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Equivariant γ-positivity of Chow rings and augmented Chow rings of matroids

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Abstract. We prove that the Chow ring and augmented Chow ring of a matroid are equivariant γ-positive under the action of any group of automorphisms of the matroid. This verifies a conjecture of Angarone, Nathanson, and Reiner. Our method gives an explicit interpretation for the coefficients of the equivariant γ-expansion, and extends the author's previous results regarding the positivity of the equivariant Charney–Davis quantity of matroids. Applying the theorems to uniform matroids, we obtain interpretations that extend Shareshian and Wachs' Schur-γ-positivity of the Eulerian and binomial Eulerian quasisymmetric functions, or equivalently, of the cohomologies of the permutahedral and the stellahedral varieties.

Keywords: matroids, Chow ring, γ-positivity, equivariant, uniform matroids, symmetric functions

1 Introduction

Given a finite sequence, there are many features that one can look for in the sequence. A sequence $\{a_i\}_{i=0}^{d}$ or a polynomial $f(t) = \sum_{i=0}^{d} a_i t^i$ of degree d is said to be *palindromic* if $a_j = a_{d-j}$ for all $j = 0, 1, \ldots, d$. It is said to be *unimodal* if there is some $0 \leq j \leq d$ such that $a_0 \leq a_1 \leq \cdots \leq a_j \geq \cdots \geq a_{d-1} \geq a_d$. A palindromic polynomial $f(t)$ can be uniquely expressed as

$$f(t) = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_k t^k (1+t)^{d-2k}.$$

We call the right-hand-side the γ-expansion of $f(t)$, and say that $f(t)$ is *γ-positive* (or *γ-nonnegative*) if $\gamma_k \geq 0$ for all k. It is not hard to see that a polynomial being γ-positive implies that it is both palindromic and unimodal. Although the γ-positivity is interesting to combinatorialists in its own right, this property has a connection to Gal's conjecture in discrete geometry, and hence has drawn much attention from researchers

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in combinatorics, discrete geometry, and commutative algebra. For more details about γ-positivity, we refer the readers to the survey [3] by Athanasiadis.

Let M be a loopless matroid on the ground set E. The *Chow ring of the matroid M* encodes information about the lattice of flats $\mathcal{L}(M)$ and is defined as

$$A(M) := \mathbb{R}[x_F : F \in \mathcal{L}(M) \setminus \{\emptyset\}]/(I + J)$$

where $I = (x_F x_{F'} : F, F' \text{ not comparable})$ and $J = (\sum_{F:i\in F} x_F : i \in E)$. The *augmented Chow ring of M* is an extension of $A(M)$ encoding both information from $\mathcal{L}(M)$ and the independence complex $\mathcal{I}(M)$, and is defined as

$$\widetilde{A}(M) := \mathbb{R}[\{x_F : F \in \mathcal{L}(M) \setminus \{[n]\}\} \cup \{y_i : i \in E\}]/(\widetilde{I} + \widetilde{J})$$

where $\widetilde{I} = (x_F x_{F'} : F, F' \text{ not comparable}) + (y_i x_F : i \notin F)$ and $\widetilde{J} = (y_i - \sum_{F:i\notin F} x_F : i \in E)$. Despite the seemingly complicated presentation, the Chow ring $A(M)$ has a very nice $\mathbb{R}$-basis called the *Feichtner–Yuzvinsky basis*

$$FY(M) := \left\{ x_{F_1}^{a_1} x_{F_2}^{a_2} \dots x_{F_\ell}^{a_\ell} : \begin{smallmatrix} \emptyset = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_\ell \text{ for } 0 \le \ell \le n \\ 1 \le a_i \le \mathrm{rk}_M(F_i) - \mathrm{rk}_M(F_{i-1}) - 1 \end{smallmatrix} \right\}, \tag{1.1}$$

which is induced by the Gröbner basis of the ideal $I + J$ found by Feichtner and Yuzvinsky [9]. There is a similar Feichtner–Yuzvinsky basis for the augmented Chow ring $\widetilde{A}(M)$,

$$\widetilde{FY}(M) := \left\{ x_{F_1}^{a_1} x_{F_2}^{a_2} \dots x_{F_\ell}^{a_\ell} : \begin{smallmatrix} \emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_\ell \text{ for } 0 \le \ell \le n \\ 1 \le a_1 \le \mathrm{rk}_M(F_1), \\ 1 \le a_i \le \mathrm{rk}_M(F_i) - \mathrm{rk}_M(F_{i-1}) - 1 \text{ for } i \ge 2 \end{smallmatrix} \right\}, \tag{1.2}$$

found by the author [14, 16] and independently by Eur, Huh, and Larson [8]. The Chow rings and augmented Chow rings of matroids play important roles in settling the long-standing Rota–Welsh conjecture and Dowling–Wilson top-heavy conjecture, respectively. They are shown to satisfy the so-called Kähler package (see [1] for Chow rings and [5] for augmented Chow rings). In this package, the Poincaré duality implies that the Hilbert series of the two rings are palindromic, and the hard Lefschetz theorem implies that the Hilbert series are unimodal. The γ-positivity of the Hilbert series was shown by Ferroni, Matherne, Stevens, and Vecchi [10], and independently by Wang [10, p.33], using the semismall decomposition introduced in [6]. However, no interpretation of the coefficients γ_k was known.

In [2], Angarone, Nathanson, and Reiner further conjectured that the matroid Chow rings, under the action of groups of automorphisms of the corresponding matroid, are equivariant γ-positive (see Section 3 for the definition). In this extended abstract, we present a proof of this conjecture using only the *Feichtner–Yuzvinsky bases* and a result in Stanley's classical paper [20] (see Theorem 2.1). Our method is motivated by the author's proof of the positivity of the equivariant Charney–Davis quantity of matroids in [15], and it gives an explicit interpretation of the γ-coefficients, which is stated in Section 3. We present the proof in Section 4. In Section 5, we apply our results to uniform matroids, in which case the Hilbert series of the (augmented) Chow rings have been studied in Hameister, Rao, and Simpson [12], Liao [15], and Hoster [13]. Our results extend Shareshian and Wachs' Schur-γ-positivity of the (binomial) Eulerian quasisymmetric functions, and equivalently the equivariant γ-positivities on the cohomologies of the permutahedral and the stellohedral varieties in [17, 18].

In a recent preprint, Stump [23] also independently gives an interpretation for the coefficients of the (non-equivariant) γ-expansion for Chow rings and augmented Chow rings of matroids using a certain evaluation of the *Poincaré extended* **ab**-*index* introduced in [7]. He interprets the γ-coefficient as the number of maximal chains in $\mathcal{L}(M)$ with fixed descent set under some R-labeling. This agrees with the result in this paper, following from Stanley [19, Theorem 3.1].

In [10], the Hilbert series of Chow rings and augmented Chow rings of matroids are called Chow and augmented Chow polynomial of matroids. In an upcoming paper [11], Ferroni, Matherne, and Vecchi generalize Chow and augmented Chow polynomials to arbitrary graded posets. They show that Chow and augmented Chow polynomials are γ-positive for Cohen–Macaulay posets using a method similar to this work.

2 Group actions on posets

For non-negative integers $a \leq b$, let $[a, b] := \{a, a+1, \ldots, b\}$; and in particular, if $a = 1$, we write $[b]$ instead of $[1, b]$.

Let P be a finite poset with a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$. If P is graded of rank n with rank function $\mathrm{rk}_P : P \longrightarrow [0, n]$, then for $S \subseteq [n-1]$, the *rank-selected subposet* of P is

$$P_S := \{x \in P \ : \ \mathrm{rk}_P(x) \in S\} \cup \{\hat{0}, \hat{1}\}.$$

Consider a group G of automorphisms of P acting on the poset P. The action of G preserves the rank of the elements in P; hence for any $S \subseteq [n-1]$, the group G permutes the maximal chains in P_S. Let us denote $\alpha_P(S)$ the permutation representation of G generated by the maximal chains in P_S. Consider the virtual representation

$$\beta_P(S) := \sum_{T \subseteq S} (-1)^{|S|-|T|} \alpha_P(T).$$

By Möbius inversion, we also have $\alpha_P(S) = \sum_{T \subseteq S} \beta_P(T)$.

Theorem 2.1 (Stanley [20]). *If P is Cohen–Macaulay, then $\beta_P(S)$ is a genuine representation of G and*

$$\beta_P(S) \cong_G \tilde{H}_{|S|-1}(P_S).$$

Remark 2.2. The dimension (rank) of $\beta_P(S)$ is given by the Möbius function $(-1)^{|S|-1}\mu(P_S)$ and is the number of maximal chains in P whose Jordan–Hölder sequences have descent set S. See Stanley [19, Theorem 3.1] or Björner [4, Theorem 2.7].

Throughout this paper, the poset P we care about is always the lattice of flats of a matroid. It is well known that the lattice of flats of a matroid is always Cohen–Macaulay.

3 Main results on equivariant γ-positivity

Let G be a finite group. For a $\mathbb{C}G$-module M, let $[M]$ be the isomorphism class of $\mathbb{C}G$-modules containing M.

Definition 3.1. The *Grothendieck ring* $R_{\mathbb{C}}(G)$ of $\mathbb{C}G$-modules is a free abelian group having the transversal of isomorphism classes of simple $\mathbb{C}G$-modules $\{[S_i],\ldots,[S_{cc(G)}]\}$ as a $\mathbb{Z}$-basis, where $cc(G)$ is the number of conjugacy classes of G, with the addition and multiplication relations given by

$$[S_i]+[S_j]:=[S_i\oplus S_j] \quad \text{and} \quad [S_i]\cdot[S_j]:=[S_i\otimes_{\mathbb{C}} S_j]$$

for $1\leq i,j\leq cc(G)$ and extended linearly over $\mathbb{C}$. Now every element A in $R_{\mathbb{C}}(G)$ has a unique expression as $a=\sum_{i=1}^{cc(G)} a_i[S_i]$. We say A is a *genuine representation* of G if $a_i\geq 0$ for all i, denoted by $A\geq_{R_{\mathbb{C}}(G)} 0$.

Given a graded $\mathbb{C}G$-module $V=\bigoplus_i V_i$, define the *equivariant Hilbert series* of V to be the formal power series

$$\mathsf{Hilb}_G(V,t)=\sum_i [V_i]t^i\in R_{\mathbb{C}}(G)[[t]].$$

Definition 3.2. For a finitely dimensional graded $\mathbb{C}G$-module $V=\bigoplus_{i=0}^d V_i$, we say that V is (G-)*equivarant γ-positive* if its equivariant Hilbert series can be expressed as

$$\mathsf{Hilb}_G(V,t)=\sum_{i=0}^{d}[V_i]t^i=\sum_{k=0}^{\lfloor\frac{d}{2}\rfloor}\gamma_k t^k(1+t)^{d-2k}$$

and the uniquely defined coefficient $\gamma_k\in R_{\mathbb{C}}(G)$ is a class of a genuine representation of G over $\mathbb{C}$ for all k, i.e. $\gamma_k\geq_{R_{\mathbb{C}}(G)} 0$ for all k.

Let M be a loopless matroid of rank r on the ground set E with the lattice of flats $\mathcal{L}(M)$. Write $A(M)_{\mathbb{C}}:=A(M)\otimes_{\mathbb{R}}\mathbb{C}$ and $\widetilde{A}(M)_{\mathbb{C}}:=\widetilde{A}(M)\otimes_{\mathbb{R}}\mathbb{C}$.

From now on, we let G be a group of automorphisms of matroid M. Consider the induced action of G on the Chow ring $A(M)_{\mathbb{C}}=\bigoplus_{i=0}^{r-1}A^i_{\mathbb{C}}$ which gives each graded piece $A^i_{\mathbb{C}}$ a $\mathbb{C}G$-module structure. The following conjecture was proposed by Angarone, Nathanson, and Reiner [2].

Conjecture 3.3 ([2, Conjecture 5.2])**.** *For any group G of automorphisms of M, the matroid Chow ring $A(M)_{\mathbb{C}} = \bigoplus_{i=0}^{r-1} A^i_{\mathbb{C}}$ is (G-)equivariant γ-positive.*

We prove Conjecture 3.3 and the equivariant γ-positivity of the augmented Chow ring $\widetilde{A}(M)$ and obtain explicit G-representations as the coefficients in the equivariant γ-expansions. Our method uses only the Feichtner–Yuzvinsky bases (1.1) and (1.2) and Theorem 2.1. For $S \subseteq [n]$, let $\mathrm{Stab}(S)$ denote the collection of subsets of S containing no consecutive integers.

Theorem 3.4. *The Chow ring $A(M)$ is equivariant γ-positive with the following γ-expansion*

$$\begin{aligned}\mathsf{Hilb}_G(A(M)_{\mathbb{C}}, t) &= \sum_{S \in \mathrm{Stab}([2,r-1])} [\tilde{H}_{|S|-1}(\mathcal{L}(M)_S)] t^{|S|}(1+t)^{r-1-2|S|} \\ &= \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} \left(\sum_{\substack{S \in \mathrm{Stab}([2,r-1]) \\ |S|=k}} [\tilde{H}_{|S|-1}(\mathcal{L}(M)_S)] \right) t^k (1+t)^{r-1-2k}.\end{aligned}$$

Theorem 3.5. *The augmented Chow ring $\widetilde{A}(M)_{\mathbb{C}}$ is equivariant γ-positve with*

$$\begin{aligned}\mathsf{Hilb}_G(\widetilde{A}(M)_{\mathbb{C}}, t) &= \sum_{S \in \mathrm{Stab}([r-1])} [\tilde{H}_{|S|-1}(\mathcal{L}(M)_S)] t^{|S|}(1+t)^{r-2|S|} \\ &= \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \left(\sum_{\substack{S \in \mathrm{Stab}([r-1]) \\ |S|=k}} [\tilde{H}_{|S|-1}(\mathcal{L}(M)_S)] \right) t^k (1+t)^{r-2k}.\end{aligned}$$

Remark 3.6. Theorem 3.4 and Theorem 3.5 generalize the author's previous results in [15], Theorem 4.9 and Theorem 4.12 respectively, regarding the equivariant Charney-Davis quantities of Chow and augmented Chow rings of matroids.

4 Proof of Theorem 3.4

In this section, we present our proof of Theorem 3.4, the equivariant γ-positivity of Chow rings of matroids. The augmented case, Theorem 3.5, can be proved in a similar way.

It is not hard to check that under the action of G, the Feichtner–Yuzvinsky bases $FY(M)$ and $\widetilde{FY}(M)$ are permutation bases for $A(M)_{\mathbb{C}}$ and $\widetilde{A}(M)_{\mathbb{C}}$, respectively.

Proof of Theorem 3.4. Recall that the permutation basis $FY(M)$ consisting of monomials $x_{F_1}^{a_1} x_{F_2}^{a_2} \dots x_{F_\ell}^{a_\ell}$ for any chains (including empty chain)

$$\emptyset \neq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_\ell \subseteq E$$

such that $1 \le a_j \le \mathrm{rk}(F_j) - \mathrm{rk}(F_{j-1}) - 1$ for all j. For a subset $S = \{s_1 < s_2 < \ldots < s_\ell\} \subseteq [r-1]$, the permutation $\mathbb{C}G$-module generated by chains in $\mathcal{L}(M)$ whose rank set is S is

$$\begin{aligned}\alpha_{\mathcal{L}(M)}(S) &= \mathbb{C}G\{F_1 \subsetneq \ldots \subsetneq F_\ell : \mathrm{rk}(F_i) = s_i \ \forall i\}.\\ &\cong \mathbb{C}G\{F_1 \subsetneq \ldots \subsetneq F_\ell \subsetneq E : \mathrm{rk}(F_i) = s_i \ \forall i\}\end{aligned}$$

Therefore, $A(M)$ as a $\mathbb{C}G$-module is the direct sum of $\alpha_{\mathcal{L}(M)}(S)$ with some multiplicities for $S \subseteq [r-1]$. Hence we have

$$\mathsf{Hilb}_G(A(M)_{\mathbb{C}}, t) = \sum_{S \subseteq [r-1]} \phi_{S,r}(t)[\alpha_{\mathcal{L}(M)}(S)] \tag{4.1}$$

for some polynomials $\phi_{S,r}(t)$. To see what the polynomial $\phi_{S,t}(t)$ is, we consider a map $f : FY(M) \longrightarrow 2^{[r-1]}$ defined by $f(x_{F_1}^{a_1} x_{F_2}^{a_2} \ldots x_{F_\ell}^{a_\ell} x_E^i) = \{\mathrm{rk}(F_1), \mathrm{rk}(F_2), \ldots, \mathrm{rk}(F_\ell)\}$ where all flats F_i are distinct and are not E. Since for any $S = \{s_1 < \cdots < s_\ell\} \subseteq [r-1]$,

$$f^{-1}(S) = \left\{ x_{F_1}^{a_1} x_{F_2}^{a_2} \ldots x_{F_\ell}^{a_\ell} x_E^i : \substack{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_\ell \subsetneq E \text{ with } \mathrm{rk}(F_j) = s_j \\ 1 \le a_j \le s_j - s_{j-1} - 1 \text{ for } j = 1,2\ldots,\ell \\ 0 \le i \le r - s_\ell - 1} \right\},$$

the polynomials that count the multiplicities are

$$\begin{aligned}\phi_{S,r}(t) &= (t + t^2 + \cdots + t^{s_1 - 1}) \ldots (t + t^2 + \cdots + t^{s_\ell - s_{\ell-1} - 1})(1 + t + \cdots + t^{r - s_\ell - 1})\\ &= t^{|S|}[s_1 - 1]_t [s_2 - s_1 - 1]_t \ldots [s_\ell - s_{\ell-1} - 1]_t [r - s_\ell]_t,\end{aligned}$$

where $[n]_t := 1 + t + \ldots + t^{n-1}$ for $n \ge 1$ and $[0]_t := 0$. Note that $\phi_{S,r}(t) = 0$ if $s_j - s_{j-1} = 1$ for some $1 \le j \le \ell$, and $[r - s_\ell]_t = 1$ if $r - 1 \in S$. Therefore, $\phi_{S,r}(t)$ takes nonzero value when $S \in \mathsf{Stab}([2, r-1])$. The identity (4.1) can be rewritten as follows:

$$\begin{aligned}\mathsf{Hilb}_G(A(M)_{\mathbb{C}}, t) &= \sum_{S \in \mathsf{Stab}([2,r-1])} \phi_{S,r}(t)[\alpha_{\mathcal{L}(M)}(S)] = \sum_{S \in \mathsf{Stab}([2,r-1])} \phi_{S,r}(t) \sum_{T : T \subseteq S} [\beta_{\mathcal{L}(M)}(T)]\\ &= \sum_{T \in \mathsf{Stab}([2,r-1])} \left(\sum_{T \subseteq S \subseteq [r-1]} \phi_{S,r}(t) \right) [\beta_{\mathcal{L}(M)}(T)].\end{aligned}$$

Surprisingly, it turns out that

$$\sum_{T \subseteq S \subseteq [r-1]} \phi_{S,r}(t) = t^{|T|}(1+t)^{r-1-2|T|}.$$

This fact and its proof will be presented later in Lemma 4.1. Combining with Theorem 2.1, which states that $\beta_{\mathcal{L}(M)}(T)$ is isomorphic to $\tilde{H}_{|T|-1}(\mathcal{L}(M)_T)$ as $\mathbb{C}G$-modules, we obtain

$$\begin{aligned}\mathsf{Hilb}_G(A(M)_{\mathbb{C}}, t) &= \sum_{T \in \mathsf{Stab}([2,r-1])} [\beta_{\mathcal{L}(M)}(T)] t^{|T|}(1+t)^{r-1-2|T|}\\ &= \sum_{T \in \mathsf{Stab}([2,r-1])} [\tilde{H}_{|T|-1}(\mathcal{L}(M)_T)] t^{|T|}(1+t)^{r-1-2|T|}.\end{aligned}$$

□

Lemma 4.1. *For $n \geq 2$ and any subset $T \in \mathrm{Stab}([2, n-1])$,*

$$\sum_{T \subseteq S \subseteq [n-1]} \phi_{S,n}(t) = t^{|T|}(1+t)^{n-1-2|T|}, \tag{4.2}$$

where

$$\phi_{S,n}(t) = t^{|S|}[s_1 - 1]_t[s_2 - s_1 - 1]_t \ldots [s_\ell - s_{\ell-1} - 1]_t[n - s_\ell]_t.$$

Note that $\phi_{S,n}(t) = 0$ if $S \notin \mathrm{Stab}([2, n-1])$.

To prove Lemma 4.1, we will use a combinatorial model motivated by Angarone, Nathanson, and Reiner's combinatorial proof of the permutation representation lift of the Poincaré duality and the Hard Lefschetz theorem in [2, Theorem 1.1].

Definition 4.2. For $n \geq 2$, consider sequences $\mathbf{w} = w_1 w_2 \ldots w_{n-1}$ in three symbols $\bullet$, $\times$, and a blank space , defined as follows:

(W1) If $w_i = \bullet$, then $w_{i-1} = \times$ (hence $w_1 \neq \bullet$ and there is no consecutive $\bullet$ in $\mathbf{w}$).

Say that $S = \{s_1 < s_2 < \ldots < s_\ell\} \subseteq \mathrm{Stab}([2, n-1])$ is the set of indices s with $w_s = \bullet$.

(W2) If there is $s_i < j < s_{i+1}$ such that $w_j = \times$, then $w_j = w_{j+1} = \ldots = w_{s_{i+1}-1} = \times$.

(W3) If there is $s_\ell < j < n$ such that $w_j = \times$, then $w_j = w_{j+1} = \ldots = w_{n-1} = \times$.

Denote by $\mathcal{W}_n$ the set of such sequences of length $n-1$. For $\mathbf{w} \in \mathcal{W}_n$, let $\mathrm{Dot}(\mathbf{w}) = \{i \in [n-1] : w_i = \bullet\}$ and let $\mathrm{cro}(\mathbf{w})$ be the number of $\times$ in $\mathbf{w}$.

Proof of Lemma 4.1. We show that (4.2) counts in two ways the sum of $t^{\mathrm{cro}(\mathbf{w})}$ for sequences $\mathbf{w}$ in $\mathcal{W}_n$ satisfying $\mathrm{Dot}(\mathbf{w}) \supseteq T$.

First, for fixed $S \supseteq T$, $S = \{s_1 < \ldots < s_\ell\} \in \mathrm{Stab}([2, n-1])$, a sequence $\mathbf{w} = w_1 \ldots w_{n-1} \in \mathcal{W}_n$ with $\mathrm{Dot}(\mathbf{w}) = S$ has $w_{s_j} = \bullet$ for all j. Between $w_{s_{j-1}}$ and w_{s_j}, we have $w_{s_j - 1} = \times$ and we can choose to place from 0 to $s_j - s_{j-1} - 1$ consecutive $\times$s on the left of $\times = w_{s_j-1}$. Hence the section between $w_{s_{j-1}}$ and w_{s_j} contributes $t[s_j - s_{j-1} - 1]_t$. In total, we have

$$\sum_{\substack{\mathbf{w} \in \mathcal{W}_n \\ \mathrm{Dot}(\mathbf{w}) = S}} t^{\mathrm{cro}(\mathbf{w})} = t^{|S|}[s_1 - 1]_t[s_2 - s_1 - 1]_t \ldots [s_\ell - s_{\ell-1} - 1]_t[n - s_\ell]_t = \phi_{S,n}(t).$$

Therefore, $\sum_{\substack{\mathbf{w} \in \mathcal{W}_n \\ \mathrm{Dot}(\mathbf{w}) \supseteq T}} t^{\mathrm{cro}(\mathbf{w})} = \sum_{T \subseteq S \subseteq [n-1]} \phi_{S,n}(t)$.

On the other hand, for any sequence $\mathbf{w} = w_1 \ldots w_{n-1} \in \mathcal{W}_n$ with $\mathrm{Dot}(\mathbf{w}) \supseteq T$, since $w_i = \bullet$ for $i \in T$, by the definition of the sequence we have $w_{i-1} = \times$ for $i \in T$. These $\times$ contribute the factor $t^{|T|}$ on the right hand side of (4.2). In the remaining $n - 1 - 2|T|$ positions, we can choose to place a $\times$ or not, which gives the factor $(1+t)^{n-1-2|T|}$ on the right-hand side of (4.2). After this process, there is a unique way to add $\bullet$ so that the sequence is in $\mathcal{W}_n$. This implies that

$$\sum_{\substack{\mathbf{w}\in\mathcal{W}_n\\ \mathsf{Dot}(\mathbf{w})\supseteq T}} t^{\mathsf{cro}(\mathbf{w})} = t^{|T|}(1+t)^{n-1-2|T|}.$$

$\square$

Example 4.3. Here is an example of the procedure in the proof of Lemma 4.1. Let $n=12$, $T=\{3,9\}\subseteq \mathsf{Stab}([2,11])$.

In the first part of the two-way counting, say $S=\{3,5,9\}\supseteq T$, so we place $\bullet$ at 3, 9 and $\times$ at 2, 8; and $\bullet$ at 5, $\times$ at 4 as follows:

$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11\\ \_ & \times & \bullet & \times & \bullet & \_ & \_ & \times & \bullet & \_ & \_ \end{array}$$
$$\Big|\quad t[2]_t \quad\Big|\quad t[1]_t \quad\Big|\quad t[3]_t \quad\Big|\quad [3]_t \quad\Big|$$

Then for example, between $w_5=\bullet$ and $w_9=\bullet$, we can have either $w_6=w_7=$blank spaces, $w_8=\times$; or $w_6=$blank space, $w_7=w_8=\times$; or $w_6=w_7=w_8=\times$. These three cases contribute $t+t^2+t^3=t[3]_t$. Hence we have

$$\sum_{\substack{\mathbf{w}\in\mathcal{W}_{12}\\ \mathsf{Dot}(\mathbf{w})=\{3,5,9\}}} t^{\mathsf{cro}(\mathbf{w})} = t[2]_t\cdot t[1]_t\cdot t[3]_t\cdot [3]_t = t^3[2]_t[1]_t[3]_t[3]_t.$$

In the second part of the two-way counting, we count sequences $\mathbf{w}$ with $w_3=w_9=\bullet$ and $w_2=w_8=\times$. We can choose each of the remaining 7 positions to be either a $\times$ or a blank space. For each choice there is exactly one way to add $\bullet$ to complete the sequence in $\mathcal{W}_{12}$. Hence this implies

$$\sum_{\substack{\mathbf{w}\in\mathcal{W}_{12}\\ \mathsf{Dot}(\mathbf{w})\supseteq\{3,9\}}} t^{\mathsf{cro}(\mathbf{w})} = t^{|\{2,8\}|}(1+t)^{|\{1,4,5,6,7,10,11\}|} = t^2(1+t)^{12-1-2\cdot 2}.$$

For example, say we choose $w_1=w_4=w_5=w_{10}=\times$, then we have

$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11\\ \times & \times & \bullet & \times & \times & \_ & \_ & \times & \bullet & \times & \_ \end{array}.$$

The only way to complete the above sequence into a sequence in $\mathcal{W}_{12}$ by adding $\bullet$ is as follows:

$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11\\ \times & \times & \bullet & \times & \times & \bullet & \_ & \times & \bullet & \times & \bullet \end{array}$$

The augmented case can be proved similarly with a slightly different combinatorial model.

5 Uniform matroids and q-uniform matroids

In this section, we apply Theorem 3.4 and Theorem 3.5 to the uniform matroid $U_{r,n}$ of rank r and the symmetric group $\mathfrak{S}_n$. In this case, the lattice of flats $\mathcal{L}(U_{r,n})$ is the rank-selected subposet $(B_n)_{[r-1]}$ of the Boolean lattice $B_n = (2^{[n]}, \subseteq)$. For the background on representation theory of $\mathfrak{S}_n$, standard Young tableaux, and symmetric functions in this section, we refer the readers to Wachs [24] and Stanley [21]. For an introduction to the permutation statistics inv, maj, etc., see Chapter 1 of Stanley [22].

Let ch be the *Frobenius characteristic map*. For a graded $\mathbb{C}\mathfrak{S}_n$-module $V = \bigoplus_i V_i$, the *graded Frobenius series* of V is defined to be $\mathsf{grFrob}(V, t) := \sum_i \mathrm{ch}(V_i) t^i$.

Let λ, μ be two partitions such that $\mu \subseteq \lambda$ (i.e. $\mu_i \leq \lambda_i$ for all i). The connected skew shape λ/μ is said to be a *ribbon* if two consecutive rows always overlap in exactly one cell. A ribbon $H_{R,n}$ can be described in terms of the number n of its cells and its *descent set* $R \subseteq [n-1]$. Given a ribbon of n cells, we number its cells from 1 to n, filling left to right in each row and filling the rows from bottommost to topmost. We call a cell i a *descent* of the ribbon if cell $i+1$ is directly above cell i. Then the collection of all its descents is called its *descent set*.

Example 5.1. Let $n = 7$. Then the diagrams of $H_{\{2,4\},7}$ and $H_{\{1,3\},7}$ are

$$\begin{array}{|c|c|c|c|c|} \hline & & 5 & 6 & 7 \\ \hline & 3 & \boxed{4} & & \\ \hline 1 & \boxed{2} & & & \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|c|} \hline & 4 & 5 & 6 & 7 \\ \hline 2 & \boxed{3} & & & \\ \hline \boxed{1} & & & & \\ \hline \end{array}.$$

Let $S^{H_{R,n}}$ be the *Specht module* of a ribbon shape, also known as the *Foulkes representation* of $\mathfrak{S}_n$, with $\mathrm{ch}(S^{H_{R,n}}) = s_{H_{R,n}}$ the *ribbon Schur function*.

Applying Theorem 3.4 to the uniform matroids $U_{r,n}$ then from [20, Theorem 4.3] or [24, Theorem 3.4.4], one obtains the graded Frobenius series of $A(U_{r,n})_{\mathbb{C}}$. Applying the operator $\prod_{i=1}^n (1-q^i)\mathrm{ps}_q(-)$, where ps_q is the stable principle specialization, to the graded Frobenius series gives the Hilbert series of the q-Uniform matroid $U_{r,n}(q)$ (See [15, Section 3] for the explanation). Note that the stable principle specialization on the ribbon Schur function gives

$$\prod_{i=1}^{n}(1-q^i)\mathrm{ps}_q(s_{H_{R,n}}) = \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathsf{DES}(\sigma)=R}} q^{\mathsf{maj}(\sigma^{-1})} = \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathsf{DES}(\sigma)=R}} q^{\mathsf{inv}(\sigma)}$$

(see [18, Theorem 4.4] and its proof for a detailed discussion of the above identity).

The following corollary generalizes Corollary 4.10 and 4.11 in the author's previous work [15]. It also extends Shareshian and Wachs' Theorem 7.3 in [17] and Corollary 3.2, Theorem 4.4, Theorem 6.1 in [18] (when $r = n$ or $n-1$).

Corollary 5.2. *For any positive integer n and $1 \leq r \leq n$,*

$$\begin{aligned}\mathsf{grFrob}(A(U_{r,n})_{\mathbb{C}}, t) &= \sum_{R \in \mathsf{Stab}([2,r-1])} s_{H_{R,n}} t^{|R|}(1+t)^{r-1-2|R|} \\ &= \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} \left(\sum_{\substack{R \in \mathsf{Stab}([2,r-1]) \\ |R|=k}} s_{H_{R,n}} \right) t^k (1+t)^{r-1-2k}\end{aligned}$$

and

$$\mathsf{Hilb}(A(U_{r,n}(q)), t) = \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} \xi_{r,n,k}(q) t^k (1+t)^{r-1-2k}$$

where

$$\xi_{r,n,k}(q) = \sum_{\sigma} q^{\mathsf{maj}(\sigma^{-1})} = \sum_{\sigma} q^{\mathsf{inv}(\sigma)}$$

and the sum runs through all $\sigma \in \mathfrak{S}_n$ for which $\mathsf{DES}(\sigma) \in \mathsf{Stab}([2, r-1])$ has k elements.

Similarly, for the augmented case, the following corollary generalizes Corollary 4.13 and 4.14 in the author's previous work in [15]. It also extends Shareshian and Wachs' Corollary 3.2, Theorem 3.4, 4.4, 4.5, and Corollary 5.4 in [18] (when $r = n$ or $n-1$).

Corollary 5.3. *For any positive integer n and $1 \leq r \leq n$,*

$$\begin{aligned}\mathsf{grFrob}(\widetilde{A}(U_{r,n})_{\mathbb{C}}, t) &= \sum_{R \in \mathsf{Stab}([r-1])} s_{H_{R,n}} t^{|R|}(1+t)^{r-2|R|} \\ &= \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \left(\sum_{\substack{R \in \mathsf{Stab}([r-1]) \\ |R|=k}} s_{H_{R,n}} \right) t^k (1+t)^{r-2k}\end{aligned}$$

and

$$\mathsf{Hilb}(\widetilde{A}(U_{r,n}(q)), t) = \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \widetilde{\xi}_{r,n,k}(q) t^k (1+t)^{r-2k},$$

where

$$\widetilde{\xi}_{r,n,k}(q) = \sum_{\sigma} q^{\mathsf{maj}(\sigma^{-1})} = \sum_{\sigma} q^{\mathsf{inv}(\sigma)}$$

and the sum runs through all $\sigma \in \mathfrak{S}_n$ for which $\mathsf{DES}(\sigma) \in \mathsf{Stab}([r-1])$ has k elements.

For a partition λ of n, denoted by $\lambda \vdash n$, let $SYT(\lambda)$ be the set of standard Young tableaux of shape λ. For $P \in SYT(\lambda)$, let $\mathsf{DES}(P)$ be the set of entries (called descents) $i \in [n-1]$ such that $i+1$ appears in a lower row than i in P, and $\mathsf{des}(P) = |\mathsf{DES}(P)|$. Let $\mathsf{grFrob}(A(U_{r,n})_{\mathbb{C}}, t) = \sum_{\lambda \vdash n} P^r_\lambda(t) s_\lambda$ and $\mathsf{grFrob}(\widetilde{A}(U_{r,n})_{\mathbb{C}}, t) = \sum_{\lambda \vdash n} \widetilde{P^r_\lambda}(t) s_\lambda$. The following result generalizes Corollary 2.41 in Athanasiadis [3].

Corollary 5.4. *For $\lambda \vdash n$, we have*

$$P^r_\lambda(t) = \sum_{\substack{P \in SYT(\lambda) \\ \mathsf{DES}(P) \in \mathsf{Stab}([2,r-1])}} t^{\mathsf{des}(P)}(1+t)^{r-1-2\mathsf{des}(P)} = \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} \xi_{r,\lambda,k}\, t^k (1+t)^{r-1-2k}$$

and

$$\widetilde{P}^r_\lambda(t) = \sum_{\substack{P \in SYT(\lambda) \\ \mathsf{DES}(P) \in \mathsf{Stab}([r-1])}} t^{\mathsf{des}(P)}(1+t)^{r-2\mathsf{des}(P)} = \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \widetilde{\xi}_{r,\lambda,k}\, t^k (1+t)^{r-2k}$$

where $\xi_{r,\lambda,k}$ (respectively, $\widetilde{\xi}_{r,\lambda,k}$) is the number of tableaux $P \in SYT(\lambda)$ for which $\mathsf{DES}(P) \in \mathsf{Stab}([2,r-1])$ (respectively, $\mathsf{Stab}([r-1])$) has k elements. In particular, $P^r_\lambda(t)$ and $\widetilde{P}^r_\lambda(t)$ are γ-positive for every r and every partition λ.

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