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The positive orthogonal Grassmannian

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Abstract. The Plücker positive region $\mathrm{OGr}_+(k, 2k)$ of the orthogonal Grassmannian emerged as the positive geometry behind the ABJM scattering amplitudes. In this paper we initiate the study of the positive orthogonal Grassmannian $\mathrm{OGr}_+(k, n)$ for general values of k, n. We determine the boundary structure of the quadric $\mathrm{OGr}_+(1, n)$ in $\mathbb{P}_+^{n-1}$ and show that it is a positive geometry. We show that $\mathrm{OGr}_+(k, 2k+1)$ is isomorphic to $\mathrm{OGr}_+(k+1, 2k+2)$ and connect its combinatorial structure to matchings on $[2k+2]$. Finally, we show that in the case $n > 2k+1$, the *positroid cells* of $\mathrm{Gr}_+(k, n)$ no longer suffice to induce a CW cell decomposition of $\mathrm{OGr}_+(k, n)$.

Keywords: Orthogonal Grassmannian, Flag variety, Positive geometry.

1 Introduction

Let $n \geq k$ be positive integers and denote by $\mathrm{Gr}(k, n)$ the *Grassmannian* of k-dimensional subspaces of $\mathbb{C}^n$. The *positive Grassmannian* $\mathrm{Gr}_+(k, n)$ is the semialgebraic set in $\mathrm{Gr}(k, n)$ where all Plücker coordinates are real and nonnegative. The matroid stratification of the Grassmannian [9] induces a natural decomposition of $\mathrm{Gr}_+(k, n)$ into the so-called *positroid cells*, see [12].

After Postnikov's landmark paper [12], the positive Grassmannian became a rich object of research in algebraic combinatorics [13, 7, 14]. Its study accelerated, in recent years, largely due to its unexpected and profound connection to Physics, in particular shallow water waves [11, 1] and scattering amplitudes in quantum field theory [4, 3, 2]. The object of study in this article is the *positive orthogonal Grassmannian* $\mathrm{OGr}_+^{\omega}(k, n)$.

Definition 1.1. Let $\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a non-degenerate symmetric bilinear form. We denote by $\mathrm{OGr}^{\omega}(k, n)$ the algebraic variety of *isotropic* k-dimensional subspaces V of $\mathbb{C}^n$ with respect to ω i.e. $\omega(x, y) = 0$ for any $x, y \in V$. The positive orthogonal Grassmannian $\mathrm{OGr}_+^{\omega}(k, n)$ is the semi-algebraic subset of $\mathrm{OGr}^{\omega}(k, n)$ where the Plücker coordinates are all real and have the same sign.

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In the special case $n = 2k$ and $\omega(x,y) = \sum_{i=1}^{2k} (-1)^{i-1} x_i y_i$, this object was first studied in the context of ABJM scattering amplitudes [10] and later connected to the Ising model in [8]. In this paper we initiate the study of $\mathrm{OGr}_+^{\omega}(k,n)$ for general values of k,n with respect to the quadratic form

$$\omega_0(x,y) = x_1y_1 - x_2y_2 + \cdots + (-1)^{n-1} x_n y_n. \tag{1.1}$$

The choice of the quadratic form ω is extremely important. For certain quadratic forms the variety $\mathrm{OGr}^{\omega}(k,n)$ has no real points, or its positive part is not full dimensional.

This article is organized as follows. In Section 2 we collect some facts on the geometry of the orthogonal Grassmannian $\mathrm{OGr}^{\omega}(k,n)$. In particular we determine the ideal of quadrics that cut out $\mathrm{OGr}(k,n)$ in $\mathbb{P}(\wedge^k \mathbb{C}^n)$, and determine a Gröbner basis for this ideal. In Section 3 we investigate $\mathrm{OGr}_+^{\omega_0}(1,n)$ with respect to the alternating form (1.1). Namely, we describe its face structure and show that it is a positive geometry. Section 4 is devoted to $\mathrm{OGr}_+^{\omega_0}(k,2k+1)$. In this section we show that $\mathrm{OGr}_+^{\omega_0}(k,2k+1)$ is isomorphic to $\mathrm{OGr}_+^{\omega_0}(k+1,2k+2)$ and we relate the face structure of $\mathrm{OGr}_+^{\omega_0}(k,2k+1)$ to matchings on $[2k+2]$. In Section 5 we initiate the study of $\mathrm{OGr}_+^{\omega_0}(k,n)$ starting with the case $k=2$. Already in this specific case, we show that the positroid cell decomposition of $\mathrm{Gr}_+(2,n)$ is no longer sufficient to induce a CW cell decomposition of $\mathrm{OGr}_+^{\omega_0}(2,n)$.

2 Commutative algebra and geometry of $\mathrm{OGr}(k,n)$

In this section we collect some facts on the algebraic variety $\mathrm{OGr}^{\omega}(k,n)$ over $\mathbb{C}$. Since all non-degenerate symmetric bilinear forms over $\mathbb{C}$ are equivalent to the standard inner product $(\cdot,\cdot)$, the varieties $\mathrm{OGr}^{\omega}(k,n)$ for different ω are isomorphic. So, in this section we may assume that ω is $(\cdot,\cdot)$, and we suppress ω and write $\mathrm{OGr}(k,n)$. Recall that the Grassmannian $\mathrm{Gr}(k,n)$ comes with $\binom{n}{k}$ *Plücker* coordinates, which we denote by p_I for any subset $I = \{i_1 < i_2 < \cdots < i_k\}$ of $[n]$.

Theorem 2.1. *The orthogonal Grassmannian* $\mathrm{OGr}(k,n)$ *is cut out in* $\mathbb{P}(\wedge^k\mathbb{C}^n)$ *by the Plücker relations and the following* $\frac{1}{2}\binom{n}{k-1}\left(\binom{n}{k-1}+1\right)$ *equations:*

$$\sum_{\ell=1}^{n} \varepsilon(I\ell)\varepsilon(J\ell)\ p_{I\ell}p_{J\ell} = 0, \quad \textit{for } I,J \in \binom{[n]}{k-1}. \tag{2.1}$$

where $\epsilon(I\ell) = (-1)^{|\{i\in I:\ i>\ell\}|}$ *denotes the sign of the permutation that sorts* $I\ell$.

Remark 2.2. In the case of the bilinear form (1.1), the equations (2.1) become:

$$\sum_{\ell=1}^{n} (-1)^{\ell-1}\epsilon(I\ell)\epsilon(J\ell)p_{I\ell}p_{J\ell} = 0, \quad \text{for} \quad I,J \in \binom{[n]}{k-1}. \tag{2.2}$$

Proposition 2.3. *The variety* $\mathrm{OGr}(k,n)$ *is empty if* $n < 2k$*. When* $n = 2k$ *it splits into two irreducible connected components, and it is irreducible when* $n > 2k$*. Moreover we have:*

$$\dim(\mathrm{OGr}(k,n)) = k(n-k) - \binom{k+1}{2} \quad \text{for } n \geq 2k.$$

Following [6], let $Y_{k,n}$ denote *Young's lattice*. This is a poset whose elements are subsets of size k in $[n]$ and the order relation in $Y_{k,n}$ is:

$$\langle i_1 < \cdots < i_k \rangle \leq \langle j_1 < \cdots < j_k \rangle \quad \text{if} \quad i_1 \leq j_1,\ i_2 \leq j_2,\ \ldots,\ i_{k-1} \leq j_{k-1} \text{ and } i_k \leq j_k.$$

We denote by $\widetilde{Y}_{k,n}$ another copy of Young's lattice. As a set $\widetilde{Y}_{k,n} = \binom{[n]}{n-k}$ and the order relation is given by:

$$[i'_1 < \cdots < i'_{n-k}] \leq [j'_1 < \cdots < j'_{n-k}] \quad \text{if} \quad i'_1 \geq j'_1,\ \ldots,\ i'_{n-k} \geq j'_{n-k}.$$

Finally we denote by $\mathcal{P}_{k,n}$ the poset which, as a set, is the disjoint union of $Y_{k,n}$ and $\widetilde{Y}_{k,n}$. All order relations in $Y_{k,n}$ and $\widetilde{Y}_{k,n}$ remain order relations in $\mathcal{P}_{k,n}$ and in addition to these relations we add $\binom{2k}{k}$ covering relations:

$$[j'_1 < \cdots < j'_{n-k}] \quad < \quad \langle i_1 < \cdots < i_k \rangle$$

whenever $\{1,2,3,\ldots,2k\} = \{i_1,\ldots,i_k\} \sqcup \{j_1,\ldots,j_k\}$ is a partition where $\{j_1,\ldots,j_k\}$ is the complement $[n] \setminus \{j'_1,\ldots,j'_{n-k}\}$. See Figure 1 for an example.

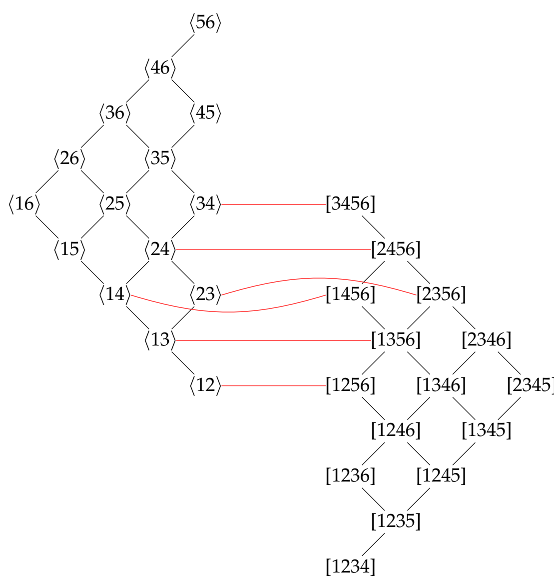


Figure 1: The poset $\mathcal{P}_{2,6}$ glued from $Y_{2,6}$ and $\widetilde{Y}_{2,6}$ using the covering relations in red.

An incomparable pair $(\langle i_1,\dots,i_k\rangle,\langle j_1,\dots,j_k\rangle)$ or $(\langle i_1,\dots,i_k\rangle,[j'_1,\dots,j'_{n-k}])$ in $\mathcal{P}_{k,n}$ yields a non-semistandard Young tableau μ of shape (k,k) or λ of shape $(n-k,k)$:

$$\mu = \begin{bmatrix} j_1 & \cdots & j_{\ell-1} & \boldsymbol{j_\ell} & \boldsymbol{j_{\ell+1}} & \cdots & \boldsymbol{j_k} \\ \boldsymbol{i_1} & \cdots & \boldsymbol{i_{\ell-1}} & \boldsymbol{i_\ell} & i_{\ell+1} & \cdots & i_k \end{bmatrix}, \quad \lambda = \begin{bmatrix} j'_1 & \cdots & j'_{\ell-1} & \boldsymbol{j'_\ell} & \boldsymbol{j'_{\ell+1}} & \cdots & \boldsymbol{j'_k} & \cdots & \boldsymbol{j'_{n-k}} \\ \boldsymbol{i_1} & \cdots & \boldsymbol{i_{\ell-1}} & \boldsymbol{i_\ell} & i_{\ell+1} & \cdots & i_k & & \end{bmatrix}. \tag{2.3}$$

The tableau μ or λ being non-semistandard means that there exists ℓ in $[k]$ such that:

$$i_1 < \cdots < i_\ell < j_\ell < \cdots < j_k \quad \text{or} \quad i_1 < \cdots < i_\ell < j'_\ell < \cdots < j'_{n-k}. \tag{2.4}$$

We pick ℓ to be the smallest index with this property. The strictly increasing sequences of integers in (2.4) are highlighted in bold in (2.3). Now consider the permutations π of the sequence $i_1 < \cdots < i_\ell < j_\ell < \cdots < j_k$ which make the first ℓ entries and the last $k-\ell+1$ entries separately increasing, and similarly, the permutations σ of the sequence $i_1 < \cdots < i_\ell < j'_\ell < \cdots < j'_{n-k}$ which make the first ℓ entries and the last $n-k-\ell+1$ entries separately increasing. Such permutations permute the bold entries in the tableaux μ and λ in (2.3) and yield

$$\begin{aligned} \pi(\mu) &= \begin{bmatrix} j_1 & \cdots & j_{\ell-1} & \pi(\boldsymbol{j_\ell}) & \pi(\boldsymbol{j_{\ell+1}}) & \cdots & \pi(\boldsymbol{j_k}) \\ \pi(\boldsymbol{i_1}) & \cdots & \pi(\boldsymbol{i_{\ell-1}}) & \pi(\boldsymbol{i_\ell}) & i_{\ell+1} & \cdots & i_k \end{bmatrix}, \\ \sigma(\lambda) &= \begin{bmatrix} j'_1 & \cdots & j'_{\ell-1} & \pi(\boldsymbol{j'_\ell}) & \pi(\boldsymbol{j'_{\ell+1}}) & \cdots & \pi(\boldsymbol{j'_k}) & \cdots & \pi(\boldsymbol{j'_{n-k}}) \\ \pi(\boldsymbol{i_1}) & \cdots & \pi(\boldsymbol{i_{\ell-1}}) & \pi(\boldsymbol{i_\ell}) & i_{\ell+1} & \cdots & i_k & & \end{bmatrix}. \end{aligned}$$

Summing over these permutations, the tableaux μ and λ yield quadrics

$$\begin{aligned} f_\mu &:= \sum_\pi \operatorname{sign}(\pi)\ \langle \pi(i_1),\dots,\pi(i_\ell), i_{\ell+1},\dots i_k\rangle\ \langle j_1,\dots,j_{\ell-1},\pi(j_\ell),\dots,\pi(j_k)\rangle, \\ f_\lambda &:= \sum_\pi \operatorname{sign}(\pi)\ \langle \pi(i_1),\dots,\pi(i_\ell), i_{\ell+1},\dots i_k\rangle\ [j'_1,\dots,j'_{\ell-1},\pi(j'_\ell),\dots,\pi(j'_k)]. \end{aligned} \tag{2.5}$$

Here, whenever $J' = \{j'_1 < \cdots < j'_{n-k}\}$ and $[n]\setminus J' = \{\bar{j}_1 < \cdots < \bar{j}_k\}$ we set

$$[j'_1,\dots,j'_{n-k}] := (-1)^{\sum_{r=1}^{n-k} j'_r}\langle \bar{j}_1,\dots,\bar{j}_k\rangle.$$

Theorem 2.4. *The quadrics in (2.5) form a Gröbner basis for the ideal $\mathrm{I}_{k,n}$ in $\mathbb{C}[p_I]$ generated by the Plücker relations and the quadratic equations in (2.1) with respect to any monomial ordering given by a linear extension of the poset $\mathcal{P}_{k,n}$.*

Proposition 2.5. *Let $n > 2k$, $m := \lfloor n/2 \rfloor$, and set $D := k(n-k) - \binom{k+1}{2}$. The degree of*

$\mathrm{OGr}(k,n)$ *in the Plücker embedding is*

$$\begin{cases} D! \cdot \left(\prod_{\substack{1\le i\le k \\ k<j\le m}} \frac{1}{(2m-i-j)(j-i)} \right) \left(\prod_{1\le i<j\le k} \frac{2}{2m-i-j} \right), & \text{if } n=2m, \\ D! \cdot \left(\prod_{1\le i\le k} \frac{2}{2m-2i+1} \right) \left(\prod_{\substack{1\le i\le k \\ k<j\le m}} \frac{1}{(2m-i-j)(j-i)} \right) \left(\prod_{1\le i<j\le k} \frac{2}{2m-i-j+1} \right), & \text{if } n=2m+1. \end{cases} \tag{2.6}$$

Theorem 2.6. *When $n > 2k$, the ideal $\mathrm{I}_{k,n}$ in $\mathbb{C}\left[p_I : I \in \binom{[n]}{k}\right]$ generated by the Plücker relations and the quadrics in (2.1) is the prime ideal of $\mathrm{OGr}(k,n)$. In particular, the degree of $I_{k,n}$ is given by (2.6).*

Remark 2.7. The ideal $\mathrm{I}_{k,2k}$ is clearly not prime since $\mathrm{OGr}(k,2k)$ has two irreducible connected components and we know that $I_{k,2k}$ cuts out $\mathrm{OGr}(k,2k)$ in $\mathbb{P}^{\binom{2k}{k}-1}$. Moreover, if $\omega = \omega_0$ is the sign alternating quadratic form in (1.1), then for any $p \in \mathrm{Gr}(k,2k)$ we have $p \in \mathrm{OGr}^{\omega_0}(k,2k)$ if and only if

$$p_I = p_{I^c} \quad \text{for all } I \in \binom{[2k]}{k} \quad \text{or } p_I = -p_{I^c} \quad \text{for all } I \in \binom{[2k]}{k}. \tag{2.7}$$

We define the *standard component* of $\mathrm{OGr}^{\omega_0}(k,2k)$ to be the connected component where $p_I = p_{I^c}$ for all $I \in \binom{[2k]}{k}$. The semialgebraic set in the standard component where all Plücker coordinates are real and have the same sign is denoted by $\mathrm{OGr}_+^{\omega_0}(k,2k)$.

3 The positive orthogonal Grassmannian $\mathrm{OGr}_+(1,n)$

In this section we study the positive geometry, in the sense of [2], of $\mathrm{OGr}(1,n)$ with the quadratic form ω_0 given by (1.1). From now on, unless specifically mentioned, we always work with ω_0. We denote by (p,q) its signature where $p = \lceil \frac{n}{2} \rceil$ and $q = \lfloor \frac{n}{2} \rfloor$.

We think of the elements of $[n]$ as vertices of a regular n-gon ordered clockwise from 1 to n. For each pair of non-empty subsets $A \subset [n] \cap (2\mathbb{Z}+1)$ and $B \subset [n] \cap 2\mathbb{Z}$, there exists a unique cycle $\sigma(A,B)$ in the symmetric group S_n such that $\sigma(A,B)$ has exactly one excedance and the support of $\sigma(A,B)$ is $A \sqcup B$. The set of such permutations[1] $\sigma(A,B)$ is denoted $\mathfrak{S}_{1,n}$. The set $\mathfrak{S}_{1,n}$ is endowed with a partial order given by:

$$\sigma(C,D) \preceq \sigma(A,B) \iff C \subseteq A \text{ and } D \subseteq B.$$

For $\sigma(A,B) \in \mathfrak{S}_{1,n}$, we denote by $\Pi_{\sigma(A,B)}$ the subset of $\mathbb{P}_+^{n-1}$ where $x_i = 0$ if and only if i is a fixed point of $\sigma(A,B)$ i.e. $i \notin A \sqcup B$. Here, $\mathbb{P}_+^{n-1}$ is simply $\mathrm{Gr}_+(1,n)$.

[1] These are decorated permutations with all fixed points having a "+" decoration.

Theorem 3.1. *The positive orthogonal Grassmannian* $\mathrm{OGr}_+(1,n)$ *is combinatorially isomorphic to the product of simplices* $\Delta_{p-1} \times \Delta_{q-1}$*. More precisely, the following hold:*

1. $\mathrm{OGr}_+(1,n) = \bigsqcup_{\sigma \in \mathfrak{S}_{1,n}} \mathrm{OGr}_+(1,n) \cap \Pi_\sigma.$
2. $\overline{\mathrm{OGr}_+(1,n) \cap \Pi_\sigma} = \bigsqcup_{\tau \preceq \sigma} \mathrm{OGr}_+(1,n) \cap \Pi_\tau.$
3. *If* $A = \{i_1 < \cdots < i_r\}$ *and* $B = \{j_1 < \cdots < j_m\}$, $\sigma = \sigma(A,B)$ *the cell* $\mathrm{OGr}_+(1,n) \cap \Pi_{\sigma(A,B)}$ *can be parameterized as follows. For each* $t_1, \ldots, t_{r-1}$ *and* $s_1, \ldots, s_{m-1}$ *in* $\mathbb{R}_{>0}$ *we get a point* $x \in \mathrm{OGr}_+(1,n) \cap \Pi_{\sigma(A,B)}$ *by setting* $x_i = 0$ *whenever* $i \notin (A \cup B)$ *and:*

$$\begin{aligned} x_{i_1} &= \frac{e^{t_1} - e^{-t_1}}{e^{t_1} + e^{-t_1}}, \quad x_{i_2} = \frac{2}{e^{t_1} + e^{-t_1}} \frac{e^{t_2} - e^{-t_2}}{e^{t_2} + e^{-t_2}}, \ \ldots, \quad x_{i_{r-1}} = \frac{2}{e^{t_{r-1}} + e^{-t_{r-1}}} \prod_{\ell=1}^{r-1} \frac{e^{t_\ell} - e^{t_\ell}}{e^{t_\ell} + e^{-t_\ell}}, \\ x_{j_1} &= \frac{e^{s_1} - e^{-s_1}}{e^{s_1} + e^{-s_1}}, \quad x_{j_2} = \frac{2}{e^{s_1} + e^{-s_1}} \frac{e^{s_2} - e^{-s_2}}{e^{s_2} + e^{-s_2}}, \ \ldots, \quad x_{j_{m-1}} = \frac{2}{e^{s_{m-1}} + e^{-s_{m-1}}} \prod_{\ell=1}^{m-1} \frac{e^{s_\ell} - e^{s_\ell}}{e^{s_\ell} + e^{-s_\ell}}. \end{aligned} \tag{3.1}$$

Example 3.2. The orthogonal Grassmannian $\mathrm{OGr}_+(1,5)$ has the same combinatorial structure as $\Delta_1 \times \Delta_2$. The poset of the boundaries of $\mathrm{OGr}_+(1,n)$ is depicted in Figure 2.

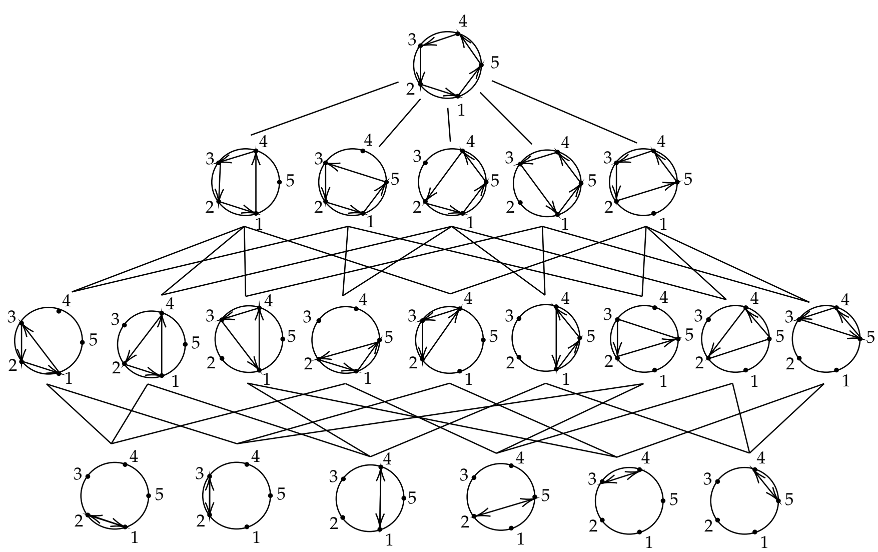


Figure 2: The Hasse diagram of the poset structure on $\mathfrak{S}_{1,5}$.

The next theorem shows that $\mathrm{OGr}_+(1,n)$ is a positive geometry. For convenience, we permute[2] the coordinates of $\mathbb{P}^{n-1}$ and write:

$$\mathrm{OGr}_+(1,n) = \left\{(y_1 : \cdots : y_n) \in \mathbb{P}_+^{n-1} \colon y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_n^2 = 0\right\}.$$

[2]Here, since $k = 1$, permuting the coordinates does not change the signs of the "minors".

Theorem 3.3. *The semi-algebraic set* $\mathrm{OGr}_+(1,n)$ *is a positive geometry. Its canonical form is:*

$$\Omega = (1 + u_{2,1}^2 + u_{3,1}^2 + \cdots + u_{p,1}^2)\ \frac{du_{2,1} \wedge du_{3,1} \wedge \cdots \wedge du_{n-1,1}}{u_{2,1}\ u_{3,1}\ \ \cdots\ \ u_{n-1,1}\ u_{n,1}^2},$$

where $u_{i,j} = x_i/x_1$ *in the projective coordinates* $(x_1 : \cdots : x_n)$ *of* $\mathbb{P}^{n-1}$.

4 The positive orthogonal Grassmannian $\mathrm{OGr}_+(k, 2k+1)$

We recall that we are working with the sign alternating form (1.1). The positroid cells of $\mathrm{Gr}_+(k,2k)$ induce a cell decomposition on the nonnegative orthogonal Grassmannian $\mathrm{OGr}_+(k,2k)$, and the cells of this decomposition are indexed by fixed-point-free involutions of $[2k]$. The face structure of $\mathrm{OGr}_+(k,2k)$ and the parametrization of its cells are studied in detail in [8, Section 5].

One of the reasons positroid cells induce a cell decomposition of the nonnegative orthogonal Grassmannian $\mathrm{OGr}_+(k,2k)$ is that, per (2.7), the latter is obtained by slicing $\mathrm{Gr}_+(k,2k)$ by a linear space. In general, one can obtain $\mathrm{OGr}_+(k,n)$ by slicing the positive flag variety with a linear space. For a subspace V in $\mathbb{C}^n$ of dimension k, we denote by $V^\perp$ its orthogonal complement with respect to the form (1.1).

Lemma 4.1. *The Hodge star map* $Gr(k,n) \to Gr(n-k,n)$, $V \to V^\perp$ *is given in Plücker coordinates by:*

$$q_J = p_{J^c}, \quad \text{for any } J \in \binom{[n]}{n-k},$$

where p_I *and* q_J *are Plücker coordinates in* $Gr(k,n)$ *and* $Gr(n-k,n)$ *respectively. In particular it restricts to an isomorphism of positive geometries between* $Gr_+(k,n)$ *and* $Gr_+(n-k,n)$.

Let $\mathcal{F}(k,n)$ be the 2-step flag variety of partial flags $V \subset W \subset \mathbb{C}^n$ where $\dim(V) = k$ and $\dim(W) = n-k$. The nonnegative part $\mathcal{F}_+(k,n)$ of $\mathcal{F}(k,n)$ is the semi-algebraic set of points $(V,W) \in \mathrm{Gr}_+(k,n) \times \mathrm{Gr}_+(n-k,n)$ such that $(V,W) \in \mathcal{F}(k,n)$. We denote by $\mathcal{D}$ the *diagonal* subset of $\mathbb{P}^{\binom{n}{k}} \times \mathbb{P}^{\binom{n}{n-k}}$ i.e.

$$\mathcal{D} := \left\{ (p,q)\colon\ p_I = q_{I^c} \quad \text{for any } I \in \binom{[n]}{k} \right\}.$$

Proposition 4.2. *The positive orthogonal Grassmannian* $\mathrm{OGr}_+(k,n)$ *is the intersection of the positive flag variety* $\mathcal{F}_+(k,n)$ *with* $\mathcal{D}$ *i.e.:*

$$\mathrm{OGr}_+(k,n) = \mathcal{F}_+(k,n) \cap \mathcal{D}. \tag{4.1}$$

This motivates the choice of the sign alternating form (1.1) in [8, 10]. However, unlike $\mathcal{F}_+(k,2k) \cong \mathrm{Gr}_+(k,2k)$, the nonnegative region $\mathcal{F}_+(k,n)$ is not well understood[3] for general k. This motivates the following question:

Problem 4.3. Study the face structure of $\mathcal{F}_+(k,n)$ and find a parametrization of its cells.

Proposition 4.4. *The homogeneous coordinate rings of the* 2*-step flag variety* $\mathcal{F}(k,2k+1)$ *and the Grassmannian* $Gr(k+1,2k+2)$ *are isomorphic.*

Theorem 4.5. *The orthogonal Grassmannians* $\mathrm{OGr}(k,2k+1)$ *and* $\mathrm{OGr}(k+1,2k+2)$ *can be identified through a linear isomorphism* $(q_J) \mapsto p_I = q_{I\cup\{2k+2\}}$*. This isomorphism restricts to an isomorphism of the positive regions* $\mathrm{OGr}_+(k,2k+1)$ *and* $\mathrm{OGr}_+(k,2k+1)$*.*

Remark 4.6. The equations that cut out $\mathrm{OGr}(k,2k+1)$ in $\mathrm{Gr}(k,2k+1)$ are all quadrics. It is remarkable that we can still describe the face structure of $\mathrm{OGr}_+(k,2k+1)$ from our understanding of the face structure of $\mathrm{OGr}_+(k+1,2k+2)$.

Example 4.7 ($\mathrm{OGr}_+(2,5)$)**.** The orthogonal Grassmannian $\mathrm{OGr}_+(2,5)$ is isomorphic to $\mathrm{OGr}_+(3,6)$. The Hasse diagram of the face poset of the latter is in [8, Figure 7]. Figure 3 gives the same Hasse diagram in the realizable permutations in $\mathrm{OGr}_+(2,5)$. These cells can be parameterized using the isomorphism in Theorem 4.5 and [8, Theorem 5.17 (i)].

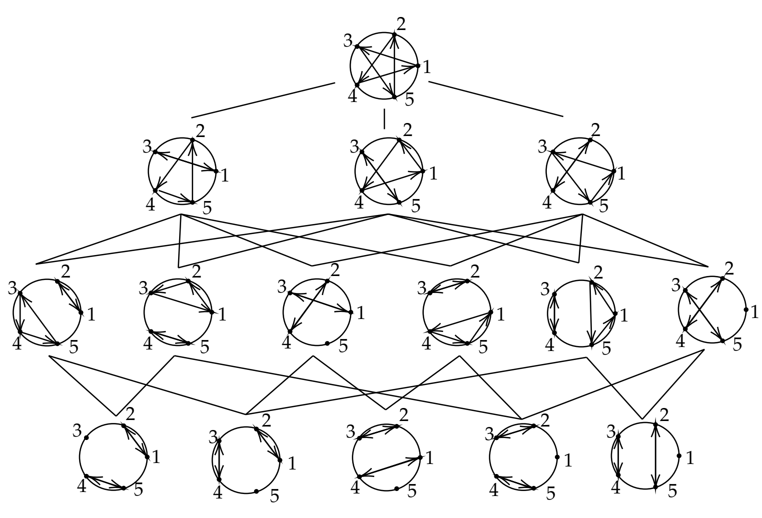


Figure 3: The face poset of $\mathrm{OGr}_+(2,5)$ matches that of $\mathrm{OGr}_+(3,6)$. See [8, Figure 7].

We finish this section by explaining how one goes from matchings τ on $[2k+2]$ to the realizable permutations in $[2k+1]$ i.e. permutations σ of $[2k+1]$ with corresponding positroid cell Π_σ such that $\Pi_\sigma \cap \mathrm{OGr}_+(k,2k+1)$ is nonempty. Let c denote the chord

[3]The Lusztig positive part of $\mathcal{F}(k,n)$ is well understood but it can be shown that the Plücker positive region $\mathcal{F}_+(k,n)$ strictly contains the Lusztig positive region when $n > 2k+1$, see [5].

in τ attached to the vertex $2k+2$ and, starting from the vertex $2k+2$, consider the largest sequence $c = c_1, c_2, \dots, c_r$ of pairwise intersecting chords of τ. Denote the $2r$ vertices of these chords by $i_1 < \cdots < i_{2r-1} < 2k+2$. Then the cell $\Pi_\tau \cap \mathrm{OGr}_+(k+1, 2k+2)$ is isomorphic to the cell $\Pi_\sigma \cap \mathrm{OGr}_+(k, 2k+1)$ where σ is the permutation of $[2k+1]$ obtained by replacing the chords $c_1, \dots, c_r$ with the unique cycle with support $\{i_1, \dots, i_{2r-1}\}$ and r excedances. See Figure 4 for an example.

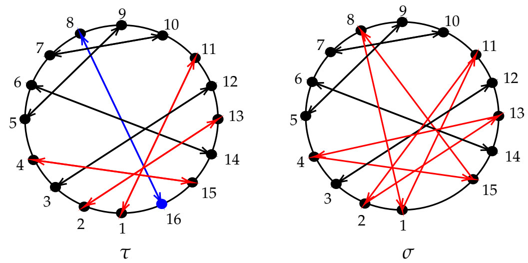


Figure 4: A matching τ of $[2k+2]$ and the corresponding permutation σ of $[2k+1]$ for $k = 7$. On the left, starting vertex 16 (in blue), the chords in red are longest sequence of chords $c_1, \dots, c_r$ that intersect pairwise. On the right, vertex 16 is deleted and the red chords turn into the unique cycle with support $\{1, 2, 4, 8, 11, 13, 15\}$ and 4 excedances.

5 What goes wrong when $n > 2k+1$ and $k > 1$?

In this section we show why positroid cells fail to induce a cell decomposition of the orthogonal Grassmannian $\mathrm{OGr}_+(k, n)$ as soon as $n > 2k+1$ and $k > 1$. Let us start with the following:

Definition 5.1. For any positroid $\mathscr{M}$ of type (k, n) and for any pair of subsets I, J of $[n]$ of size $k-1$ we define the following two subsets of $[n]$:

$$A^{\pm}_{IJ}(\mathscr{M}) = \{\ell \in [n] \colon I\ell, J\ell \in \mathscr{M} \text{ and } (-1)^{\ell-1}\epsilon_{I\ell}\epsilon_{J\ell} = \pm 1\}.$$

We say that $\mathscr{M}$ is an orthopositroid if for any $I, J \in \binom{[n]}{k-1}$ we have:

$$A^{+}_{IJ}(\mathscr{M}) = \emptyset \quad \iff \quad A^{-}_{IJ}(\mathscr{M}) = \emptyset.$$

Example 5.2. Let $n = 5$ and consider the two following positroids:

$$\mathscr{M}_1 = \{\{1,2\}, \{1,4\}, \{2,5\}, \{4,5\}\} \text{ and } \mathscr{M}_2 = \{\{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}\}.$$

We then have $A^{+}_{24}(\mathscr{M}_1) = \emptyset$ and $A^{-}_{24}(\mathscr{M}_1) = \{2\}$. So $\mathscr{M}_1$ is ***not*** an orthopositroid. One can check that $\mathscr{M}_2$ is an orthopositroid.

The motivation behind this definition is that if X is a point in $\mathrm{OGr}_+(k,n)$ and $\mathscr{M}_X$ is its associated positroid then $\mathscr{M}_X$ is necessarily an orthopositroid in the sense of Definition 5.1. This is because the Plücker coordinates of X satisfy the equations (2.2).

Conjecture 5.3. For each orthopositroid $\mathscr{M}$ of type (k,n), there exists X in $\mathrm{OGr}_+(k,n)$ such that $\mathscr{M}_X = \mathscr{M}$.

Since we will show that positroid cells do not give a cell decomposition of $\mathrm{OGr}_+(k,n)$, we refrain from elaborating on the realizability of orthopositroids for general k.

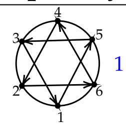
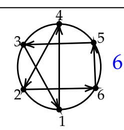
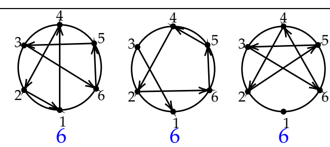
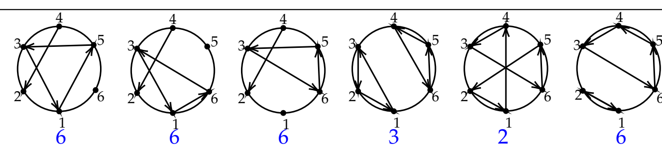
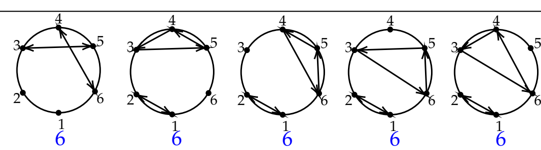
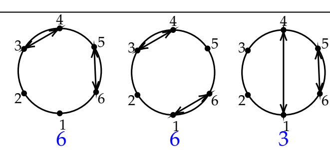
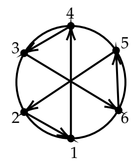
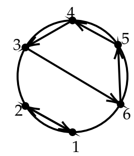
dim.	permutations up to cyclic symmetry	#
5		1
4		6
3		18
2		29
1		30
0		15

Table 1: The 99 realizable permutations in $\mathrm{OGr}_+(2,6)$, organized by dimension.

Let us start with $\mathrm{OGr}_+(2,6)$. An exhaustive computation shows that, out of all the positroids $\mathscr{M}$ (or decorated permutations σ) of type $(2,6)$, there are exactly 99 orthopositroids (or admissible permutations). We list them in Table 1. Let us focus on the following two orthopositroid cells in $\mathrm{OGr}_+(2,6)$:



σ



τ

Let $C_\sigma := \Pi_\sigma \cap \mathrm{OGr}_+(2,6)$ and $C_\tau := \Pi_\tau \cap \mathrm{OGr}_+(2,6)$ be the two positroid cells in $\mathrm{OGr}_+(2,6)$ corresponding to σ and τ respectively. We start by giving generic matrices M_σ, M_τ that parametrize the points of C_σ, C_τ respectively:

$$M_\sigma = \begin{bmatrix} 1 & 1 & 0 & 0 & -x & -x \\ 0 & 0 & 1 & 1 & y & y \end{bmatrix} \quad \text{and} \quad M_\tau = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & a & b & c \end{bmatrix}$$

for $x, y > 0$ and $a, b, c > 0$ such that $1 + b^2 = a^2 + c^2$.

The closure $\overline{C_\tau}$ of C_τ has the combinatorial type of a square and the closure $\overline{C_\sigma}$ of the cell C_σ has the combinatorial type of a triangle. The edges of the latter are given by:

$$e_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & b & b \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 1 & 1 & b & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & b & b \end{bmatrix} \quad b \geq 0.$$

The edge e_1 is one of the diagonals of the "square" C_τ. So the cell C_σ glues with the cell C_τ as in Figure 5.

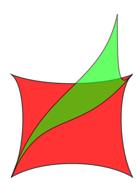


Figure 5: A cartoon of the cell C_σ (in green) glued to the cell C_τ (in red).

This shows that the positroid cells are not enough to induce a CW cell decomposition on $\mathrm{OGr}_+(2,6)$. In general this problem arises as soon as $n > 2k+1$. This is because whenever $n > 2k+1$ we have $n - 6 \geq 2(k-2)$, so we can extend a 2×6 matrix in $\mathrm{OGr}_+(2,6)$ by a $(k-2) \times (n-6)$ as follows

$$\left[\begin{array}{cccccccccccc|cccccc}
1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
0 & 0 & 1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
0 & \cdots & \cdots & & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & & & (0) & & & \\
\vdots & \cdots & \cdots & \cdots & 0 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 & & & & & & \\
 & & & & & & & & & & & & * & * & * & * & * & * \\
 & & & & & & (0) & & & & & & * & * & * & * & * & *
\end{array}\right].$$

We can then realize each positroid cell in $\mathrm{OGr}_+(2,6)$ as some positroid cell of $\mathrm{OGr}_+(k,n)$ and the same problem as above arises again. This highlights the need for new combinatorics to give a CW cell decomposition of $\mathrm{OGr}_+(k,n)$ when $n > 2k+1$ and $k > 1$.

Problem 5.4. Find a cell decomposition for $\mathrm{OGr}_+(k,n)$ when $n > 2k+1$ and describe the combinatorics behind its face poset.

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