal{W}_\lambda(\mathbb{C}^r) = \mathrm{Hom}_{\mathrm{SL}_r}\left(\bigotimes_{i=1}^n \bigwedge^{\lambda_i} \mathbb{C}^r, \mathbb{C}\right)$ be the space of tensor invariants of multidegree λ. This space is spanned by invariants coming from SL_r *webs*, certain planar graphs

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embedded in a disk. For $r \leq 4$, bases of *web invariants* are known due to Kuperberg [10] ($r = 2, 3$) and Gaetz, Pechenik, Pfannerer, Striker, and Swanson [8] ($r = 4$).

Postnikov defined a *boundary measurement map* linking Plücker coordinates to *dimer covers* (also called *almost perfect matchings*) on *plabic graphs* [16]. This map associates to a *network* N (a plabic graph G along with a choice of edge weights in $\mathbb{C}^\times$) a point $\widehat{X}(N)$ in $\widehat{\mathrm{Gr}}(k,n)$. Lam [11], and later Fraser–Lam–Le [7], extended the boundary measurement map using r-dimer covers[1] on plabic graphs. Each r-dimer cover of G with boundary condition $\lambda \in \mathbb{N}^n$ gives rise to an SL_r web. The r-fold boundary measurement $\mathrm{Web}_r(N;\lambda)$ is a linear combination of the web invariants of all r-dimer covers of G, with coefficients coming from the edge weights of N; see Section 2.

Work of Scott [17] provides a cluster algebra structure for $\mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]$, which is given explicitly in terms of plabic graphs. A plabic graph G determines an initial seed of this cluster algebra, consisting of one Plücker coordinate for each face of G. By [12, 13], the boundary measurement map can be used to write the images of Plücker coordinates under an important automorphism, the *twist map* τ of [1], as a Laurent polynomial in the initial seed. Theorem 3.4 in [4] explains that by switching from using edge weights to *face weights*, using the face labeling of a plabic graph, we can now give Laurent expansions of twists of Plücker coordinates. In our Theorem 2.1, we give a wide generalization of this result.

Inspired by the interaction between the twist automorphism and the face weights, we define a twisted version $\mathrm{Web}_r^\tau(N;\lambda)$ of Fraser–Lam–Le's higher boundary measurement map $\mathrm{Web}_r(N;\lambda)$; see (2.3). As seen in Section 2, a key feature of $\mathrm{Web}_r(N;\lambda)$ is that it can be used to define an isomorphism between the dual space $\mathcal{W}_\lambda(\mathbb{C}^r)^*$ and $\mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]_\lambda$ when $\sum_i \lambda_i = kr$, thereby providing a bilinear pairing $\langle \cdot, \cdot \rangle : \mathcal{W}_\lambda(\mathbb{C}^r) \otimes \mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]_\lambda \to \mathbb{C}$. We call this the *FLL pairing*. In Corollary 2.2, by studying pairings of the form $\langle \mathrm{Web}_r^\tau(N;\lambda), f \rangle$ for $f \in \mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]_\lambda$, we show that $\mathrm{Web}_r^\tau(N;\lambda)$ functions as a twisted higher boundary measurement map.

In Section 3, we turn our attention to the cases when $n = kr$ and $\lambda = (1, \ldots, 1) \in \mathbb{N}^n$. In this setting, we have $\mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]_\lambda \cong \mathcal{W}_\lambda(\mathbb{C}^k)$, so FLL duality provides a pairing between $\mathcal{W}_\lambda(\mathbb{C}^r)$ and $\mathcal{W}_\lambda(\mathbb{C}^k)$. For $k \leq 4$, it is natural to compute the *web immanants* dual under $\langle \cdot, \cdot \rangle$ to the known SL_k web bases of [10] and [8]. This was done by Fraser [6] for $k = 2$ and by Fraser–Lam–Le for $(k,n) = (3,6)$ and $(3,9)$. Remarkably, in each of these cases, the web immanants are themselves SL_r web invariants. In Theorem 3.1, we extend these results by computing the dual basis to Kuperberg's SL_3 web basis when $(k,n) = (3,12)$. Basis webs for $\mathcal{W}_\lambda(\mathbb{C}^3)$ (resp. $\mathcal{W}_\lambda(\mathbb{C}^4)$) are in bijection with standard Young tableaux (SYTs) of rectangular shape 3×4 (resp. 4×3), and our results show that in all but one case, the web immanants are again basis web invariants whose tableau is the transpose of the tableau indexing the original basis web. This duality is depicted in Tables 1 and 2. As an application, we obtain combinatorial expansion formulas for twists

[1]Multi-sets of edges incident to each internal vertex (resp. boundary vertex i) exactly r (resp. λ_i) times.

of elements of $\mathbb{C}[\widehat{\mathrm{Gr}}(3,12)]_\lambda$ and $\mathbb{C}[\widehat{\mathrm{Gr}}(4,12)]_\lambda$ (Theorem 3.3), extending the results of [4].

Part of the motivation of our work is to provide expansion formulas for *cluster variables* in $\mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]$. In a cluster algebra, the cluster variables are a distinguished set of generators defined recursively through *mutation*. Cluster variables sit in certain overlapping sets called *clusters*, which in geometric examples corresponds to a transcendence basis of the coordinate ring. By [12, Proposition 8.10], the twist map on $\mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]$ sends cluster variables to cluster variables, up to multiplication by *frozen* variables. For $k = 2$, every Grassmannian cluster variable is a Plücker coordinate, and the twist[2] sends the Plücker coordinate $\Delta_{a,b}$ to $\Delta_{a+1,b+1}$. However, in Grassmannian cluster algebras with $k \geq 3$ and $n \geq 6$, some cluster variables are more complicated polynomials in Plücker coordinates, and even some Plücker coordinates only appear as (factors of) twists if applied to these higher degree non-Plücker cluster variables.

For most values of k and n, $\mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]$ has infinitely many cluster variables, for which there is no known parametrization. However, in the case $k = 3$, an influential conjecture of Fomin and Pylyavskyy [5] posits that the cluster variables in $\mathbb{C}[\widehat{\mathrm{Gr}}(3,n)]$ are exactly the basis web invariants which are both indecomposable and *arborizable*. In our analysis of the degree 4 web invariants in $\mathbb{C}[\widehat{\mathrm{Gr}}(3,n)]$, we verify that the Fomin–Pylyavskyy conjecture is consistent with enumerative conjectures of [3] (see Section 4).

2 Twists and r-dimer covers

Let G be a plabic graph which represents the top cell of $\widehat{\mathrm{Gr}}(k,n)$, and let $\lambda \in \mathbb{N}^n$. Let $\mathcal{D}_{r,\lambda}(G)$ denote the set of all r-dimer covers of G such that the number of edges in D incident to boundary vertex i is exactly λ_i. Each r-dimer D gives rise to a *r-weblike subgraph* of G, which is in turn uniquely associated to a tensor invariant $\mathbf{D} \in \mathcal{W}_\lambda(\mathbb{C}^r)$. Given a (k,n) *network* N, which assigns nonzero complex weights to each edge in G, we define the edge weight $\mathrm{ewt}_N(D) := \prod_{e \in D} \mathrm{wt}(e)$, where D is a multiset of edges. Then the *FLL tensor invariant* is defined as $\mathrm{Web}_r(N;\lambda) := \sum_{D \in \mathcal{D}_{r,\lambda}(G)} \mathrm{ewt}_N(D)\mathbf{D} \quad \in \mathcal{W}_\lambda(\mathbb{C}^r)$.

Let $\sum_i \lambda_i = kr$. By [7, Theorem 4.8], for any φ in the dual space $(\mathcal{W}_\lambda(\mathbb{C}^r))^*$, there exists a unique $f \in \mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]$ which makes the following diagram commute. In particular, the *immanant map*, $\mathrm{Imm} : (\mathcal{W}_\lambda(\mathbb{C}^r))^* \to \mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]_\lambda$, defined by setting $\mathrm{Imm}(\varphi)$ to be this unique function f, is an isomorphism. Here, $\widehat{X}$ is Postnikov's boundary measurement map.

$$\begin{array}{ccc} \{(k,n)\text{ networks}\} & \xrightarrow{\mathrm{Web}_r(-;\lambda)} & \mathcal{W}_\lambda(\mathbb{C}^r) \\ {\scriptstyle \widehat{X}}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \widehat{\mathrm{Gr}}(k,n) & \overset{\exists! f =: \mathrm{Imm}(\varphi)}{\dashrightarrow} & \mathbb{C} \end{array}$$

[2]We follow the conventions of [4], and use right twists, which are variants of Marsh–Scott (left) twists.

The definition of the immanant map provides us with the *FLL pairing* $\langle\cdot,\cdot\rangle$ discussed in the introduction. When the element of $\mathcal{W}_\lambda(\mathbb{C}^r)$ is an FLL tensor, the following equation [7, Equation 5.17] follows from the definition:

$$\langle \mathrm{Web}_r(N;\lambda), f\rangle = f(\widehat{X}(N)). \tag{2.1}$$

Let $\mathcal{S} = (S(1),\ldots,S(n))$ be a list of subsets $S(i) \subseteq [r]$ such that $S(1)\cup S(2)\cup\cdots\cup S(n) = \{1^k,\ldots,r^k\}$ as a multiset. A *consistent labeling* of web W with boundary label condition $\mathcal{S}$ is an assignment of a subset of $[r]$ to each edge of W such that boundary vertex i is incident to edges labeled with the elements in $S(i)$, and each internal vertex is incident to exactly one instance of i for each $i \in [r]$. An edge with multiplicity m is labeled with a size m subset of $[r]$. Let $a(\mathcal{S};W)$ be the number of consistent labelings of W with boundary label condition $\mathcal{S}$. In the following, given an SL_r web W, we let $\mathbf{W}$ denote the associated tensor invariant in $\mathcal{W}_\lambda(\mathbb{C}^r)$ using the convention in [7, (4.1)]. Pairing a web invariant with a product of Plücker coordinates reduces to counting consistent labelings [7, Equation 5.16]:

$$\langle \mathbf{W},\ \Delta_{I_1}\cdots\Delta_{I_r}\rangle = a(\mathcal{S};W). \tag{2.2}$$

Given a plabic graph G and an r-dimer cover $D \in \mathcal{D}_{r,\lambda}(G)$, let $F(G)$ denote the set of all faces of G, $I_{\mathrm{f}} \in \binom{[n]}{k}$ the k-subset labeling face f, W_{f} the number of white vertices bordering face f, and D_{f} the number of non-boundary edges of f used in D. We define *face weights* of D, both with respect to the plabic graph G, and to a network N with underlying graph G: $\mathrm{fwt}_G(D) := \prod_{\mathrm{f}\in F(G)} \Delta_{I_{\mathrm{f}}}^{rW_{\mathrm{f}}-D_{\mathrm{f}}-r} \in \mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]_\lambda$ and $\mathrm{fwt}_N(D) := (\mathrm{fwt}_G(D))(\widehat{X}(N)) \in \mathbb{C}$. Notice that the former is a Laurent monomial in Plücker coordinates, while the latter is a complex number. We use the face weights to define a *twisted FLL tensor invariant* $\mathrm{Web}_r^\tau(N;\lambda)$:

$$\mathrm{Web}_r^\tau(N;\lambda) := \sum_{D\in\mathcal{D}_{r,\lambda}(G)} \mathrm{fwt}_N(D)\mathbf{D} \quad \in \mathcal{W}_\lambda(\mathbb{C}^r). \tag{2.3}$$

Corollary 2.2 will justify this terminology.

Our first theorem gives Laurent expansions for the twist automorphism τ of an element of $\mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]_\lambda$, with coefficients given by the FLL pairing. This is a generalization of results from [4, 12].

Theorem 2.1. Let $f \in \mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]_\lambda$ and let G be a plabic graph representing the top cell of $\widehat{\mathrm{Gr}}(k,n)$. The twist of f is given by

$$\tau(f) = \sum_{D\in\mathcal{D}_{r,\lambda}(G)} \langle \mathbf{D}, f\rangle\, \mathrm{fwt}_G(D). \tag{2.4}$$

Proof Sketch. Since τ is a ring endomorphism, it suffices to prove the claim when f is a product $\Delta_{I_1}\cdots\Delta_{I_r}$ of Plücker coordinates, as these span $\mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]$. When $r=1$, the claim follows from combining (2.2) with [4, Theorem 3.4].

In the inductive step, to show the claim holds for f a degree $r > 1$ product of Plücker coordinates, we can split f into a product of lower degree products $f = f'f''$; it suffices to let f' be degree 1 and f'' be degree $r-1$. Let λ be such that $f \in \mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]_\lambda$ and let $\mathcal{S} = (S(1),\ldots,S(n))$ such that $S(i) = \{j \mid i \in I_j\} \subset [r]$. Define $\lambda', \lambda'', \mathcal{S}', \mathcal{S}''$ similarly with respect to f', f''. We use (2.2) and the fact that τ is a ring endomorphism to rewrite $\tau(f)$ in terms of dimers $D' \in \mathcal{D}_{1,\lambda'}(G)$ and $D'' \in \mathcal{D}_{r-1,\lambda''}(G)$. Setting $A(\mathcal{S};D)$ to be the set of consistent labelings enumerated by $a(\mathcal{S};D)$, the crux of the argument is showing a straightforward bijection between $\{A(\mathcal{S}';D') \times A(\mathcal{S}'';D'') \mid D' \cup D'' = D\}$ and $A(\mathcal{S};D)$ and noting that the definition of fwt_G is multiplicative with respect to this map. □

As a corollary, we obtain a "twisted version" of (2.1) by evaluating each side of (2.4) at the point $\widehat{X}(N) \in \widehat{\mathrm{Gr}}(k,n)$.

Corollary 2.2. For N a (k,n) network and $f \in \mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]_\lambda$, we have

$$\langle \mathrm{Web}_r^\tau(N;\lambda), f\rangle = (\tau(f))(\widehat{X}(N)).$$

In the cases where web bases $\mathcal{B}$ and $\mathcal{B}^*$ for $\mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]_\lambda$ and $\mathcal{W}_\lambda(\mathbb{C}^r)$, respectively, are known, Theorem 2.1 gives a method for computing explicit Laurent expansion formulas for twists. In the next section, we will exhibit dual bases in the setting when $k=3$, $r=4$, $n=12$, and $\lambda = (1,\ldots,1) \in \mathbb{N}^{12}$.

3 Computing web duals

For each of the known web bases ($k = 2,3,4$) [10, 8] and $\lambda = (1,\ldots,1)$, there is a bijection between the basis webs W for $\mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]_\lambda$ and SYTs $T(W)$ of shape $k \times r$ [8, 9, 18]. These bijections satisfy several nice properties. Promotion on $T(W)$ corresponds to rotation of W [8, 15], and evacuation on $T(W)$ corresponds to reflection of W [8, 14]. For an SYT T, define the *row word* $w(T) = w_1 \cdots w_n$ to be the lattice word such that $w_i \in [k]$ is the row of T containing i. This word determines a boundary condition $\mathcal{S}(w(T)) = (\{w_1\},\ldots,\{w_n\})$. For all basis webs W, we have $a(\mathcal{S}(w(T(W))),W) = 1$. Furthermore, for all other words v on the multiset $\{1^r,\ldots,k^r\}$ such that $a(\mathcal{S}(v);W) \neq 0$, we have $v > w(T(W))$ in lexicographic order [8].

Fraser–Lam–Le [7] observed that in the small cases $(k,n) = (3,6)$ and $(3,9)$, the web bases are dual, and duality corresponds to transposing the SYTs associated to basis webs. We will show here that this correspondence also holds for $(k,n) = (3,12)$.

Let $\lambda = (1,\ldots,1) \in \mathbb{N}^{12}$, and let $\mathcal{B}$ denote Kuperberg's non-elliptic web basis for the 462-dimensional $\mathbb{C}$-vector space $\mathbb{C}[\widehat{\mathrm{Gr}}(3,12)]_\lambda$. The dual basis $\mathcal{B}^*$ for the space $\mathbb{C}[\widehat{\mathrm{Gr}}(4,12)]_\lambda \cong \mathcal{W}_\lambda(\mathbb{C}^4) \cong (\mathbb{C}[\widehat{\mathrm{Gr}}(3,12)]_\lambda)^*$ is shown in Tables 1 and 2. Each cell in the table depicts a basis web $X_i \in \mathcal{B}$ (right) and its dual $W_i = X_i^* \in \mathcal{B}^*$ (left) as well as their corresponding row words. The 462 basis webs in $\mathcal{B}$ can be grouped into 32 orbits up to rotation and reflection. We depict one representative $W_i \in \mathcal{B}^*$ for each of these orbits. For the remainder of this abstract, for $1 \leq i \leq 32$, we will let $W_i \in \mathcal{B}^*$ and $X_i \in \mathcal{B}$ denote the webs depicted in cell i of the table. In this abstract, we only consider duality up to a sign, and we omit considerations of tagged webs.

Theorem 3.1. The bases $\mathcal{B}$ and $\mathcal{B}^*$, as depicted in Tables 1 and 2, are dual with respect to the FLL pairing (possibly up to a sign). In other words, $\langle \mathbf{W}_i, \mathbf{X}_i \rangle = \pm 1$, and $\langle \mathbf{W}_i, \mathbf{X}_j \rangle = 0$ for all $i \neq j$. Moreover, the corresponding tableaux satisfy $T(W_i) = T(X_i)^t$ (where "t" denotes transpose).

To prove Theorem 3.1, we fix an element $W_i \in \mathcal{B}^*$ and evaluate $\langle \mathbf{W}_i, \mathbf{X}_j \rangle$ for all $X_j \in \mathcal{B}$. We can dramatically reduce our casework through a set of lemmas concerning the existence of a consistent labeling of a web with certain boundary conditions.

For $i < j \in [n]$, we say a web W has a *fork at* (i, j) if boundary vertices i and j are each connected to a common interior vertex by a single edge. Our critical observation is the following lemma:

Lemma 3.2. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\sum_i \lambda_i = kr$. Let W be an SL_r web and X an SL_k web, each with boundary conditions given by λ. If both W and X have forks at (i, j), then $\langle \mathbf{W}, \mathbf{X} \rangle = 0$.

Proof Sketch. If X has a fork at (i, j), then by an inductive argument using the skein relations [2] for SL_k webs, there exists an expansion of $\mathbf{X}$ as a sum of products of Plücker coordinates, $\mathbf{X} = \sum_\ell \mathbf{X}_\ell$ with $\mathbf{X}_\ell = \prod_{m=1}^r \Delta_{I_{\ell,m}}$, such that without loss of generality $i, j \in I_{\ell,1}$ for each ℓ. Thus each $\mathbf{X}_\ell$ prescribes a boundary label condition $\mathcal{S} = (S_1, \ldots, S_n)$ such that $S_i \cap S_j \neq \emptyset$. If W also has a fork at (i, j), then there is no consistent labeling of W with this boundary condition, i.e. $a(\mathcal{S}; W) = 0$. Thus $\langle \mathbf{W}, \mathbf{X}_\ell \rangle = 0$ for all ℓ, and by bilinearity $\langle \mathbf{W}, \mathbf{X} \rangle = \sum_\ell \langle \mathbf{W}, \mathbf{X}_\ell \rangle = 0$. □

We eliminated pairs of webs whose pairing is 0 by Lemma 3.2, using the fact that a web X_j has fork at $(\ell, \ell + 1)$ if and only if i is in the descent set of the corresponding row word [8]. Moreover, we prove other technical lemmas concerning the existence of consistent labelings and skein relations to identify more pairs W, X with $\langle \mathbf{W}, \mathbf{X} \rangle = 0$. For the specific case when $r = 4$, $k = 3$, some small examples are shown in Figure 1. Lastly, we checked the pairing of W with the subset of $\mathcal{B}$ not yet considered, by expanding each remaining $\mathbf{X}_j$ into a sum of products of Plücker coordinates and using (2.2).

For all but four of the 462 basis webs, our dual basis $\mathcal{B}^*$ for $\mathbb{C}[\widehat{\mathrm{Gr}}(4, 12)]_\lambda$ coincides with the basis given by the growth algorithm of [8, Section 5]. The exceptional cases are the four rotations of the highly symmetric SL_3 web $X_{16} = \Delta_{1,2,3}\Delta_{5,6,7}\Delta_{9,10,11}\Delta_{4,8,12}$.

Under the bijection of [8], the transposed tableau $T(X_{16})^t$ corresponds to a web with a hexagonal "benzene face" where the edges alternate between single and double edges (this web is one of the summands of W_{16}). As explained in [8, Section 6], tableau $T(X_{16})^t$

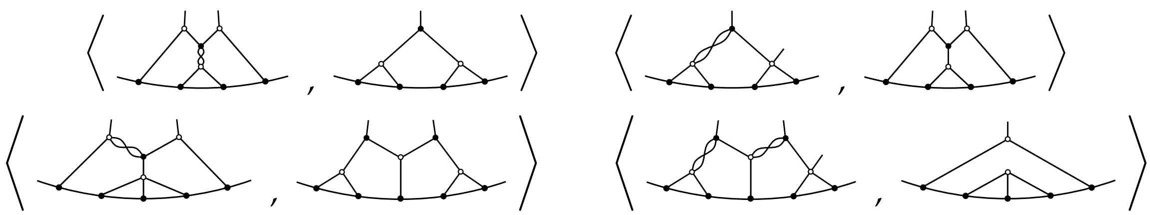

Figure 1: For $r = 4$, $k = 3$ pairings with these boundary configurations evaluate to 0.

in fact corresponds to the entire benzene-move-equivalence class of this web, where *benzene moves* interchange the single and double edges bounding the hexagon. The authors of [8] break this symmetry by choosing the *top* representative (maximal among the benzene-move-equivalence class). However, this is not a reflection-invariant choice, losing a property for $k = 4$ that held for web bases when $k = 2$ or $k = 3$. By setting $W_{16} = X^*_{16}$ to be the difference of two webs, our basis has the desirable property of being both rotation- and reflection-invariant, as W_{16} and its reflection are in fact equal as web invariants.

As an application of Theorems 2.1 and 3.1, we obtain combinatorial expansion formulas for twists of functions in $\mathbb{C}[\widehat{\mathrm{Gr}}(3,12)]_\lambda$ and $\mathbb{C}[\widehat{\mathrm{Gr}}(4,12)]_\lambda$, $\lambda = (1,\ldots,1)$, in terms of higher dimer covers. In these settings, given a web-like subgraph $\mathbf{D}$, we can use skein relations from [5, 8] to write $\mathbf{D}$ as a linear combination of basis web invariants, $\mathbf{D} = \sum_W C^D_W \mathbf{W}$. Then, Theorem 3.1 allows us to identify the coefficients in Theorem 3.3 for a web invariant as coming from these web expansions.

Theorem 3.3. Let $n = 12$, $k \in \{3,4\}$, and $\lambda = (1,\ldots,1) \in \mathbb{N}^{12}$. Let $\mathbf{Y} \in \mathbb{C}[\widehat{\mathrm{Gr}}(k,12)]_\lambda$ be the web invariant associated with an SL_k-basis web Y (excluding $Y = X_{16}$ and $Y = W_{16}$)[3]. Let G be a reduced plabic graph for the top cell of $\widehat{\mathrm{Gr}}(k,12)_{\geq 0}$. Then, we can express $\tau(\mathbf{Y}) = \sum_{D \in \mathcal{D}_{r,\lambda}(G)} C^D_{Y^*} \,\mathrm{fwt}_G(D)$, where Y^* is the dual basis web associated to Y.

Example 3.4. Let G be the reduced plabic graph for the top cell of $\widehat{\mathrm{Gr}}(3,12)_{\geq 0}$ appearing in Figure 2 (top). Consider the basis web invariant $\mathbf{X}_{28} \in \mathbb{C}[\widehat{\mathrm{Gr}}(3,12)]_{(1,\ldots,1)}$ associated to the degree four SL_3 web X_{28} (the 28th representative of Table 2). We can expand $\mathbf{X}_{28}$ in terms of Plücker coordinates and use [4, Proposition 4.3] to write its twist,

$$\tau(\mathbf{X}_{28}) = [\Delta_{1,2,12}\Delta_{2,3,4}\Delta_{4,5,6}\Delta_{6,7,8}\Delta_{8,9,10}\Delta_{10,11,12}](\Delta_{1,7,11}\Delta_{3,5,9} - \Delta_{1,9,11}\Delta_{3,5,7})$$

which is a degree two non-Plücker cluster variable multiplied by a degree six frozen monomial. Thus, as an application of Theorem 3.3, the Laurent expansion

$$\tau(\mathbf{X}_{28}) = \frac{\Delta_{1,2,12}\Delta_{2,3,4}\Delta_{4,5,6}\Delta_{6,7,8}\Delta_{8,9,10}\Delta_{10,11,12}(\Delta_{1,3,5}\Delta_{1,9,11}\Delta_{5,7,9} + \Delta_{1,5,11}\Delta_{1,7,9}\Delta_{3,5,9})}{\Delta_{1,5,9}}$$

[3]We can in fact give combinatorial formulas in these cases with a small modification, but for sake of brevity we omit this.

relative to this choice of initial seed agrees with the face weights of the only two quadruple dimer covers of this plabic graph, as illustrated in Figure 2 (top), that admit a 4-weblike subgraph, see Figure 2 (bottom), that expands via SL_4 skein relations to include W_{28}, the dual of X_{28}, in its support (and with coefficient 1). Note that the 4-weblike subgraph on the right has forbidden squares in its move equivalence class which allow us to apply the SL_4-skein relations.

The pairing between the web bases for $\mathbb{C}[\widehat{\mathrm{Gr}}(3,12)]_{(1,\ldots,1)}$ and $\mathbb{C}[\widehat{\mathrm{Gr}}(4,12)]_{(1,\ldots,1)}$, along with the observations in the Appendix in [7], lead us to believe that this phenomenon will persist more generally. Given n,k,r with $n = kr$ and a $k \times r$ SYT T and its transpose T^t, we expect there are corresponding web basis elements W and W^*, related by web duality, associated to $\mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]$ and $\mathbb{C}[\widehat{\mathrm{Gr}}(r,n)]$, respectively, such that the Laurent expansion of the degree r SL_k web invariant $\mathbf{W}$ in the cluster algebra for $\mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]$ may be expressed as in Theorem 2.1 using the dual web W^*.

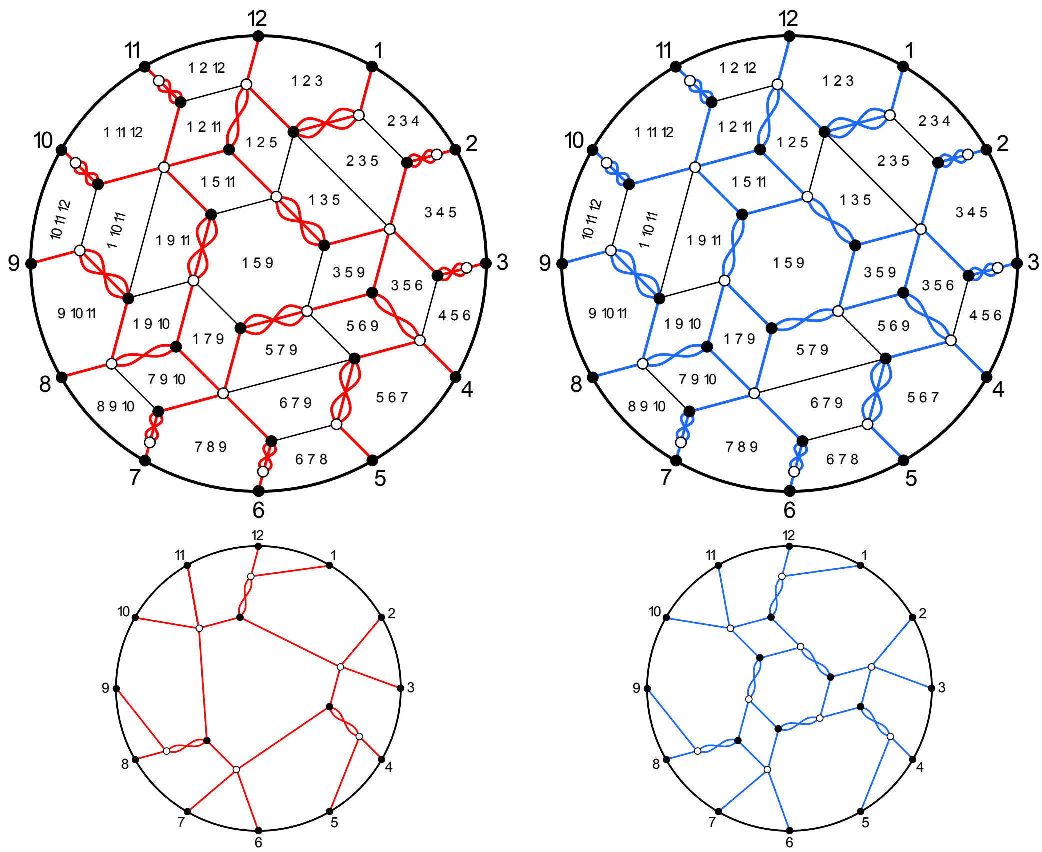

Figure 2: (Top): Two 4-dimer covers on a plabic graph G for $\mathbb{C}[\widehat{\mathrm{Gr}}(3,12)]$; (Bottom): The 4-weblike subgraphs equivalent to these choices of 4-dimer covers.

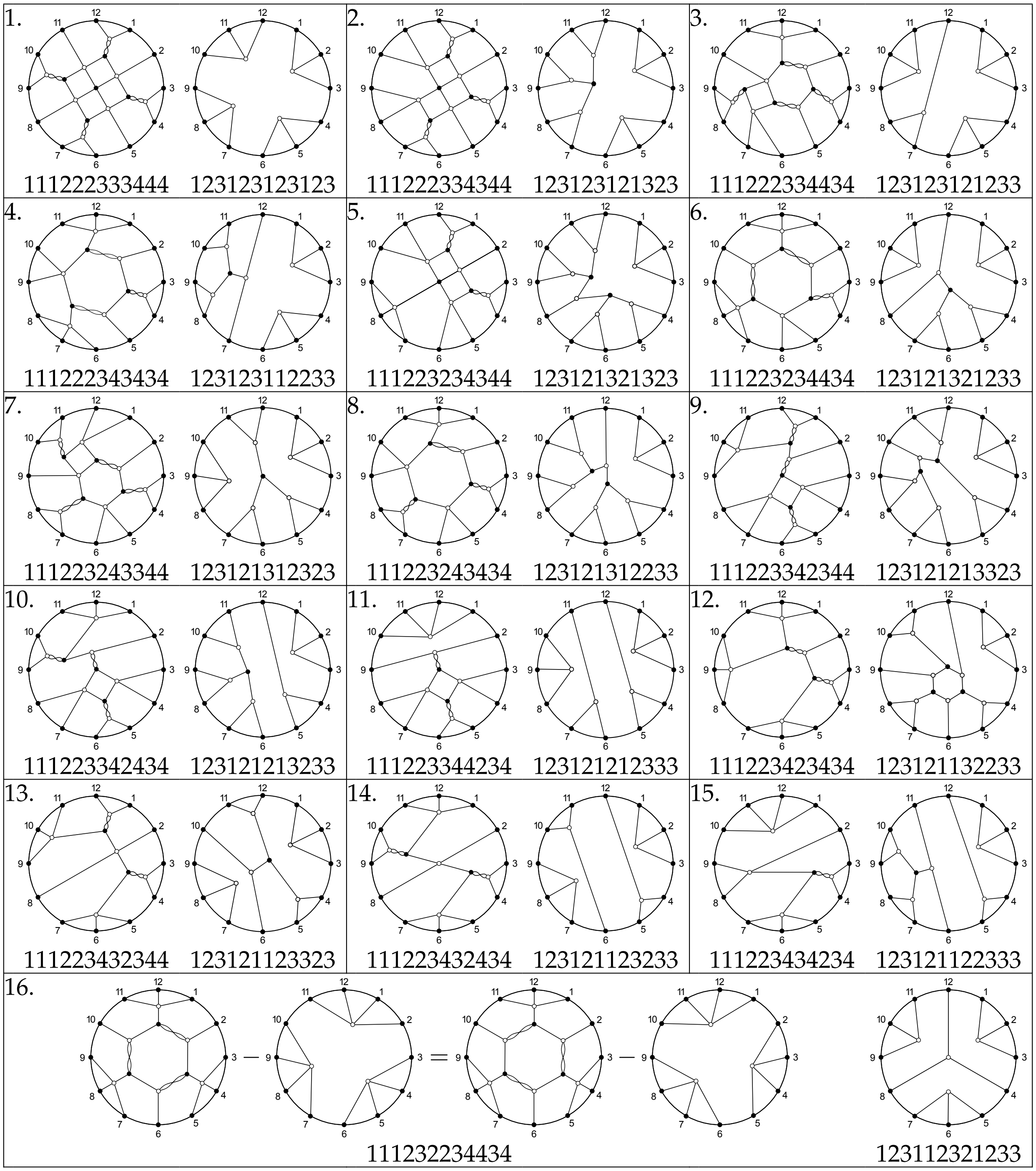

Table 1: Dual bases for $\mathcal{W}_\lambda(\mathbb{C}^4)$ and $\mathcal{W}_\lambda(\mathbb{C}^3)$, $\lambda = (1,\ldots,1)$. Cell i depicts basis elements $W_i \in \mathcal{W}_\lambda(\mathbb{C}^4)$ (left) and $X_i \in \mathcal{W}_\lambda(\mathbb{C}^3)$ (right), as well as the row words of their associated standard Young tableaux. Following [8], we depict edges of multiplicity > 1 as *hourglass edges*.

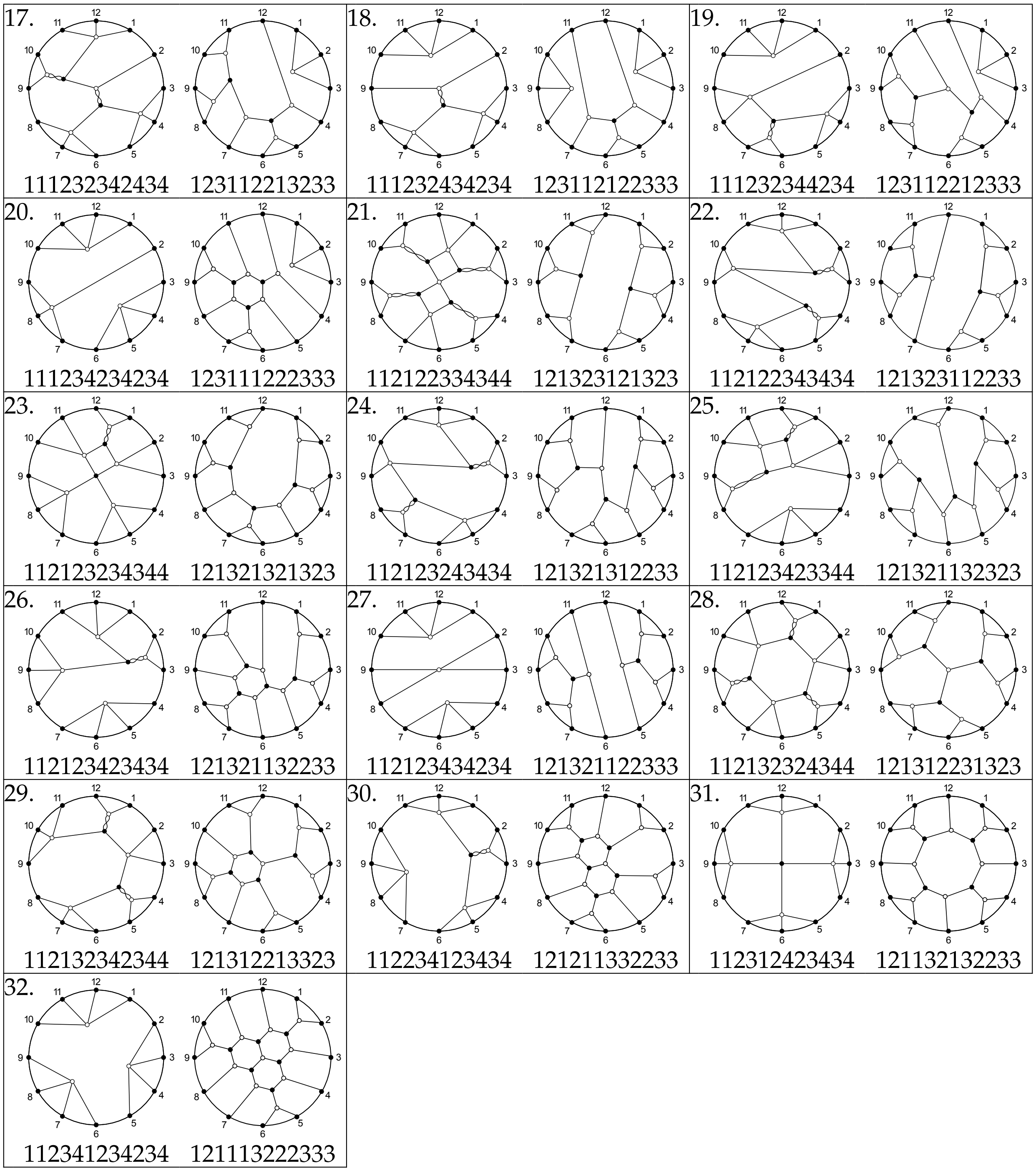

Table 2: Continued from Table 1.

4 Enumeration of cluster variables

In [3], the authors use high-performance computing to conjecture the number of cluster variables of small degrees in $\mathbb{C}[\widehat{\mathrm{Gr}}(3,n)]$ and $\mathbb{C}[\widehat{\mathrm{Gr}}(4,n)]$. In particular, letting $N_{k,n,r}$ be the number of cluster variables of degree r in $\mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]$, the authors conjecture[4]

$$N_{3,n,4} = 288\binom{n}{9} + 400\binom{n}{10} + 264\binom{n}{11} + 52\binom{n}{12}; \tag{4.1}$$

$$N_{4,n,3} = 174\binom{n}{9} + 855\binom{n}{10} + 1285\binom{n}{11} + 123\binom{n}{12}. \tag{4.2}$$

Fomin–Pylyavskyy [5] conjectured that the cluster variables in $\mathbb{C}[\widehat{\mathrm{Gr}}(3,n)]$ are exactly the non-elliptic webs which are both indecomposable and *arborizable*. That is, these webs have an alternate representation as a (possibly non-planar) tree, via skein relations on tensor diagrams.

By enumerating all arborizable, indecomposable basis webs of degree 4 in $\mathbb{C}[\widehat{\mathrm{Gr}}(3,n)]$, we have found that the conjectured formula (4.1) is consistent with the Fomin–Pylyavskyy conjecture, providing evidence in favor of both conjectures. When $n = kr$, a basis web in $\mathbb{C}[\widehat{\mathrm{Gr}}(k,kr)]_{(1,\dots,1)}$ is arborizable and indecomposable if and only if it is a tree. In the web basis for $\mathbb{C}[\widehat{\mathrm{Gr}}(3,12)]_{(1,\dots,1)}$, the only trees are the dihedral orbits of webs X_{23}, X_{24}, X_{25}, and X_{28}, as shown in Table 2. There are exactly 52 distinct rotations and reflections of these webs, matching the coefficient of $\binom{n}{12}$ in (4.1). We recovered the other coefficients by *clasping* adjacent boundary vertices together to obtain a web for $\mathbb{C}[\widehat{\mathrm{Gr}}(3,11)]$, $\mathbb{C}[\widehat{\mathrm{Gr}}(3,10)]$, or $\mathbb{C}[\widehat{\mathrm{Gr}}(3,9)]$, and checking whether the resulting web is arborizable and indecomposable.

More generally, let $T_{k,r}$ denote the number of web invariants in $\mathbb{C}[\widehat{\mathrm{Gr}}(k,kr)]_{(1,\dots,1)}$ whose web diagrams are trees. By the Fomin–Pylyavskyy conjecture, $T_{3,r}$ is conjecturally equal to the coefficient of $\binom{n}{3r}$ in $N_{3,n,r}$. By considering a bijection between plabic trees in $\mathbb{C}[\widehat{\mathrm{Gr}}(3,3r)]_{(1,\dots,1)}$ and a certain family of binary trees, we obtain the following formula.

Proposition 4.1. $T_{3,r} = \binom{4r-3}{r-1}\frac{2}{3r-1}$. [5]

In addition, we observe exactly 123 trees in the basis for $\mathbb{C}[\widehat{\mathrm{Gr}}(4,12)]_{(1,\dots,1)}$, matching the coefficient of $\binom{n}{12}$ in (4.2).

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[4]The 52 is a 48 in [3, Conjecture 3.1], but has been confirmed by one of the authors of [3] to be a typo.

[5]This sequence is OEIS A069271, with indexing shifted by 1.

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