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Standard Monomials for Positroid Varieties

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Abstract. We give an explicit characterization of the standard monomials for positroid varieties with respect to the Hodge degeneration and give a Gröbner basis. Furthermore, we show that promotion and evacuation biject standard monomials of a positroid variety with those of its cyclic shifts and w_0-reflection, respectively. The connection to promotion allows us to identify standard monomials of a positroid variety with Lam's cyclic Demazure crystal. Using a recurrence on the Hilbert series, we give an inductive formula for the characters of cyclic Demazure modules, solving a problem posted by Lam.

Keywords: positroid varieties, promotion, evacuation, Gröbner basis, standard monomial theory.

1 Introduction

Classical work of Hodge [4] described a particular set of bases for the homogeneous coordinate rings of the Grassmannian $\mathrm{Gr}(k, n)$ and its Schubert varieties under the Plücker embedding. Building on Hodge's ideas, Seshadri initiated the study of standard monomial theory (SMT), with the aim of giving standard monomial bases for the space of global sections of line bundles over a (generalized) flag variety G/P. The foundation of SMT was built in the works of Lakshmibai, Musili, and Seshadri, [17, 10, 8, 9]. The tools developed in SMT yield a wide range of applications, such as derivations of geometric properties (Cohen–Macaulayness, normality, singularities, cohomological vanishings, etc.) of Schubert varieties and character formulas for Demazure modules.

The main goal of our paper is to extend the work of Hodge to *positroid varieties* in $\mathrm{Gr}(k, n)$, based on works of Knutson–Lam–Speyer. Positroid varieties are the closed strata of the positroid stratification, the common refinement of the cyclically permuted Bruhat decompositions. These are also exactly the projections of Richardson varieties from the complete flag variety to $\mathrm{Gr}(k, n)$, and are the only compatibly split subvarieties with respect to the standard Frobenius splitting on the Grassmannian [6].

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Using the standard monomial theory of Richardson varieties [7, 2], Knutson–Lam–Speyer [6] described the Stanley–Reisner complexes of positroid varieties under the *Hodge degeneration*, a special kind of Gröbner degeneration. However, they did not give an explicit description (i.e., without referring to the chains in the Bruhat order of G/B) of the standard monomials. In this extended abstract, we present an explicit description of standard monomials for positroid varieties, analogous to Hodge's original description based on semistandard Young tableaux (Theorem A). This approach yields several applications in algebra, combinatorics, and representation theory. More specifically:

- an explicit Gröbner basis for positroid varieties with respect to the Hodge degeneration, analogous to the classical "straightening relations" (Theorem B).
- a connection between the *promotion* (resp. *evacuation*) on rectangular semistandard Young tableaux and rotations (resp. reflections) of positroid varieties (Theorem C).
- a formula for characters of *cyclic Demazure modules*, resolving a problem posted by Lam [11] (Theorem D).

2 Background

2.1 Grassmannians and Plücker embeddings

For $k \leq n \in \mathbb{Z}_{>0}$ and an algebraically closed field $\mathbf{k}$ of characteristic 0, the **Grassmannian** $\mathrm{Gr}(k,n)$ is

$$\mathrm{Gr}(k,n) = \{W \subseteq \mathbf{k}^n : \dim(W) = k\}.$$

For $W \in \mathrm{Gr}(k,n)$ and $\{w_1, \ldots, w_k\}$ a chosen basis of W. The **Plücker embedding** is the map $\iota : \mathrm{Gr}(k,n) \to \mathbb{P}(\Lambda^k(\mathbf{k}^n))$ sending W to $[w_1 \wedge \cdots \wedge w_k]$.

Set $R(k,n) := \mathbf{k}[[\mathbf{a}] : \mathbf{a} \in \binom{[n]}{k}]$ to be the homogeneous coordinate ring of the projective space $\mathbb{P}(\Lambda^k(\mathbf{k}^n))$. We extend the notation of $[\mathbf{a}]$ to all sequences $\mathbf{a} = (a_1, \ldots, a_k) \in [n]^k$, where we use the convention that for any permutation $\sigma \in \mathfrak{S}_n$,

$$[a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(k)}] = \mathrm{sign}(\sigma) \cdot [a_1, \ldots, a_k]. \tag{2.1}$$

In particular, this implies that $[\mathbf{a}] = 0$ if $a_i = a_j$ for some $i \neq j \in [n]$.

The defining ideal $\mathcal{J}$ of $\mathrm{Gr}(k,n)$ as a projective subvariety of $\mathbb{P}(\Lambda^k(\mathbf{k}^n))$ is generated by the following **straightening relations**:

Definition 1 ([20]). *Let $s \in [k], \alpha \in \binom{[n]}{s-1}, \beta \in \binom{[n]}{k+1}$ and $\gamma \in \binom{[n]}{k-s}$ where elements in α, β, γ are in increasing order. The straightening relation attached to α, β, γ is*

$$\sum_{I \in \binom{[k+1]}{s}} (-1)^{sign(I)} [\alpha_1, \ldots, \alpha_{s-1}, \beta_{i'_1}, \ldots, \beta_{i'_{k-s+1}}] \cdot [\beta_{i_1}, \ldots, \beta_{i_s}, \gamma_1, \ldots, \gamma_{k-s}]$$

where $I = \{i_1 < \cdots < i_s\} \subset [k+1]$, $\{i'_1 < \cdots < i'_{k-s+1}\} := [k+1] \setminus I$, *and* $sign(I) := \sum_{j=1}^{s} i_j - \binom{s+1}{2}$.

We will often[1] represent a monomial $\mathbf{m} = \prod_{i=1}^{d} [\mathbf{a}^{(i)}]$ as a $k \times d$ tableau, where column i is strictly increasing with entries in $\mathbf{a}^{(i)}$ for all i.

Example 2. *An example of a straightening relation in* $\mathrm{Gr}(4,8)$ *arises from the product* $[1,2,6,7] \cdot [3,4,5,8]$. *Here* $\alpha = \{1,2\}$, $\beta = \{3,4,5,6,7\}$, *and* $\gamma = \{8\}$. *The monomials are obtained by permuting the elements in* β *(colored in red).*

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 6 & 5 \\ \hline 7 & 8 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline 7 & 8 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 7 \\ \hline 6 & 8 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 7 \\ \hline 6 & 8 \\ \hline \end{array} \cdots$$

Definition 3. *A tableau is called* ***semistandard*** *if it is strictly increasing along columns and weakly increasing along rows.*

The homogeneous coordinate ring of $\mathrm{Gr}(k,n)$ is $\mathbf{k}[\mathrm{Gr}(k,n)] = R(k,n)/\mathcal{J}$. It is the direct sum of its graded pieces:

$$\mathbf{k}[\mathrm{Gr}(k,n)] = \bigoplus_{d=0}^{\infty} \mathbf{k}[\mathrm{Gr}(k,n)]_d,$$

where each $\mathbf{k}[\mathrm{Gr}(k,n)]_d$ is a finite dimensional $\mathbf{k}$-vector space spanned by the monomials $\{\prod_{i=1}^{d} [\mathbf{a}^{(i)}] : \mathbf{a}^{(i)} \in \binom{[n]}{k} \text{ for all } i\}$. The following classical theorem is originally due to Hodge [4] (c.f. [3, Lemma 7.2.3]).

Theorem 4. *The monomials* $\prod_{i=1}^{d} [\mathbf{a}^{(i)}]$ *that correspond to semistandard tableaux form a basis of* $\mathbf{k}[\mathrm{Gr}(k,n)]_d$.

Let $\mathcal{P} = (\binom{[n]}{k}, \leq)$ be the poset where

$$[c_1, \ldots, c_k] \leq [d_1, \ldots, d_k] \text{ if and only if } c_i \leq d_i \text{ for all } i = 1, \ldots, k.$$

Let $<_\omega$ be a degree revlex monomial order on the polynomial ring $R(k,n) := \mathbf{k}[[\mathbf{a}] : \mathbf{a} \in \binom{[n]}{k}]$ where the Plücker variables are ordered by some linear extension of $\mathcal{P}$.

Definition 5. *For a subvariety* $X \subseteq \mathrm{Gr}(k,n)$, *let* $\mathcal{J}_X$ *be its defining ideal under the Plücker embedding. A monomial* $\mathbf{m} = \prod_{i=1}^{d} [\mathbf{a}^{(i)}]$ *is called a* ***standard monomial*** *for* X *if* $\mathbf{m} \notin \mathrm{In}_\omega(\mathcal{J}_X)$.

[1] Sometimes we need to work with tableaux with permuted entries. In these cases, it is important to keep in mind that we may sort the columns as long as we keep track of signs.

For each $d \in \mathbb{N}$, the degree d standard monomials $\{\mathbf{m} : \deg(\mathbf{m}) = d, \mathbf{m} \notin \mathrm{In}_\omega(\mathcal{J}_X)\}$ form a basis of $\mathbf{k}[X]_d$. This basis is also known as the **standard monomial basis**. The monomials in Theorem 4 are standard monomials by the following:

Theorem 6 ([20]). *The straightening relations in Definition 1 form a Gröbner basis of $\mathcal{J}$ under the term order $<_\omega$. The initial ideal is*

$$\mathrm{In}_\omega(\mathcal{J}) = \langle [\mathbf{a}][\mathbf{b}] : \mathbf{a}, \mathbf{b} \textit{ not comparable in } \mathcal{P} \rangle.$$

2.2 Positroid varieties

We first recall a few sets of combinatorial objects that index positroid varieties.

Define the set of **bounded affine permutations** to be

$$\mathrm{Bound}(k,n) := \{f : \mathbb{Z} \to \mathbb{Z} | f(i+n) = f(i) + n, i \leq f(i) \leq i+n \text{ for all } i \in \mathbb{Z}\}. \quad (2.2)$$

We will sometimes write $f \in \mathrm{Bound}(k,n)$ as $f = [f(1)\ f(2) \cdots f(n)]$. The partial order on $\mathrm{Bound}(k,n)$ is inherited from the Bruhat order on affine permutations.

Given $V \in \mathrm{Gr}(k,n)$, let $\widetilde{V}$ denote a choice of $k \times n$ matrix such that $\mathrm{rowspan}(\widetilde{V}) = V$. We write $\widetilde{V} = [v_1, \cdots, v_n]$ where v_i is the ith column. We extend the sequence $v_1 \cdots v_n$ by setting $v_i = v_{i+n}$ for all $i \in [n]$, and denote by $\widetilde{V}_{[i,j]}$ the matrix with column vectors $v_i, \cdots, v_j$. Consider the affine permutation $f_{\widetilde{V}} : \mathbb{Z} \to \mathbb{Z}$ given by

$$f_{\widetilde{V}}(i) = \min\{j \geq i : v_i \in \mathrm{span}(v_{i+1}, \cdots, v_j)\}. \quad (2.3)$$

It is known that $f_V \in \mathrm{Bound}(k,n)$ and that all of $\mathrm{Bound}(k,n)$ can arise this way. Moreover, $f_{\widetilde{V}}$ only depends on the $V := \mathrm{rowspan}(\widetilde{V})$, so we may define f_V for $V \in \mathrm{Gr}(k,n)$.

The **open positroid variety** associated to a bounded affine permutation f is $\Pi_f^\circ := \{V \in \mathrm{Gr}(k,n) : f_V = f\}$ and the **positroid variety** is its Zariski closure $\Pi_f = \overline{\Pi_f^\circ}$. In fact, positroid varieties stratify the Grassmannian: $\Pi_f = \bigsqcup_{f' \geq f} \Pi_{f'}^\circ$.

Let $\chi : \mathrm{Gr}(k,n) \to \mathrm{Gr}(k,n)$ be the **cyclic rotation** such that for $V \in \mathrm{Gr}(k,n)$,

$$\widetilde{V} = [v_1, \cdots, v_n], \quad \chi(\widetilde{V}) := [v_n, v_1, \cdots, v_{n-1}], \quad \chi(V) := \mathrm{rowspan}(\chi(\widetilde{V})). \quad (2.4)$$

We abuse the notation and define the corresponding cyclic shift on $\mathrm{Bound}(k,n)$ as

$$\chi(f)(i) = f(i-1) + 1.$$

Lemma 7. *For $f \in \mathrm{Bound}(k,n)$, $\chi(\Pi_f) = \Pi_{\chi(f)}$.*

An equivalent way to define positroid varieties is through **cyclic rank matrices** as defined in [5, Corollary 3.12]. For any $f \in \mathrm{Bound}(k,n)$, write f as the $\infty \times \infty$ matrix with 1's at positions $(i, f(i))$ and 0's everywhere else. Let $r(f)$ be the infinite periodic matrix defined for all $i \in \mathbb{Z}$, $i \le j \le i+n$,

$$r(f)_{i,j} = |[i,j]| - \#\{\text{number of 1s in } f\text{'s matrix weakly southwest of } (i,j)\}.$$

Let $V \in \mathrm{Gr}(k,n)$ so that $f_V = f$, then $r(f)_{i,j} = \mathrm{rank}(v_i, \cdots, v_j)$. Furthermore,

$$\Pi_f^\circ = \{U \in \mathrm{Gr}(k,n) : \mathrm{rank}(\widetilde{U}_{[i,j]}) = r(f)_{i,j} \text{ for all } i \in \mathbb{Z}, j \in [i, i+n]\}$$

and Π_f is obtained by replacing "=" with "$\le$".

For any $\alpha \in [0, n-1]$ and $m < n$, define the cyclic interval

$$[\alpha+1, \alpha+m]^\circ = \begin{cases} \{i : \alpha+1 \le i \le \alpha+m\} & \text{if } \alpha+m \le n \\ \{i : \alpha+1 \le i \le n \text{ or } 1 \le i \le (\alpha+m) \bmod n\} & \text{if } \alpha+m > n \end{cases}.$$

We call the first kind of cyclic interval an **interval** and the second kind a **wrapped-around interval**. In the wrapped-around case, we sometimes write $[\alpha+1, \alpha+m-n]^\circ := [\alpha+1, \alpha+m]^\circ$.

For any $S \subset [n]$ and any $r \in \mathbb{N}$, define

$$\Pi_{S \le r} = \{V \in \mathrm{Gr}(k,n) : \mathrm{rank}(\widetilde{V}_S) \le r\},$$

where $\mathrm{rank}(\widetilde{V}_S)$ is the rank of the submatrix of $\widetilde{V}$ with column index S.

Lemma/Definition 8. *If S is a cyclic interval, we say $\Pi_{S\le r}$ is a **basic positroid variety**. This is indeed an instance of a positroid variety.*

For $f \in \mathrm{Bound}(k,n)$, the **essential set of** f is:

$$ess(f) := \{(i,j) : i \in \mathbb{Z}, j \in [i, i+n], f(i-1) > j, f^{-1}(j+1) < i, f(i) \le j, f^{-1}(j) \ge i\}.$$

The following statement, which follows from [6, Theorem 5.1] is crucial for our main results.

Proposition 9. *Every positroid variety is the scheme-theoretic intersection of basic positroid varieties:*

$$\Pi_f = \bigcap_{(i,j)\in ess(f)} \Pi_{[i,j]^\circ \le r(f)_{i,j}}.$$

For the purpose of this paper, We only need the existence of essential sets in the context of Proposition 9 rather than its precise description.

Example 10. *Set $k = 3$, $n = 6$. Let f be the bounded affine permutation*

$$\cdots [5, 2, 4, 7, 9, 12] \cdots$$

where $f(1) = 5$. Then $\Pi_f = \Pi_{[2]\le 0} \cap \Pi_{[2,4]\le 1} \cap \Pi_{[1,5]\le 2}$.

Let $w_0 : \mathrm{Gr}(k,n) \to \mathrm{Gr}(k,n)$ be the reflection map such that for

$$\widetilde{V} = [v_1, \ldots, v_n], \quad w_0(\widetilde{V}) = [v_n, \ldots, v_1].$$

It is straightforward to see that $w_0(\Pi_{[i,j]\leq r}) = \Pi_{[n+1-j,n+1-i]\leq r}$, hence

$$w_0(\Pi_f) = \bigcap_{(i,j)\in ess(f)} \Pi_{[n+1-j,n+1-i]^\circ \leq r(f)_{i,j}}.$$

Lemma/Definition 11. *The image $w_0(\Pi_f)$ is again a positroid variety. Define f^* to be the bounded affine permutation such that $\Pi_{f^*} = w_0(\Pi_f)$.*

3 Standard monomials for positroid varieties

Let $B(k,n,d)$ denote the set of rectangular semistandard tableaux of shape $k \times d$ with entries $\leq n$. We write $\mathbf{m} \in B(k,n,d)$ both for the monomial and its tableau. For a positroid variety Π_f, let $\mathcal{J}_f := \mathcal{J}_{\Pi_f}$ and

$$B_f(k,n,d) := \{\mathbf{m} \in B(k,n,d) : \mathbf{m} \notin \mathrm{In}(\mathcal{J}_f)\}.$$

be the set of degree-d standard monomials for Π_f.

The main theorem of this section is an explicit combinatorial description of $B_f(k,n,d)$.

Theorem A. *A monomial $\mathbf{m} = \prod_{i=1}^{d}[\mathbf{a}^{(i)}] \in B_f(k,n,d)$ if and only if $\mathbf{m} \in B(k,n,d)$ and*

1. *when $\Pi_f = \Pi_{S\leq r}$ for some interval S, $\mathbf{m}$ does not contain a* ***generalized antidiagonal*** *for $S \leq r$ (Definition 12);*
2. *when $\Pi_f = \Pi_{S\leq r}$ for some wrapped-around interval S, $\mathbf{m}^\vee$ does not contain a generalized antidiagonal for $S^\vee \leq r^\vee$ (Construction 14);*
3. *when $\Pi_f = \bigcap_i \Pi_{S_i\leq r_i}$, $\mathbf{m} \in \bigcap_i B_{f_i}(k,n,d)$ where Π_{f_i} is the basic positroid variety $\Pi_{S_i\leq r_i}$.*

We note that for (3), any positroid variety can be written as a finite intersection of basic poistroid varieties $\Pi_{S_i\leq r_i}$ (see Proposition 9).

Let $\mathbf{m} = \prod_{i=1}^{d}[\mathbf{a}^{(i)}]$ be a standard monomial of $\mathrm{Gr}(k,n)$ where the Plücker variables $\mathbf{a}^{(1)} \leq \mathbf{a}^{(2)} \leq \cdots \leq \mathbf{a}^{(d)}$ are sorted using the partial order in the Plücker poset.

Definition 12. *Fix an interval $S = [\alpha+1, \alpha+m]$ in $[n]$ and some $r < m$. A* ***generalized antidiagonal*** *of $\mathbf{m} \in B(k,n,d)$ for the rank condition $S \leq r$ is a vertical strip in the tableau $\mathbf{m}$ of size $r+1$ with entries in S that are strictly increasing from NE to SW.*

Example 13. *We illustrate the definition above with a small example. Let $n = 5$, $k = 3$, and consider the interval positroid variety $\Pi_{[2,4]\le 2}$. The following $\mathbf{m} \in B(k,n,d)$ contains a generalized antidiagonal for the rank condition $[2,4] \le 2$:*

$$\begin{array}{|c|}\hline \color{red}{2} \\ \hline \color{red}{3} \\ \hline \color{red}{4} \\ \hline \end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & \color{red}{2} \\ \hline 2 & \color{red}{3} \\ \hline \color{red}{4} & 5 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & \color{red}{2} \\ \hline 3 & \color{red}{3} \\ \hline \color{red}{4} & 5 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & \color{red}{2} \\ \hline \color{red}{3} & 4 \\ \hline \color{red}{4} & 5 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 1 & \color{red}{2} \\ \hline 2 & \color{red}{3} & 4 \\ \hline \color{red}{4} & 5 & 5 \\ \hline \end{array}\,.$$

In each monomial, the entries in the generalized antidiagonal are highlighted. By part 1 of Theorem A, these are not standard monomials for $\Pi_{[2,4]\le 2}$.

To study the standard monomials for a basic positroid variety $\Pi_{S\le r}$ where S is a wrapped-around interval, we need the following contruction.

Construction 14. *Let $S = [\alpha+1, \alpha+m]^\circ$ be a cyclic interval and $r < m$. Define the following:*

- $S^\vee := [\alpha+m+1, \alpha]^\circ$.
- $r^\vee := n-k-m+r$
- *For any monomial $\mathbf{m} = \prod_{i=1}^d [\mathbf{a}^{(i)}] \in R(k,n)$, define $\mathbf{m}^\vee$ to be $\prod_{i=1}^d [\mathbf{a}^{(i)\vee}] \in R(n-k,n)$, where $\mathbf{a}^{(i)\vee} := [n] \setminus \mathbf{a}^{(i)}$ for all $i \in [d]$.*
- *For any polynomial $g \in R(k,n)$, define $g^\vee \in R(n-k,n)$ to be the polynomial obtained from g by applying $\vee$ to each monomial summand.*

Proposition 15. *A monomial $\mathbf{m} \in \mathrm{In}_\omega(\mathcal{J}_{S\le r})$ if and only if $\mathbf{m}^\vee \in \mathrm{In}_\omega(\mathcal{J}_{S^\vee \le r^\vee})$.*

Example 16. *Fix $\mathrm{Gr}(4,6)$ and let $S = [2,5]$ and $r = 3$. Then $S^\vee = [6] \setminus S = [6,1]^\circ$ and $r^\vee = 6-4-4+3 = 1$. Given a monomial $\mathbf{m} \in \mathrm{In}_\omega(\mathcal{J}_{S\le r}) \subset R(4,6)$ (with the generalized antidiagonal highlighted), we obtain $\mathbf{m}^\vee \in \mathrm{In}_\omega(\mathcal{J}_{S^\vee \le r^\vee}) \subset R(2,6)$ by taking complements of each of the Plücker coordinates dividing $\mathbf{m}$. For instance,*

$$\begin{array}{|c|}\hline \color{red}{2} \\ \hline \color{red}{3} \\ \hline \color{red}{4} \\ \hline \color{red}{5} \\ \hline \end{array} \overset{\textit{complement}}{\longleftrightarrow} \begin{array}{|c|}\hline 1 \\ \hline 6 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & \color{red}{2} \\ \hline 2 & 2 & \color{red}{3} & 4 \\ \hline 3 & \color{red}{4} & 5 & 5 \\ \hline \color{red}{5} & 6 & 6 & 6 \\ \hline \end{array} \overset{\textit{complement}}{\longleftrightarrow} \begin{array}{|c|c|c|c|}\hline 4 & 3 & 2 & 1 \\ \hline 6 & 5 & 4 & 3 \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 3 & 4 & 5 & 6 \\ \hline \end{array}$$

Using the Proposition below, we prove part (3) of Theorem A.

Proposition 17. *Let $\mathbf{m} := \prod_{i=1}^d [\mathbf{a}^{(i)}] \in B(k,n,d)$, where $\mathbf{a}^{(1)} \le \cdots \le \mathbf{a}^{(d)}$. Then there exists a unique minimal positroid variety Π_f such that $\mathbf{m} \in B_f(k,n,d)$. In other words, if $\Pi_{f'}$ is a positroid variety such that $\mathbf{m} \in B_{f'}(k,n,d)$, then $\Pi_f \subseteq \Pi_{f'}$.*

Proof sketch for part (3) of Theorem A: Notice first that if $\Pi_f \subseteq \Pi_{f'}$ then $\mathcal{J}_f \supseteq \mathcal{J}_{f'}$ and thus $B_f(n,k,d) \subseteq B_{f'}(n,k,d)$. This proves $B_f(k,n,d) \subseteq \bigcap_i B_{f_i}(n,k,d)$. For $\mathbf{m} \in \bigcap_i B_{f_i}(n,k,d)$, by Proposition 17, there is a unique minimal $\Pi_g \subseteq \bigcap_i \Pi_{f_i} = \Pi_f$ such that $\mathbf{m} \in B_g(n,k,d)$. Therefore $\mathbf{m} \in B_f(n,k,d)$.

4 Gröbner bases for positroid varieties

Using straightening relations and our characterizations for standard monomials, we obtain an explicit Gröbner basis for positroid varieties.

Theorem B. *Let $\mathcal{J}_f$ be the defining ideal for the positroid variety Π_f. $\mathcal{J}_f$ has a Gröbner basis with respect to $<_\omega$ (see bottom of p.4) consisting of straightening relations (Definition 1), and*

(1) when $\Pi_f = \Pi_{S\le r}$ for some interval S, $\sum_{w\in\mathfrak{S}_{r+1}}(-1)^{\ell(w)}w\cdot\mathbf{m}$ where $\mathbf{m}$ minimally contains a generalized antidiagonal for $S\le r$ and $\mathfrak{S}_{r+1}$ acts on $\mathbf{m}$ by permuting entries in the generalized antidiagonal;

(2) when $\Pi_f = \Pi_{S\le r}$ for some wrapped-around interval S, $\sum_{w\in\mathfrak{S}_{r^\vee+1}}(-1)^{\ell(w)}(w\cdot\mathbf{m}^\vee)^\vee$ where $\mathbf{m}^\vee$ minimally contains a generalized antidiagonal for $S^\vee\le r^\vee$;

(3) when $\Pi_f = \bigcap_i \Pi_{S_i\le r_i}$, the union of the Gröbner basis for each $\Pi_{S_i\le r_i}$.

Example 18. *Let $n=5$, $k=3$, $S=[2,4]$, $r=2$, and consider the basic positroid variety $X_{[2,4]\le 2}$. The generators of the Gröbner basis of $\mathcal{J}_{[2,4]\le 2}$ which are not Plücker relations for $\mathrm{Gr}(3,5)$ consist of the following:*

$$\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\,,\quad \begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline 4&5\\\hline\end{array}-\begin{array}{|c|c|}\hline 1&2\\\hline 2&4\\\hline 3&5\\\hline\end{array}\,,\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline 4&5\\\hline\end{array}-\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 3&5\\\hline\end{array}\,,\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 4&5\\\hline\end{array}-\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 4&5\\\hline\end{array}\,,$$

$$\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3&4\\\hline 4&5&5\\\hline\end{array}-\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&4&4\\\hline 3&5&5\\\hline\end{array}-\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2&4\\\hline 4&5&5\\\hline\end{array}\,.$$

We see above that there is a single degree 3 generator of $\mathrm{gb}(\mathcal{J}_{[2,4]\le 2})$. Therefore, we also know that we have a degree 3 generator of $\mathrm{gb}(\mathcal{J}_{[5,1]^\circ\le 1})$:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 3&4&5\\\hline\end{array}-\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&3&5\\\hline\end{array}-\begin{array}{|c|c|c|}\hline 1&3&3\\\hline 2&4&5\\\hline\end{array}\,.$$

5 Promotion on standard monomials

Our next application concerns *promotion* and *evacuation* introduced by Schützenberger [15, 16]. Evacuation appears in the theory of canonical bases where the bijection by Berenstein-Zelevinsky [1] between canonical bases and semistandard tableaux sends Lusztig's involution to evacuation. Stembridge [18, 19] observed the $q=-1$ *phenomenon*, meaning that the number of fixed points under an involution is the value of a polynomial $f(q)$ at $q=-1$. Motivated by the work of Stembridge, Reiner–Stanton–White [12] introduced the *cyclic sieving phenomenon* for the enumeration of fixed points under cyclic group actions. Rhoades [13] showed that promotion on $B(k,n,d)$ exhibits the cyclic sieving phenomenon, where the number of fixed tableaux after applying promotion a times is the value of a polynomial at the roots of unity ξ_n^a. More recently, these operations are of interest in the study of dynamical algebraic combinatorics (see e.g., the survey [14] and the references therein).

We relate promotion and evacuation to the cyclic and reflection symmetry of positroid varieties, respectively.

Definition 19. *Let* ***promotion*** *be the map* $\mathrm{prom} : B(k,n,d) \longrightarrow B(k,n,d)$ *defined as follows:*

- *If* $\mathbf{m} \in B(k,n,d)$ *does not contain* n*, then increase each entry of* $\mathbf{m}$ *by* 1*.*
- *If* $\mathbf{m}$ *contains* n*, replace each* n *with* $\bullet$ *and perform the* ***jeu de taquin (jdt)*** *slide:*

$$\begin{array}{|c|c|}\hline a & c \\ \hline b & \bullet \\ \hline \end{array} \longrightarrow \begin{cases} \begin{array}{|c|c|}\hline a & \bullet \\ \hline b & c \\ \hline \end{array} & \text{if } b \le c \text{ or } a,b \text{ do not exist,} \\ \\ \begin{array}{|c|c|}\hline a & c \\ \hline \bullet & b \\ \hline \end{array} & \text{if } b > c \text{ or } a,c \text{ do not exist} \end{cases}$$

until no longer possible. Replace each $\bullet$ *with* 0 *and increase all entries by* 1*.*

Definition 20. *Let* ***evacuation*** *be the map* $\mathrm{evac} : B(k,n,d) \to B(k,n,d)$ *defined by:*

1. *replace every entry* x *with* $n+1-x$*,*
2. *rotate the tableau by* 180°*.*

Our main theorem of this section is the following.

Theorem C. *Promotion (resp. evacuation) bijects the set of standard monomials of a positroid variety* Π_f *and those of its cyclic shift* $\chi(\Pi_f)$ *(resp. its reflection* $\Pi_{f^*} = w_0 \cdot \Pi_f$*).*

Since prom and evac are both bijections from $B(k,n,d)$ to $B(k,n,d)$, Theorem C is equivalent to showing prom (resp. evac) bijects $\mathrm{In}(\mathcal{J}_f)$ with $\mathrm{In}(\mathcal{J}_{\chi(f)})$ (resp. $\mathrm{In}(\mathcal{J}_{f^*})$). Here we give an example to see how promotion (resp. evacuation) maps monomials in $\mathrm{In}(\mathcal{J}_f)$ to $\mathrm{In}(\mathcal{J}_{\chi(f)})$ (resp. $\mathrm{In}(\mathcal{J}_{f^*})$).

Example 21. *Consider the basic positroid variety* $\Pi_f = \Pi_{[2,4]\le 1} \subset \mathrm{Gr}(3,7)$*, then* $\chi(\Pi_f) = \Pi_{[3,5]\le 2}$ *and* $w_0 \cdot \Pi_f = \Pi_{[4,6]\le 2}$*. For* $\mathbf{m} = [1,3,4]\cdot[2,5,7] \in \mathrm{In}(\mathcal{J}_f)$*, we have* $\mathrm{prom}(\mathbf{m}) = [1,2,3]\cdot[3,4,5] \in \mathrm{In}(\mathcal{J}_{\chi(f)})$ *and* $\mathrm{evac}(\mathbf{m}) = [1,3,6]\cdot[4,5,7] \in \mathrm{In}(\mathcal{J}_{f^*})$*. We demonstrate the process below:*

$$\mathrm{prom} : \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 7 \\ \hline \end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \bullet \\ \hline \end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \bullet \\ \hline 4 & 5 \\ \hline \end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bullet & 3 \\ \hline 4 & 5 \\ \hline \end{array} \to \begin{array}{|c|c|}\hline \bullet & 2 \\ \hline 1 & 3 \\ \hline 4 & 5 \\ \hline \end{array} \to \begin{array}{|c|c|}\hline 0 & 2 \\ \hline 1 & 3 \\ \hline 4 & 5 \\ \hline \end{array} \to \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline \end{array}.$$

$$\mathrm{evac} : \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 7 \\ \hline \end{array} \to \begin{array}{|c|c|}\hline 7 & 6 \\ \hline 5 & 3 \\ \hline 4 & 1 \\ \hline \end{array} \to \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 & 5 \\ \hline 6 & 7 \\ \hline \end{array}$$

6 Hilbert series and cyclic Demazure characters

Motivated by the cyclic symmetry of the Grassmannian and its positroid varieties, Lam introduced cyclic Demazure modules $V_f(d\omega_k)$ as the cyclic intersections of Demazure modules. They are isomorphic to the space of sections $H^0(\Pi_f, \mathcal{L}_{d\omega_k})$ as T-modules.

Let ω_k be the k-th fundamental weight, $\mathcal{L}_{\omega_k}$ the associated line bundle on $\mathrm{Gr}(k,n)$ and $\mathcal{L}_{d\omega_k} := \mathcal{L}_{\omega_k}^{\otimes d}$. Let $\mathrm{res}_f : H^0(\mathrm{Gr}(k,n), \mathcal{L}_{d\omega_k}) \to H^0(\Pi_f, \mathcal{L}_{d\omega_k})$ be the restriction map.

Proposition 22. *The map* res_f *is surjective and* $H^0(\Pi_f, \mathcal{L}_{d\omega_k}) \cong \mathbf{k}[\Pi_f]_d$ *as T-modules.*

We establish a recurrence on the multigraded Hilbert series of positroid varieties with the grading induced by the standard T-action on the Grassmannian. This allows us to give an inductive formula for the T-character of cyclic Demazure modules, answering a question of Lam [11, Problem 24].

Theorem D. *For* $f \in \mathrm{Bound}(k,n)$*, if* $d = 0$*, then* $\mathrm{ch}(V_f(d\omega_k)) = 1$*. Otherwise,*

$$\mathrm{ch}(V_f(d\omega_k)) = \mathbf{t}^{\mathrm{top}(f)}\mathrm{ch}(V_f((d-1)\omega_k) + \sum_{f' \in \mathcal{L}_0(f) \cap \mathrm{Bound}(k,n)} (-1)^{\ell(f')-\ell(f)+1} V_{f'}(d\omega_k).$$

Here $\mathrm{top}(f) \in \binom{[n]}{k}$ corresponds to the smallest Plücker coordinate that does not vanish on Π_f, $\mathbf{t}^{\mathrm{top}(f)} := \prod_{i \in \mathrm{top}(f)} t_i$, and $\mathcal{L}_0(f)$ is derived from Lenart's K-theoretic Monk's rule for Grothendieck polynomials defined below.

Definition 23. *Let* $\Gamma_0(f)$ *denote the set of saturated chains in the Bruhat order of affine permutations:*

$$f \lessdot f t_{a_1,b_1} \lessdot f t_{a_1,b_1} t_{a_2,b_2} \lessdot \cdots \lessdot f t_{a_1,b_1} t_{a_2,b_2} \cdots t_{a_m,b_m},$$

where $a_i \le 0, b_i \in [1,n]$*, and the pairs* (a_i, b_i) *satisfies either* $(b_i > b_{i+1})$ *or* $(b_i = b_{i+1}$ *and* $a_i < a_{i+1})$*. Let* $\mathcal{L}_0(f)$ *be the set of permutations that are endpoints of chains in* $\Gamma_0(f)$*.*

Proposition 24 ([5, Corollary 7.3])**.** *Let* $C(f) := \{f' \in \mathrm{Bound}(k,n) : f \lessdot f' = f t_{a,b}, a \le 0 < b\}$*. Then the following formula holds scheme-theoretically:*

$$\{[\mathrm{top}(f)] = 0\} \cap \Pi_f = \bigcup_{f' \in C(f)} \Pi_{f'}. \tag{6.1}$$

Consider the $\mathbb{Z}^n$-grading on the polynomial ring $R(k,n)$ such that for any $\mathbf{a} \in \binom{[n]}{k}$, the degree of the Plücker variable $[\mathbf{a}]$ is $(d_1, \cdots, d_n)$ where $d_i = 0$ if $i \notin \mathbf{a}$ and $d_i = 1$ if $i \in \mathbf{a}$. For $\mathbf{m} \in B_f(k,n,d)$, define $\mathrm{content}(\mathbf{m}) := (c_1, \cdots, c_n)$ where c_i counts the number of appearance of i in $\mathbf{m}$. Set $\mathbf{t}^{\mathrm{content}(\mathbf{m})} := \prod_{i=1}^n t_i^{c_i}$.

Key Observation: By definition, the Hilbert series

$$\mathrm{Hilb}(R/\mathcal{J}_f; \mathbf{t}) = \sum_{d=0}^{\infty} \sum_{\mathbf{m} \in B_f(k,n,d)} \mathbf{t}^{\mathrm{content}(\mathbf{m})},$$

and the degree d part of $H(R/\mathcal{J}_f;\mathbf{t})$ is the character of $V_f(d\omega_k)$. Since $[\mathrm{top}(f)]$ is not a zero divisor of $R/\mathcal{J}_f$, the Hilbert series of the LHS of (6.1) is

$$\mathrm{Hilb}(R/(\mathcal{J}_f+\langle[\mathrm{top}(f)]\rangle);\mathbf{t}) = (1-\mathbf{t}^{\mathrm{top}(f)})\,\mathrm{Hilb}(R/\mathcal{J}_f;\mathbf{t}). \tag{6.2}$$

For the RHS, we have

Proposition 25.

$$\mathrm{Hilb}(R/(\bigcap_{f'\in C(f)}\mathcal{J}'_f);\mathbf{t}) = \sum_{f'\in\mathcal{L}_0(f)\cap\mathrm{Bound}(k,n)} (-1)^{\ell(f')-\ell(f)+1}\,\mathrm{Hilb}(R/\mathcal{J}'_f;\mathbf{t}),$$

The proof of the above proposition relies on proving an isomorphism of posets of varieties, generated by taking the closure of the "intersect and decompose" operation on the varieties appearing in the Monk's rule for positroid and matrix Schubert varieties.

Theorem D then follows from taking the degree d piece of the Hilbert series on both sides of (6.1) by combining (6.2) and Proposition 25.

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