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Quilts of Alternating Sign Matrices

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Abstract. In this extended abstract, we present new objects, *quilts of alternating sign matrices* with respect to two given posets. For example, the rank function on all submatrices of a matrix gives rise to a quilt with respect to two Boolean lattices. When the two posets are chains, a quilt is equivalent to an alternating sign matrix and its corresponding corner sum matrix. Quilts also generalize the monotone Boolean functions counted by the Dedekind numbers. They form a distributive lattice with many beautiful properties and contain many classical and well known sublattices, such as the lattice of matroids of a given rank and ground set. While enumerating quilts is hard in general, we prove two major enumerative results, when one of the posets is an antichain and when one of them is a chain. We also give some bounds for the number of quilts when one poset is the Boolean lattice.

Keywords: Alternating sign matrices, Bruhat order, Dedekind numbers, posets

1 Introduction

The *rank* of a matrix is fundamental in mathematics, science, and engineering. The notion of rank can also be associated to graphs, matroids, and partial orders. One can refine the rank function to submatrices, subgraphs, etc. as well to get a family of ranks to associate to each object. We observe that such families always follow certain Boolean growth rules leading to the concept of a generalized rank function, which we call a quilt. The goal of this paper is to consider families of generalized rank functions and their connection with the well-studied alternating sign matrices (ASMs). We present some applications and some related enumeration results.

The application that motivated us to study quilts comes from the geometry of spanning line configurations and an analog of Bruhat order based on a cell decomposition of that space used by Pawlowski and Rhoades [12]. They posed the problem of characterizing the covering relations in the poset given by containment of cell closures. While that problem is still open, the lattice of quilts on a chain poset with a Boolean poset naturally contains the poset, now also called the medium roast Fubini–Bruhat order [2]. Hence,

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we can use quilts to study the vanishing flag minors defining the cell closures in the spanning line configurations. See the full length version of this abstract [1] for details. Note that due to length limitations, we omit all proofs in this extended abstract.

2 Background

We recall some history and background to motivate the main new concepts. See [3] for more background.

If a matrix has k rows and n columns, we say it has *size* $k \times n$. An *alternating sign matrix* (or ASM for short) is a matrix of size $k \times n$ with entries in $\{-1, 0, 1\}$ such that in each row and each column the non-zero entries alternate, the leftmost non-zero entry in every row and the bottommost non-zero entry in every column is 1, if $k \leq n$, the rightmost non-zero entry in every row is 1, and if $k \geq n$, the topmost non-zero entry in every column is 1. In particular, if $n = k$, the non-zero entries of every row and every column alternate and begin and end with 1. Note that what we call an ASM is typically called a *rectangular* or *truncated* ASM in the literature; we will instead emphasize that we have a *square* ASM when $n = k$. Denote the set of all ASMs of size $k \times n$ by $\mathrm{ASM}_{k,n}$. For example, every permutation matrix is a square ASM, but already for $n = 3$ there are more than $n!$ square ASMs. A very famous result tells us that $|\mathrm{ASM}_{n,n}| = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$. See Example 4.5 for the asymptotics of $|\mathrm{ASM}_{k,n}|$.

Given $A = (a_{i,j}) \in \mathrm{ASM}_{k,n}$, we can define a new matrix $C(A) = (c_{ij})$, called its *corner sum matrix*, or CSM for short, of size $(k+1) \times (n+1)$ by setting $c_{ij} = 0$ if $i = k+1$ or $j = 1$, and $c_{ij} = \sum_{i'=i}^{k} \sum_{j'=1}^{j} a_{i',j'}$ otherwise. For an example, see (2.4). In the resulting matrix $C(A)$, the entries change by 0 or 1 when moving to the right or up, and the bottom row and the leftmost column always consist of 0's. Furthermore, if $k \geq n$, the top row consists of $C_n = \{0, 1, \ldots, n\}$, and if $k \leq n$, the rightmost column consists of $C_k = \{0, 1, \ldots, k\}$. Conversely, given a matrix B, with rows numbered $0, 1, \ldots, k$ and starting at the bottom, and columns numbered $0, 1, \ldots, n$ and starting on the left, satisfying these properties, the $k \times n$ matrix $A = (a_{ij})$ given by $a_{i,j} = b_{i,j} - b_{i-1,j} - b_{i,j-1} + b_{i-1,j-1}$ for all $i \in [k]$ and $j \in [n]$ is an ASM. Therefore, we can equivalently define CSMs directly as follows using the notation $(i,j) \lessdot (i',j')$ to mean the covering relation in $C_k \times C_n$.

Definition 2.1. A (k,n)*-corner sum matrix (CSM)* is a map $f \colon C_k \times C_n \longrightarrow \mathbb{N}$ satisfying:

- $f(i,0) = 0$ for $i = 0, \ldots, k$, $f(0,j) = 0$ for $j = 0, \ldots, n$,
- $f(k,n) = \min\{k,n\}$, and
- if $(i,j) \lessdot (i',j')$ in $C_k \times C_n$, then $f(i',j') \in \{f(i,j), f(i,j)+1\}$.

Let $\mathrm{CSM}_{k,n}$ denote the set of all (k,n)-CSMs. We refer to the third condition in the definition of a CSM as a *Boolean growth rule*, which is a central concept in this paper.

One more way to think of ASMs/CSMs is as monotone triangles (MTs). Given a CSM $f\colon C_k \times C_n \longrightarrow \mathbb{N}$, record the positions of *jumps* in each row, denoted by

$$J_f(i) = \{j \in [n]\colon f(i,j) = f(i,j-1)+1\}. \tag{2.1}$$

Then, the *monotone triangle* (MT) corresponding to f is the "triangular array" of jump sequences with rows $J_f(k), \ldots, J_f(2), J_f(1)$ (from top to bottom). The rows of the monotone triangle are *interlacing* in the sense that if $J_f(i) = \{s_1 < s_2 < \cdots < s_p\}$ and $J_f(i+1) = \{t_1 < t_2 < \cdots < t_q\}$ then either $p = q-1$ and

$$t_1 \le s_1 \le t_2 \le s_2 \le \cdots \le s_{q-1} \le t_q, \tag{2.2}$$

or $p = q$ and

$$t_1 \le s_1 \le t_2 \le s_2 \le \cdots \le t_q \le s_q. \tag{2.3}$$

Clearly, the original CSM can be recovered from its interlacing jump sets by the Boolean growth property, so there are easy bijections between the ASMs, CSMs, and MTs for given k, n. For an example with $k = 5$ and $n = 6$, consider

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad C(A) = \begin{bmatrix} 0 & 1 & 2 & 3 & 3 & 4 & 5 \\ 0 & 1 & 1 & 2 & 3 & 3 & 4 \\ 0 & 0 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad MT(A) = \begin{array}{ccccccccc} 1 & & 2 & & 3 & & 5 & & 6 \\ & 1 & & 3 & & 4 & & 6 & \\ & & 2 & & 3 & & 5 & & \\ & & & 3 & & 4 & & & \\ & & & & 3. & & & & \end{array} \tag{2.4}$$

Remark 2.2. Terwilliger introduced a poset Φ_n on the subsets of $[n]$ with covering relations given by $S \lessdot T$ whenever S and T are interlacing in the sense of (2.2). The poset Φ_n contains the Boolean lattice B_n as a subposet. He showed maximal chains in Φ_n are in bijection with $\mathrm{ASM}_{n,n}$, just as the maximal chains of B_n are in bijection with S_n [17, Theorem 3.4]. Building on this work, Hamaker and Reiner [7] showed that Φ_n is a shellable poset, introduced a notion of descents for monotone triangles, and connected them to a generalization of the Malvenuto–Reutenauer Hopf algebra of permutations.

3 Dedekind Maps and Quilts

In this section, we introduce a generalization of the Dedekind numbers, which count the number of monotone increasing Boolean functions [11, A000372]. Such functions are closely related to the CSMs defined in Definition 2.1 and are natural precursors to the notion of a quilt defined later in this section. Assume all posets below are finite, ranked, and have a unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$.

Definition 3.1. A *Dedekind map of rank k* on a poset P is a surjective map $f\colon P \to C_k$ satisfying $x \lessdot y \Rightarrow f(y) \in \{f(x), f(x)+1\}$. The set of all Dedekind maps of rank k on P is denoted by $D_k(P)$, their union by $D(P)$, and we write $d_k(P) = |D_k(P)|$ and $d(P) = |D(P)| = \sum_k d_k(P)$ for the *kth Dedekind number of P* and *Dedekind number of P*, respectively.

Example 3.2. The rank function of a matroid on ground set $[n]$ of rank k is a Dedekind map of rank k on the Boolean lattice B_n.

Observe that for a Dedekind map of rank k, $f\colon P \to C_k$, the following conditions are satisfied: $f(\hat{0}) = 0$; $f(\hat{1}) = k$; and if $x \lessdot y$, then $f(y) \in \{f(x), f(x)+1\}$ (Boolean growth). Consequently, $f(x) \leq \operatorname{rank} x$ for all $x \in P$. The k-Dedekind number of a chain is $d_k(C_n) = \binom{n}{k}$. For the antichain poset $A_2(j)$, we have $d_1(A_2(j)) = 2^j$.

Remark 3.3. Given $f \in D_1(P)$, the set of minimal elements satisfying $f(x) = 1$ is a nonempty antichain in $P \setminus \{\hat{0}\}$; so $d_1(P)$ counts the number of antichains in P (except for $\emptyset$ and $\{\hat{0}\}$), which is a #P-complete problem [13]. In particular, $d_1(B_n) + 2$ is the classical Dedekind number and is notoriously difficult to compute. The exact value for $n = 9$ was first computed in 2023, thirty years after the value for $n = 8$ [8].

Lemma 3.4. *For any poset P and $k \geq 1$, we have $d_k(P) \leq d_1(P)^k$.*

Every column of a CSM of size $(k+1) \times (n+1)$ can be seen as a Dedekind map on C_k and every row as a Dedekind map on C_n. As one reads left to right in columns or bottom to top in rows, another Boolean growth rule must hold. This second type of Boolean growth rule gives rise to the following graphs.

Definition 3.5. Let $G_D(P)$ denote the *Dedekind graph of P*, defined as the directed graph with vertex set given by the Dedekind maps in $D(P)$ and an edge from f to g if $g(x) \in \{f(x), f(x)+1\}$ for all $x \in P$. The *restricted Dedekind graph of P*, $G'_D(P)$, is the directed graph with vertex set $D(P)$ and an edge from f to g if $g(\hat{1}_P) = f(\hat{1}_P) + 1$ and $g(x) \in \{f(x), f(x)+1\}$ for all $x \in P$.

Proposition 3.6. *For any $1 \leq k \leq n$, the map between the set $CSM_{k,n}$ and the set of walks in the Dedekind graph $G_D(C_n)$ from its unique sink to a vertex in $D_k(C_n)$ determined by the consecutive list of columns is a bijection.*

Recall the definition of interlacing sets and monotone triangles from Section 2. The Dedekind graph of a chain C_n also respects the interlacing conditions. The next statement follows from Proposition 3.6. See Proposition 3.13 for more connections with the interlacing conditions.

Corollary 3.7. *The restricted Dedekind graph $G'_D(C_n)$ is isomorphic to the directed graph on B_n, with an edge from S to T if $|S| = |T| - 1$ and the sets S and T are interlacing: $t_1 \leq s_1 \leq t_2 \leq s_2 \leq \cdots \leq s_{|T|-1} \leq t_{|T|}$. Similarly, the Dedekind graph $G_D(C_n)$ is isomorphic to the directed graph on B_n, edges as above plus an edge from S to T whenever $|S| = |T|$ and $t_1 \leq s_1 \leq t_2 \leq s_2 \leq \cdots \leq t_{|T|} \leq s_{|T|}$.*

The following is the main definition of this paper. It generalizes the definition of a CSM in Definition 2.1.

Definition 3.8. Let P and Q be finite ranked posets with least and greatest elements. A *quilt of alternating sign matrices of type* (P, Q) is a map $f\colon P \times Q \longrightarrow \mathbb{N}$ satisfying:

- $f(x, \hat{0}_Q) = 0$ for all $x \in P$, $f(\hat{0}_P, y) = 0$ for all $y \in Q$,
- $f(\hat{1}_P, \hat{1}_Q) = \min\{\operatorname{rank} P,\ \operatorname{rank} Q\}$, and
- if $(x, y) \lessdot (x', y')$ in $P \times Q$, then $f(x', y') \in \{f(x, y), f(x, y) + 1\}$ (Boolean growth).

We will also call such a map an *ASM quilt* or just a *quilt* for short. The set of all quilts of type (P, Q) will be denoted by $\operatorname{Quilts}(P, Q)$.

Remark 3.9. A quilt of type (C_k, C_n) is a CSM on $C_k \times C_n$, so there is also a corresponding ASM and MT. Similarly, for any $f \in \operatorname{Quilts}(P, Q)$ and any pair of maximal chains $\hat{0}_P = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{k-1} \lessdot x_k = \hat{1}_P$, $\hat{0}_Q = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_{n-1} \lessdot y_n = \hat{1}_Q$ in P and Q, the map $(i, j) \mapsto f(x_i, y_j)$ is a CSM on $C_k \times C_n$, which again has a corresponding ASM and MT. So we can think of quilts as encoding collections of alternating sign matrices, one for each pair of maximal chains in the two posets, appropriately "pieced" together like the fabric of a quilt.

Example 3.10. Let M be a $k \times n$ matrix of full rank. The function $f_M : B_k \times B_n \longrightarrow \mathbb{N}$ given by setting $f_M(I, J)$ to be the rank of the submatrix of M in rows I and columns J is a quilt of type (B_k, B_n).

The following lemma also shows that the entire rank function of the smaller ranked poset is encoded in each quilt. This justifies our claim that quilts generalize rank functions of posets.

Lemma 3.11. *Let* $f \in \operatorname{Quilts}(P, Q)$*. If* $\operatorname{rank} P \geq \operatorname{rank} Q$*, then* $f(\hat{1}_P, y) = \operatorname{rank}_Q y$ *for all* $y \in Q$*. If* $\operatorname{rank} P \leq \operatorname{rank} Q$*, then* $f(x, \hat{1}_Q) = \operatorname{rank}_P x$ *for all* $x \in P$*.*

There is a natural partial order on $\operatorname{Quilts}(P, Q)$: we say that $f \leq g$ if $f(x, y) \leq g(x, y)$ for all $x \in P$, $y \in Q$. For $P = C_k$, $Q = C_n$, this is the well-known partial order on the set of CSMs or ASMs generalizing Bruhat order on the symmetric group [10]. We will call $\operatorname{Quilts}(P, Q)$ the *quilt lattice*, as justified by the following.

Theorem 3.12. *Let* P, Q *be finite ranked posets with least and greatest elements. The poset* $\operatorname{Quilts}(P, Q)$ *is a distributive lattice ranked by*

$$\operatorname{quiltrank} f = \sum_{x \in P,\, y \in Q} f(x, y) - \sum_{x \in P,\, y \in Q} f_{\hat{0}}(x, y),$$

where $f_{\hat{0}}(x, y) = \max\{0, \operatorname{rank} x + \operatorname{rank} y - \max\{n, k\}\}$ *is the least element of* $\operatorname{Quilts}(P, Q)$*. The greatest element of* $\operatorname{Quilts}(P, Q)$ *is* $f_{\hat{1}}(x, y) = \min\{\operatorname{rank} x,\ \operatorname{rank} y\}$.

As mentioned in the introduction, our motivating example comes from vanishing flag minors where the posets are the Boolean lattice and the other one is a chain. We will call a quilt in $\mathrm{Quilts}(P, C_n)$ or $\mathrm{Quilts}(C_n, P)$ a *chain quilt*, a quilt in $\mathrm{Quilts}(P, A_2(j))$ or $\mathrm{Quilts}(A_2(j), P)$ an *antichain quilt*, a quilt in $\mathrm{Quilts}(P, B_n)$ or $\mathrm{Quilts}(B_n, P)$ a *Boolean quilt*.

There are three important ways to think of a chain quilt $f \in \mathrm{Quilts}(P, C_n)$. One is to see it as a sequence of Dedekind maps in $D(P)$ that correspond with a walk in the Dedekind graph $G_D(P)$, generalizing Proposition 3.6. Another is to say that f maps an arbitrary $x \in P$ to the sequence $(f(x,0), f(x,1), \ldots, f(x,n))$ of length $n+1$. This sequence has the property that every two consecutive elements are either equal or they differ by one. We also have $f(y,i) \in \{f(x,i), f(x,i)+1\}$ when $x \lessdot y$. The element $\hat{0}_P$ is mapped to the zero sequence, and the sequence corresponding to $\hat{1}_P$ ends with $\min\{\mathrm{rank}\, P,\ n\}$.

Another equivalent, and probably even more intuitive, way to represent a chain quilt $f : P \times C_n \longrightarrow \mathbb{N}$ is to say that it is a map that sends $x \in P$ to the set of *jumps of f at x*, $J_f(x) = \{i \colon f(x,i) = f(x,i-1)+1\} \subseteq [n]$. We will call this the *monotone triangle (MT) form* of the quilt f. It is easy to go back from the MT form of a quilt: given $J : P \longrightarrow B_n$, then $f(x,i) = |J(x) \cap [i]|$ defines $f \colon P \times C_n \longrightarrow \mathbb{N}$. The Boolean growth condition translates into adjacent sets interlacing for quilts in MT form, see Proposition 3.13.

The first picture on the left in Figure 1 shows an element $f \in \mathrm{Quilts}(B_3, C_2)$, where

$$f(\{2\},1) = f(\{1,2\},1) = f(\{2,3\},1) = f(\{1,2,3\},1) = 1$$

and $f(T,1) = 0$ for all other subsets T, while $f(\{2,3\},2) = f(\{1,2,3\},2) = 2$. The second picture is the MT form of f, where we omit braces and commas for the sets. The third picture in Figure 1 represent a chain quilt $g \in \mathrm{Quilts}(B_3, C_5)$. Note how the top element of f is $01\ldots n$, and the rightmost element of every sequence in the picture for g is equal to its rank, as stated in Lemma 3.11. The fourth picture is the MT form for g.

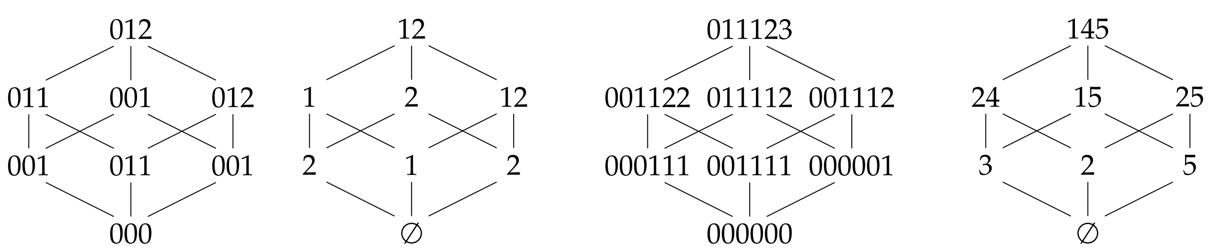

Figure 1: Two ways to visualize quilts of types (B_3, C_2) and (B_3, C_5).

Proposition 3.13. *Take $f \in \mathrm{Quilts}(P, C_n)$. For all $x, y \in P$ with $x \lessdot y$, the sets $S = J_f(x)$ and $T = J_f(y)$ are interlacing as in (2.2) and (2.3). When $n \le \mathrm{rank}\, P$, $J_f(\hat{1}_P) = [n]$. When $n \ge \mathrm{rank}\, P$, we have $|J_f(x)| = \mathrm{rank}\, x$ for all $x \in P$.*

The general problem of enumerating quilts is hard, as the following theorem shows.

Theorem 3.14. *Computing* $|\operatorname{Quilts}(P,Q)|$ *for general* P *and* Q *is a #P-complete problem.*

Even though computing $|\operatorname{Quilts}(P,Q)|$ for general P and Q is out of reach, we do have a simple upper bound. Let $b(P) = \sum_{x\in P} \operatorname{rank} x$.

Theorem 3.15. *If* $\operatorname{rank} P \le \operatorname{rank} Q$, *then* $|\operatorname{Quilts}(P,Q)| \le d_1(Q)^{b(P)}$.

Observe from the definitions that for all posets P, Q with $\operatorname{rank} P = k$ and $\operatorname{rank} Q = n$, the map $\iota\colon \operatorname{CSM}_{k,n} \longrightarrow \operatorname{Quilts}(P,Q)$ given by $\iota(f)(x,y) = f(\operatorname{rank} x,\ \operatorname{rank} y)$ is a lattice embedding. The lattice $\operatorname{Quilts}(C_k, B_n)$, $n \ge k$, also contains (among others) the following three overlapping subposets:

- Matroids on ground set $[n]$ with rank k are embedded into $\operatorname{Quilts}(C_k, B_n)$ via the map that sends a matroid on $[n]$ to the quilt $f : C_k \times B_n \longrightarrow \mathbb{N}$ with $f(i,T) = \min\{i,\ \operatorname{rank} T\}$, where $\operatorname{rank} T$ is the cardinality of the largest independent set contained in T.
- The rank functions of flag matroids $\mathcal{M}$ on the ground set $[n]$ with ranks $\mathbf{k} = (k_1,\dots,k_s)$ and rank functions $\operatorname{rank}_1,\dots,\operatorname{rank}_s$ can also be encoded as a Boolean-chain quilt. Specifically, the embedding into $\operatorname{Quilts}(C_{k_s}, B_n)$ is given by $f_{\mathcal{M}}(i,X)$ is either $\min\{i, \operatorname{rank}_1 X\}$ if $1 \le i \le k_1$ or $\min\{\operatorname{rank}_j X + i - k_j, \operatorname{rank}_{j+1} X\}$ if $k_j < i \le k_{j+1}, 1 \le j \le s-1$.
- The medium roast Fubini–Bruhat order embeds into $\operatorname{Quilts}(C_k, B_n)$ via northerly rank conditions [2].

Theorem 3.16. *Let* P *and* Q *be finite ranked posets with least and greatest elements. If* φ *is an (involutive) antiautomorphism of* P, $\operatorname{rank} P \ge \operatorname{rank} Q$, *then* $\Phi\colon \operatorname{Quilts}(P,Q) \to \operatorname{Quilts}(P,Q)$, *where* $\Phi f(x,y) = \operatorname{rank} y - f(\varphi(x),y)$, *is an (involutive) antiautomorphism of the lattice* $\operatorname{Quilts}(P,Q)$. *Given an involutive antiautomorphism* $\varphi\colon P \to P$, *there is an action of the dihedral group* D_4 *acting on* $\operatorname{Quilts}(P,P)$ *that sends the horizontal reflection of the square to* Φ *and the diagonal reflection of the square to* Σ. *If* $\operatorname{rank} P \ge 2$, *the action is faithful.*

Take posets P_1 and P_2 with the same rank. The *disjoint union* $P_1 + P_2$ is the poset we get by "merging" $\hat{0}_{P_1}$ with $\hat{0}_{P_2}$ and $\hat{1}_{P_1}$ with $\hat{1}_{P_2}$, and adding the other elements of P_1 and P_2 without any new relations. For example, $A_2(j_1) + A_2(j_2)$ is isomorphic to $A_2(j_1 + j_2)$. Write jP for the disjoint union of j copies of P. For example, $A_2(j) = jC_2$.

Proposition 3.17. *Assume that* $\operatorname{rank} P_1 = \operatorname{rank} P_2 \ge \operatorname{rank} Q$. *Then the map*

$$\Theta\colon \operatorname{Quilts}(P_1 + P_2, Q) \longrightarrow \operatorname{Quilts}(P_1, Q) \times \operatorname{Quilts}(P_2, Q)$$

defined by $f \mapsto (f_1, f_2)$, $f_i(x_i, y) = f(x_i, y)$ *for* $x_i \in P_i, y \in Q$, *is an isomorphism of lattices.*

Corollary 3.18. *For* $k \ge n$ *and arbitrary positive integer* j, $|\operatorname{Quilts}(jC_k, C_n)| = |\operatorname{ASM}_{k\times n}|^j$. *For any* i, j, n, *we have* $|\operatorname{Quilts}(iC_n, jC_n)| = |\operatorname{ASM}_{n\times n}|^{ij}$.

4 Enumerative results

4.1 Enumeration of antichain quilts

We next consider the case of counting the number of quilts of type (P, Q) when Q is an antichain poset. The enumeration is in terms of the number of antichains in convex cut sets of P. We say that a subset S of a poset P is *convex* if $x, y \in S$ implies $[x, y] \subseteq S$. We say that S is a *cut set* if it intersects every maximal chain in P. If you have a convex cut set C, it makes sense to say that an element $x \in P \setminus C$ is *above* C or *below* C: x lies on a maximal chain, the maximal chain intersects C in some element x', and x is above C if $x > x'$ and below C if $x < x'$. This is well defined, as $x' < x < x''$ for $x', x'' \in C$ would imply $x \in C$. For example, if $\operatorname{rank} P \geq 2$, then $C = P \setminus \{\hat{0}_P, \hat{1}_P\}$ is a convex cut set, $\hat{0}_P$ is below C, and $\hat{1}_P$ is above C.

Recall from Remark 3.3 that $d_1(P)$ counts the number of nonempty antichains in P other than $\{\hat{0}\}$. Such antichains are in bijection with antichains in $P \setminus \{\hat{0}_P, \hat{1}_P\}$. For any $S \subseteq P$, denote by $\alpha_P(S)$ the number of antichains in S. Then, we have $\alpha_P(P \setminus \{\hat{0}_P, \hat{1}_P\}) = d_1(P)$. Given two infinite sequences (a_n) and (b_n), we write $a_n \sim b_n$ to mean $a_n / b_n \to 1$ as n goes to infinity.

Theorem 4.1. *Take a ranked poset P with least and greatest elements,* $\operatorname{rank} P \geq 2$*, and $j \geq 1$. We have*

$$|\operatorname{Quilts}(P, A_2(j))| = \sum_C \alpha_P(C)^j, \tag{4.1}$$

where the sum is over all subsets C of $P \setminus \{\hat{0}_P, \hat{1}_P\}$ that are convex cut sets of P. In particular, as j goes to infinity, we have $|\operatorname{Quilts}(P, A_2(j))| \sim d_1(P)^j$.

Corollary 4.2. *We have $|\operatorname{Quilts}(C_k, A_2(j))| = \sum_{i=2}^{k} (k+1-i) i^j$ for $j \geq 1$, $k \geq 2$.*

4.2 Enumeration of chain quilts

Recall that $b(P) = \sum_{x \in P} \operatorname{rank} x$. If $f \in \operatorname{Quilts}(P, C_{b(P)})$, we say that $i \in [b(P)]$ is a *jump for f* if there exists $x \in P$ so that $f(x, i) = f(x, i-1) + 1$. If the set of jumps of f is equal to $[m]$, we say that f is *m-fundamental for P*. A *standard* quilt is one that is $b(P)$-fundamental. Let $F_m(P)$ be the set of all m-fundamental quilts for P. Let $S(P) = F_{b(P)}$ and $F(P) = \bigcup_m F_m(P)$. Observe that a chain quilt is m-fundamental if and only if its MT form contains precisely the elements $1, \ldots, m$. In particular, it is standard if and only if its MT form contains exactly one of each of $1, \ldots, b(P)$.

Theorem 4.3. *For a fixed poset P of rank k with least and greatest elements and any integer $n \geq k$, the number of chain quilts of type (P, C_n) is given by a polynomial in n. Namely,*

$$|\operatorname{Quilts}(P, C_n)| = \sum_{m=k}^{b(P)} |F_m(P)| \binom{n}{m} \sim \frac{|S(P)|}{b(P)!} \cdot n^{b(P)}. \tag{4.2}$$

Example 4.4. Consider $P = B_2$. We have $b(P) = 4$, and there are four 2-fundamental, five 3-fundamental, and two 4-fundamental (standard) quilts, presented here in MT form.

$$\begin{array}{ccc} & 12 & \\ / & & \backslash \\ 1 & & 1 \\ \backslash & & / \\ & \emptyset & \end{array} \quad \begin{array}{ccc} & 12 & \\ / & & \backslash \\ 1 & & 2 \\ \backslash & & / \\ & \emptyset & \end{array} \quad \begin{array}{ccc} & 12 & \\ / & & \backslash \\ 2 & & 1 \\ \backslash & & / \\ & \emptyset & \end{array} \quad \begin{array}{ccc} & 12 & \\ / & & \backslash \\ 2 & & 2 \\ \backslash & & / \\ & \emptyset & \end{array} \quad \begin{array}{ccc} & 13 & \\ / & & \backslash \\ 1 & & 2 \\ \backslash & & / \\ & \emptyset & \end{array} \quad \begin{array}{ccc} & 13 & \\ / & & \backslash \\ 2 & & 1 \\ \backslash & & / \\ & \emptyset & \end{array} \quad \begin{array}{ccc} & 13 & \\ / & & \backslash \\ 2 & & 2 \\ \backslash & & / \\ & \emptyset & \end{array} \quad \begin{array}{ccc} & 13 & \\ / & & \backslash \\ 2 & & 3 \\ \backslash & & / \\ & \emptyset & \end{array} \quad \begin{array}{ccc} & 13 & \\ / & & \backslash \\ 3 & & 2 \\ \backslash & & / \\ & \emptyset & \end{array} \quad \begin{array}{ccc} & 14 & \\ / & & \backslash \\ 2 & & 3 \\ \backslash & & / \\ & \emptyset & \end{array} \quad \begin{array}{ccc} & 14 & \\ / & & \backslash \\ 3 & & 2 \\ \backslash & & / \\ & \emptyset & \end{array}$$

Therefore it follows from Theorem 4.3 that for $n \geq 2$,

$$|\operatorname{Quilts}(B_2, C_n)| = 4\binom{n}{2} + 5\binom{n}{3} + 2\binom{n}{4} = \frac{n^4}{12} + \frac{n^3}{3} + \frac{5n^2}{12} - \frac{5n}{6} \sim 2 \cdot \frac{n^4}{4!}.$$

Example 4.5. We can think of a standard quilt for $P = C_k$ as a monotone triangle (in the classical sense) in which all numbers $1, \dots, \binom{k+1}{2}$ appear. After an up-down reflection and a 45° rotation, we get a shifted standard Young tableau of shape $(k, k-1, \dots, 1)$. For example, for $k = 3$, we get monotone triangles $\begin{array}{ccccc} 1 & & 3 & & 6 \\ & 2 & & 5 & \\ & & 4 & & \end{array}$ and $\begin{array}{ccccc} 1 & & 4 & & 6 \\ & 2 & & 5 & \\ & & 3 & & \end{array}$ and shifted standard Young tableaux $\begin{array}{ccc} 1 & 2 & 4 \\ & 3 & 5 \\ & & 6 \end{array}$ and $\begin{array}{ccc} 1 & 2 & 3 \\ & 4 & 5 \\ & & 6 \end{array}$. The hook-length formula for shifted standard Young tableaux [18] gives (for fixed k and $n \to \infty$) $|\operatorname{ASM}_{k,n}| \sim \frac{\prod_{i=0}^{k-1}(2i)!}{\prod_{i=0}^{k-1}(k+i)!} \cdot n^{\binom{k+1}{2}}$.

There is in fact one more way to compute $|\operatorname{Quilts}(P, C_n)|$ and prove the polynomiality property via the Transfer-Matrix Method [15, Theorem 4.7.2] using the adjacency matrix $A_D(P)$ of the Dedekind graph of P defined in Definition 3.5.

Theorem 4.6. *For a finite poset P of rank $k \geq 1$ with least and greatest elements, we have*

$$\sum_{n=k}^{\infty} |\operatorname{Quilts}(P, C_n)| x^n = (I - xA_D(P))^{-1}_{1,d(P)} = \frac{(-1)^{d(P)-1}}{(1-x)^{d(P)}} \det T(P), \tag{4.3}$$

where $T(P)$ is the transfer-matrix $I - xA_D(P)$ with the the column indexing the unique source and the row indexing the unique sink in $G_D(P)$ removed. In particular, the sequence $0, 0, \dots, 0$, $|\operatorname{Quilts}(P, C_k)|, |\operatorname{Quilts}(P, C_{k+1})|, \dots$ is given by a polynomial of degree $< d(P)$. Furthermore,

$$\sum_{n=0}^{k-1} |\operatorname{Quilts}(P, C_n)| x^n = \sum_{i=1}^{d(P)-1} (I - xA'_D(P))^{-1}_{1,i} = \sum_{i=1}^{d(P)-1} (-1)^{i-1} \det T'(P)_i, \tag{4.4}$$

where $T'(P)_i$ is the matrix $I - xA'_D(P)$ with the source column and i-th row removed.

There is an easy bijection between $k \times n$ ASMs and monotone triangles with all possible length k top row sequences. Such a top row sequence will be denoted by $(a_1, \dots, a_k)$ with $1 \leq a_1 < a_2 < \dots < a_k \leq n$. Fischer proved that the cardinality of $\operatorname{MT}(a_1, \dots, a_k)$, the set of monotone triangles with top row $(a_1, \ldots, a_k)$, is a polynomial in variables $a_1, \ldots, a_k$, and she also found an explicit (operator) formula for $|\operatorname{MT}(a_1, \ldots, a_k)|$, see [6]. The definition can be extended to arbitrary chain quilts: given a poset P of rank k and $1 \leq a_1 < a_2 < \cdots < a_k \leq n$, define $\operatorname{MT}_P(a_1, \ldots, a_k)$ as the set of quilts $f \in \operatorname{Quilts}(P, C_n)$ for which $J_f(\hat{1}_P) = \{a_1, \ldots, a_k\}$. Let $J_f(\hat{1}_P)_i$ denote the i-th largest element of the set.

Theorem 4.7. *For a finite poset P of rank k with least and greatest elements, we have*

$$|\operatorname{MT}_P(a_1, \ldots, a_k)| = \sum_{f \in F(P)} \prod_{i=2}^{k} \binom{a_i - a_{i-1} - 1}{J_f(\hat{1}_P)_i - J_f(\hat{1}_P)_{i-1} - 1}, \tag{4.5}$$

4.3 Enumeration of Boolean quilts

Exact enumeration of Dedekind maps for B_n and Boolean quilts is at least as difficult as finding a formula for the Dedekind numbers. However, some bounds can be given. For example, we can construct $2^{\binom{n}{\lfloor n/2 \rfloor}}$ 1-Dedekind maps on B_n by taking $f(T) = 0$ for $|T| < \lfloor n/2 \rfloor$, $f(T) = 1$ for $|T| > \lfloor n/2 \rfloor$, $f(T) \in \{0, 1\}$ for $|T| = \lfloor n/2 \rfloor$. It follows that $d_1(B_n) \geq 2^{\binom{n}{\lfloor n/2 \rfloor}}$. Kleitman [9] proved the upper bound $d_1(B_n) \leq 2^{(1+c \ln n/\sqrt{n})\binom{n}{\lfloor n/2 \rfloor}}$ for some constant c. We use that result for the following.

Lemma 4.8. *There exists a constant $c > 0$ so that $d_k(B_n) \leq 2^{k(1+c \ln n/\sqrt{n})\binom{n}{\lfloor n/2 \rfloor}}$ for all $1 \leq k \leq n$. Furthermore, for every $\varepsilon > 0$, $d_k(B_n) \geq 2^{(k-\varepsilon)\binom{n}{\lfloor n/2 \rfloor}}$ for large enough n.*

Theorem 4.9. *Let P be a finite ranked poset with least and greatest elements. There exists a constant $c > 0$ so that $2^{\binom{n}{\lfloor n/2 \rfloor}} \leq |\operatorname{Quilts}(P, B_n)| \leq 2^{b(P)(1+c \ln n/\sqrt{n})\binom{n}{\lfloor n/2 \rfloor}}$ if $n \geq \operatorname{rank} P$. If $n \geq 2 \operatorname{rank} P$, we have the improved lower bound $|\operatorname{Quilts}(P, B_n)| \geq d_1(P)^{\binom{n}{\lfloor n/2 \rfloor}}$. In particular,*

$$2^{\binom{k}{\lfloor k/2 \rfloor}\binom{n}{\lfloor n/2 \rfloor}} \leq |\operatorname{Quilts}(B_k, B_n)| \leq 2^{k 2^{k-1}(1+c \ln n/\sqrt{n})\binom{n}{\lfloor n/2 \rfloor}} \qquad \text{for } n \geq 2k.$$

Remark 4.10. Theorem 4.9 guarantees that for a poset P, there are positive numbers A_P and B_P such that $(\ln |\operatorname{Quilts}(P, B_n)|)/\binom{n}{\lfloor n/2 \rfloor} \in [A_P, B_P]$ for $n \geq \operatorname{rank} P$. It is natural to ask if the limit $L(P) = \lim_{n \to \infty} \ln |\operatorname{Quilts}(P, B_n)|/\binom{n}{\lfloor n/2 \rfloor}$ exists. By the last part of the theorem, if $L(B_k)$ exists, it must be in the interval $\left[\binom{k}{\lfloor k/2 \rfloor} \ln 2, k 2^{k-1} \ln 2\right]$. We do not have enough data to state an explicit conjecture, but we believe that the limit does indeed exist; if we had to venture a guess as to what this number would be, we would say $L(P) = b(P) \ln 2$. In other words, we believe that $2^{b(P)\binom{n}{\lfloor n/2 \rfloor}}$ is the best estimate for $|\operatorname{Quilts}(P, B_n)|$ among functions of the form $C^{\binom{n}{\lfloor n/2 \rfloor}}$.

5 Final remarks

In this section, we discuss some possible avenues for future research.

Representability. Call a quilt $f \in \mathrm{Quilts}(B_k, B_n)$ *representable* if there exists a matrix $A \in \mathbb{R}^{k\times n}$, $\operatorname{rank} A = \min\{k, n\}$, so that $f(I, J)$ is equal to the rank of the matrix obtained by taking rows in I and columns in J in the matrix A. It would be interesting to characterize the representable quilts of type (B_k, B_n).

Quilt polytopes. There are beautiful results about the polytopes of alternating sign matrices, matroids, and flag matroids, see e.g. [16] and [4]. In 2018, Sanyal–Stump [14] defined the *Lipschitz polytope* of a poset P, denoted $\mathcal{L}(P)$, as the set of functions $f \in \mathbb{R}^P$ such that $0 \le f(a) \le 1$ for all minimal elements $a \in P$ and $0 \le f(y) - f(x) \le 1$ for all $x \lessdot y$ in P. Therefore, the vertices of the Lipschitz polytopes are closely related to the Dedekind maps on P. This variation on Boolean growth leads us to define $\mathcal{L}(P, Q)$ for a pair of finite ranked posets P, Q with least and greatest as the set of functions $f \in \mathbb{R}^{P\times Q}$ satisfying the boundary conditions $f(x, \hat{0}_Q) = 0$ for all $x \in P$, $f(\hat{0}_P, y) = 0$ for all $y \in Q$, $f(\hat{1}_P, \hat{1}_Q) = \min\{\operatorname{rank} P,\ \operatorname{rank} Q\}$, and $0 \le f(x', y') - f(x, y) \le 1$ if $(x, y) \lessdot (x', y')$ in $P \times Q$ (bounded growth). Thus, integer lattice points of $\mathcal{L}(P, Q)$ are exactly the quilts of type (P, Q). What is the Ehrhart polynomial for these generalized Lipschitz polytopes? What more can be said about these polytopes?

Enumeration. As we stated in the introduction, one of the most fascinating facts in the area is that there is a product formula for the number of square ASMs. Corollary 3.18 gives a simple generalization of this statement. Is there a simple formula for $|\mathrm{Quilts}(P, P)|$ for some family of posets $P \neq jC_n$? Can we at least find asymptotic formulas for $|\mathrm{Quilts}(P_n, P_n)|$ for some nice families of posets P_n, or upper and lower estimates? Can we improve the bounds for $|\mathrm{Quilts}(P, B_n)|$ beyond Theorem 4.9?

Generalizing ASM. The literature on permutations and alternating sign matrices provide a rich source of problems for quilts, some of which are mentioned in the introduction and Remark 2.2. What can be said about the interlacing Boolean lattice with both types of interlacing conditions? Following Hamaker–Reiner [7], what are the descents for quilts? Can we generalize their result on shellability to an (appropriately defined) Dedekind poset? Is there a Hopf algebra interpretation for quilts and an analog of the shuffle product? See also the work of Cheballah–Giraudo–Maurice, who defined a Hopf algebra with basis given by alternating sign matrices [5].

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