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The h^*-polynomials of type C hypersimplices

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Abstract. We study the Ehrhart theory of hypersimplices of type C, as introduced by Lam and Postnikov for general crystallographic root systems. The h^*-polynomials of classical hypersimplices are known to relate to various Eulerian statistics on the symmetric group. In this paper, we introduce a new statistic and partial order on signed permutations, which we use to derive explicit formulas for the h^*-polynomials of type C hypersimplices. Additionally, we explore connections with other statistics, including flag-excedances and circular descents, flag-descents, and Coxeter descents.

Keywords: hypersimplices, h^*-polynomial, alcoved polytopes, signed permutations

1 Introduction

Let $P \subset \mathbb{R}^n$ be a d-dimensional polytope with vertices on a lattice $\Lambda \subset \mathbb{R}^n$. Ehrhart [3] showed that the function $L_P = L_P^\Lambda : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} : r \mapsto |rP \cap \Lambda|$ is a polynomial of degree d in r. Later, Stanley [10] showed that the polynomial $h_P^*(t)$, defined from $L_P(r)$ by

$$\sum_{r \geq 0} L_P(r) t^r = \frac{h_P^*(t)}{(1-t)^{d+1}},$$

has *only* nonnegative integer coefficients. The polynomial $h_P^*(t)$ is known as the **(Ehrhart) h^*-polynomial** of P. Understanding the coefficients of the h^*-polynomial for arbitrary polytopes P is a difficult problem that has motivated substantial research in recent years. The sum of the coefficients, however, is simple to understand: it equals the (normalized) volume of P. In this paper, we study the h^*-polynomial of *type C hypersimplices*.

For positive integers $k < n$, the (type A) **hypersimplex** $\Delta_{n,k} \subset \mathbb{R}^n$ is the polytope with vertices all the $(0,1)$-vectors with exactly k ones. Hypersimplices are ubiquitous objects in algebraic combinatorics; for instance, they appear as the image of the Grassmannian under the moment map. It is a classical result, implicit in work of Laplace, that the volume of the hypersimplex $\Delta_{n,k}$ is given by the **Eulerian number** $A_{n-1,k-1}$, which counts permutations $w \in \mathfrak{S}_{n-1}$ having exactly $k-1$ **descents** (indices i such that $w_i > w_{i+1}$).

In a series of influential papers, Lam and Postnikov [6, 7] define generalized hypersimplices for any crystallographic root system Φ. They also show that the volume of

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these polytopes is determined by the distribution of a new statistic on the corresponding Coxeter group W_Φ, called **circular descent number**; see Section 3.2. For the root system of type A, both the generalized hypersimplex $\Delta_{\Phi,k}$ and the circular descent statistic are equivalent to the classical definitions. However, for other root systems, the distribution of circular descents does not agree with the corresponding Eulerian distribution.

The **Φ-hypersimplex** $\Delta_{\Phi,k}$ is determined by inequalities of the form $0 \leq \langle x, \alpha \rangle \leq 1$ for all simple roots α and $k-1 \leq \langle x, \theta \rangle \leq k$, where θ is the highest root of Φ. Explicitly, for the root system of type C_n with simple roots $2e_1$, $e_2 - e_1$, $e_3 - e_2, \ldots, e_n - e_{n-1}$,

$$\Delta_{C_n,k} := \{x \in \mathbb{R}^n \mid 0 \leq 2x_1\,,\, x_2 - x_1\,,\, \ldots\,,\, x_n - x_{n-1} \leq 1 \text{ and } k-1 \leq 2x_n \leq k\}, \tag{1.1}$$

for $k = 1, \ldots, 2n-1$. The vertices of $\Delta_{C_n,k}$ lie on the half-integer lattice $\Lambda := \frac{1}{2}\mathbb{Z}^n$, with respect to which we compute its Ehrhart and h^*-polynomial.

Our results involve the **half-open hypersimplex** $\Delta'_{C_n,k}$, obtained by replacing the weak inequality $k-1 \leq 2x_n$ in Equation (1.1) by the strict inequality $k-1 < 2x_n$; except at $k=1$, for which $\Delta'_{C_n,1} := \Delta_{C_n,1}$. The first one expresses the h^*-polynomial of $\Delta'_{C_n,k}$ in terms of basc, the **big ascent statistic** that we introduce in Definition 3.1, and cdes, the circular descent statistic of Lam and Postnikov. Let $\mathfrak{B}_n$ denote the hyperoctahedral group, or group of signed permutations, and consider the subset $X_n := \{w \in \mathfrak{B}_n \mid w^{-1}(1) > 0\}$.

Theorem 1.1. *For $n \geq 1$ and $k \geq 1$, the h^*-polynomial of the half-open hypersimplex $\Delta'_{C_n,k}$ is*

$$h^*_{\Delta'_{C_n,k}}(t) = \sum_{\substack{w \in X_n: \\ \operatorname{cdes}(w^{-1}) = k}} t^{\operatorname{basc}(w)}.$$

Our proof relies on the *alcove triangulation* of $\Delta_{C_n,k}$ (see Section 2) and a **partial order on X_n** whose cover relations are determined by the big ascent set of a signed permutation. Crucially, $(X_n, \leq)$ serves as a combinatorial model of a certain **interval of the weak order** on the alcoves of the affine reflection arrangement of type C (see Theorem 3.8).

The half-open hypersimplices partition the **fundamental parallelepiped**

$$\Pi_{C_n} := \{x \in \mathbb{R}^n \mid 0 \leq 2x_1\,,\, x_2 - x_1\,,\, \ldots\,,\, x_n - x_{n-1} \leq 1\}. \tag{1.2}$$

Since all these polytopes have the same dimension, Theorem 1.1 implies

$$h^*_{\Pi_{C_n}}(t) = \sum_k h^*_{\Delta'_{C_n,k}}(t) = \sum_{w \in X_n} t^{\operatorname{basc}(w)}. \tag{1.3}$$

The parallelepiped $(\Pi_{C_n}, \frac{1}{2}\mathbb{Z}^n)$ is integrally equivalent to the box $([0,1] \times [-1,1]^{n-1}, \mathbb{Z}^n)$. By studying the triangulation obtained by slicing this box with the linear Coxeter arrangement of type BC_n, we find an alternative formula for the h^*-polynomial of Π_{C_n}:

$$h^*_{\Pi_{C_n}}(t) = \sum_{w \in X_n} t^{\operatorname{des}_W(w)}, \tag{1.4}$$

where des_W denotes the usual Coxeter descent statistic of $\mathfrak{B}_n$.

A natural question arises by comparing Equations (1.3) and (1.4):

Is there a formula for the h^*-polynomial of $\Delta'_{C_n,k}$ that uses the Coxeter descent statistic of the hyperoctahedral group?

The following result gives a positive answer to this question.

Theorem 1.2. *For all $n \geq 1$ and $k \geq 1$,*

$$h^*_{\Delta'_{C_n,k}}(t) = \sum_{\substack{w \in X_n: \\ \mathrm{fexc}(w)=k-1}} t^{\mathrm{des}_W(w)},$$

where fexc *denotes the* flag-excedance *statistic of Foata and Han [4].*

The definition of fexc and a sketch of the proof of this result appear in Section 4. Evaluating at $t = 1$, we find a new formula for the volume of the type C hypersimplices that is fundamentally different from that of Lam and Postnikov in Equation (2.2).

Corollary 1.3. *For all $n \geq 1$ and $k \geq 1$, $\mathrm{Vol}(\Delta_{C_n,k}) = |\{w \in X_n \mid \mathrm{fexc}(w) = k-1\}|$.*

A type A analog of Theorem 1.2 was first conjectured by Stanley and later proved by Li in [8], where she also proves a type A analog of Theorem 1.1. A consequence of their formula, which can also be derived by slicing the unit cube with the type A Coxeter arrangement, is that the h^*-polynomial of the type A fundamental parallelepiped $\Pi_{A_{n+1}}$ is the Eulerian polynomial $A_n(t) = \sum_k A_{n,k}t^k$, whose exponential generating function is

$$\mathrm{Eul}_A(t, x) := \sum_{n \geq 0} A_n(t)\frac{x^n}{n!} = \frac{t-1}{t - e^{x(t-1)}}.$$

By explicitly computing the Ehrhart series of $\Pi_{C_{n+1}}$, we extend this result to type C.

Theorem 1.4. *The exponential generating function for the h^*-polynomials of the type C fundamental parallelepipeds is*

$$\sum_{n \geq 0} h^*_{\Pi_{C_{n+1}}}(t)\frac{x^n}{n!} = e^{3x(t-1)}\mathrm{Eul}_A(t, 2x)^2.$$

In light of (1.3), Theorem 1.4 can also be interpreted as the generating function for the big ascent statistic on the sequence of posets $\{(X_n, \leq)\}_{n\geq 1}$. In Section 5.2, we explore the limiting behavior of the posets X_n. In a precise sense, the *limit* of X_n is the **lattice of strict integer partitions**. In [1], the authors show that the type A analog of these posets converges to Young's lattice of integer partitions in the limit. Finally, in Sections 5.1 and 6.1, we establish further relations between X_n and the type BC Eulerian numbers.

2 Hypersimplices and the alcove triangulation

This section provides an overview of key aspects of the work by Lam and Postnikov in [6, 7] that are central to the discussion in this paper. Consider an n-dimensional real Euclidean space V with a nondegenerate inner product $\langle \cdot, \cdot \rangle$. Let $\Phi \subset V$ be an irreducible crystallographic root system of rank n, with a fixed set of **simple roots** $\{\alpha_1, \ldots, \alpha_n\} \subset \Phi$. The corresponding set of **positive roots** is denoted $\Phi^+ := \Phi \cap \mathbb{Z}_{\geq 0}\{\alpha_1, \ldots, \alpha_n\}$, and it is partially ordered by setting $\alpha \leq \beta$ if and only if $\beta - \alpha \in \mathbb{Z}_{\geq 0}\{\alpha_1, \ldots, \alpha_n\}$. This relation is the **root order** on Φ^+, and it has a unique maximum element θ, called the **highest root** of Φ. The **Coxeter number** of Φ is $h(\Phi) := a_1 + \cdots + a_n + 1$, where $\theta = a_1\alpha_1 + \cdots + a_n\alpha_n$. Finally, the **fundamental coweights** $\{\omega_1, \ldots, \omega_n\}$ form the basis of V dual to the simple roots, so $a_i = \langle \theta, \omega_i \rangle$.

Affine Coxeter arrangement and alcoves The **affine Coxeter arrangement** associated with Φ is the collection of hyperplanes $\mathcal{H}_\Phi := \{H_{\alpha,k} \mid \alpha \in \Phi^+, k \in \mathbb{Z}\}$, where $H_{\alpha,k} \subset V$ is the hyperplane with equation $\langle x, \alpha \rangle = k$. The **affine Coxeter group** $\widetilde{W}_\Phi$ is the group of affine transformations of V generated by reflections across the hyperplanes in $\mathcal{H}_\Phi$. The **finite Coxeter group** W_Φ is the subgroup of $\widetilde{W}_\Phi$ generated by reflections across the linear hyperplanes $H_{\alpha,0}$. See Figure 1 for a picture of $\mathcal{H}_\Phi$ when Φ is of type C.

The complement in V of the hyperplanes in $\mathcal{H}_\Phi$ is the disjoint union of bounded convex open sets, with their closures being the **alcoves** of the arrangement. The **fundamental alcove** is $A_\circ := \{x \in V \mid 0 \leq \langle x, \alpha \rangle \leq 1 \text{ for all } \alpha \in \Phi^+\} = \mathrm{Conv}\{0, \frac{\omega_1}{a_1}, \ldots, \frac{\omega_n}{a_n}\}$.

Let $\leq$ denote the **weak order** on the alcoves of $\mathcal{H}_\Phi$ with base region $A_\circ$. That is, $A \leq A'$ if and only if every hyperplane $H \in \mathcal{H}_\Phi$ separating $A_\circ$ and A also separates $A_\circ$ and A'. The group $\widetilde{W}_\Phi$ acts faithfully and transitively on the collection of alcoves, allowing us to identify elements $w \in \widetilde{W}_\Phi$ with alcoves $w \cdot A_\circ$. Under this identification, the weak order on alcoves and the left weak order on $\widetilde{W}_\Phi$ as a Coxeter group agree.

Generalized hypersimplices The **fundamental parallelepiped** $\Pi_\Phi \subset V$ is the polytope

$$\Pi_\Phi := \{x \in V \mid 0 \leq \langle x, \alpha_i \rangle \leq 1 \text{ for all } i = 1, \ldots, n\}.$$

It is bounded by hyperplanes of the affine arrangement $\mathcal{H}_\Phi$, and is therefore a union of alcoves of this arrangement[1]. These alcoves form an interval $\mathcal{I}_\Phi$ of the weak order. See for instance the work of Gashi, Schedler, and Speyer [5], where, among other results, they compute the Hilbert polynomial of these intervals.

The **Φ-hypersimplices** are the polytopes obtained by slicing Π_Φ with the hyperplanes of $\mathcal{H}_\Phi$ orthogonal to the highest root θ. Explicitly,

$$\Delta_{\Phi,k} := \{x \in \Pi_\Phi \mid k - 1 \leq \langle x, \theta \rangle \leq k\}, \quad \text{for } k = 1, \ldots, h(\Phi) - 1.$$

[1] Polytopes that are unions of alcoves of $\mathcal{H}_\Phi$ are called **alcoved polytopes** in [6, 7].

Set $\alpha_0 = -\theta$ and, for each $w \in W_\Phi$ and $i = 0, 1, \dots, n$, let $d_i(w) \in \{0, 1\}$ be 1 if and only if $w(\alpha_i) \in -\Phi^+$. The **circular descent number** of $w \in W_\Phi$ is

$$\mathrm{cdes}(w) := d_0(w) + a_1 d_1(w) + a_2 d_2(w) + \cdots + a_n d_n(w). \tag{2.1}$$

Theorem 9.3 of [7] shows that

$$\frac{|W_\Phi|}{n! \cdot a_1 \dots a_n} \mathrm{Vol}(\Delta_{\Phi,k}) = |\{w \in W_\Phi \mid \mathrm{cdes}(w) = k\}|. \tag{2.2}$$

The type C setting Let $e_1, \dots, e_n$ denote the canonical basis of $V = \mathbb{R}^n$. Let Φ_{C_n} denote the **root system of type C** with system of simple roots

$$\alpha_1 = 2e_1, \quad \alpha_2 = e_2 - e_1, \quad \dots, \quad \alpha_{n-1} = e_{n-1} - e_{n-2}, \quad \text{and} \quad \alpha_n = e_n - e_{n-1}.$$

The corresponding highest root is $\theta = 1\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + 2\alpha_n = 2e_n$, and so $h(\Phi_{C_n}) = 1 + 2 + \cdots + 2 + 1 = 2n$. The fundamental coweights are

$$\omega_1 = \tfrac{1}{2}(e_1 + e_2 + \cdots + e_n), \quad \omega_2 = e_2 + \cdots + e_n, \quad \dots, \quad \omega_{n-1} = e_{n-1} + e_n, \quad \omega_n = e_n.$$

Therefore, the lattice generated by the vertices of $A_\circ$ is $\Lambda = \mathbb{Z}\{\frac{\omega_i}{a_i}\}_i = \frac{1}{2}\mathbb{Z}^n$; which contains the vertices of all the alcoves of $\mathcal{H}_{\Phi_{C_n}}$. Every alcove is thus **unimodular** with respect to Λ, in particular they all have normalized volume 1. This also occurs in type A, but not in type B or D; see Section 6.

Explicit equations for the type C hypersimplices and fundamental parallelepiped appear in the Introduction; respectively in Equations (1.1) and (1.2). The associated finite Coxeter group is the hyperoctahedral group $W_{\Phi_C} \cong \mathfrak{B}_n$; see Section 3.1 for more details.

3 Combinatorial model

3.1 Big ascents and a new partial order

Given a positive integer n, write $[n] := \{1, \cdots, n\}$ and $[\![n]\!] := \{-n, \cdots, -1, 1, \cdots, n\}$. We also write $\bar{i} := -i$ for all integers i. A **signed permutation** of length n is a permutation w of the set $[\![n]\!]$ that satisfies $w(\bar{i}) = \overline{w(i)}$ for all $i \in [\![n]\!]$. Under composition, they form the **hyperoctahedral group**, denoted $\mathfrak{B}_n$. The **window notation** of a signed permutation $w \in \mathfrak{B}_n$ is the word $w_1 w_2 \dots w_n \in [\![n]\!]^n$, where $w_i = w(i)$. Similarly, write $w_{\bar{i}} = \overline{w_i} = w(\bar{i})$. The **complete notation** of $w \in \mathfrak{B}_n$ is the word $w_{\bar{n}} \cdots w_{\bar{1}} w_1 \cdots w_n$.

Endow $[\![n]\!]$ with the usual order

$$\bar{n} < \cdots < \bar{2} < \bar{1} < 1 < 2 < \cdots < n.$$

Write $i \ll j$ to indicate that $i < k < j$ for some $k \in [\![n]\!]$. For instance, $4 \ll 8$ and $\overline{2} \ll 1$, but $\overline{1} \not\ll 1$. We use i^+ to denote the **cyclic successor** of $i \in [\![n]\!]$ in the order above. For example, $\overline{1}^+ = 1$, $2^+ = 3$, and $n^+ = \overline{n}$. We introduce a new statistic on signed permutations.

Definition 3.1. We say that $w \in \mathfrak{B}_n$ has a **big ascent** at position $i \in \{\overline{1}\} \cup [n]$ if $w_i \ll w_{i^+}$. Let $\mathrm{BAsc}(w)$ denote the set of big ascents of w and $\mathrm{basc}(w) := |\mathrm{BAsc}(w)|$. See Figure 1 for many examples of big ascents.

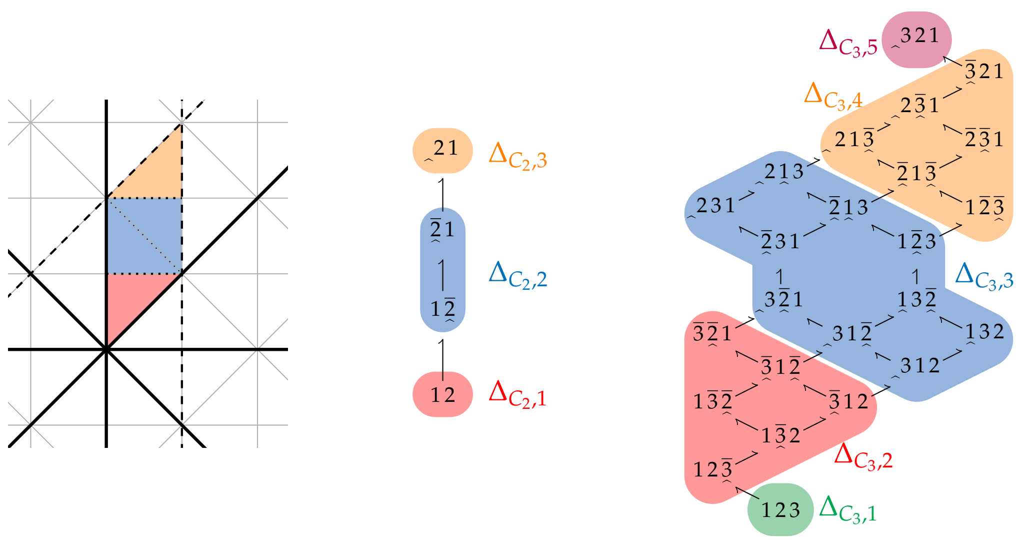

Figure 1: The affine arrangement $\mathcal{H}_{\Phi_{C_2}}$ (left) with the fundamental parallelepiped Π_{C_2} subdivided into the three hypersimplices $\Delta_{C_2,k}$. The poset $(X_n, \leq)$ for $n = 2$ (center) and $n = 3$ (right) with the big ascent positions $i \in \{\overline{1}\} \cup [n]$ underlined with a little wedge "$\frown$". Elements $w \in X_n$ are grouped by the value of $\mathrm{cdes}(w^{-1})$. Therefore, as an application of Theorem 1.1, we can read from the figure that $h^*_{\Delta'_{C_{3,3}}}(t) = 5t + 5t^2$.

We use this new statistic to construct our combinatorial model of the poset $\mathcal{I}_{C_n}$ of alcoves inside the fundamental parallelepiped.

Definition 3.2. For $n \geq 1$, let X_n be the following collection of signed permutations:

$$X_n := \{w \in \mathfrak{B}_n \mid w^{-1}(1) \in [n]\}.$$

That is, X_n is the set of signed permutations $w \in \mathfrak{B}_n$ such that 1 appears in its window notation. In addition, define the binary relation $\rightharpoonup$ on X_n as follows. We say $u \rightharpoonup w$, with $w = w_1 w_2 \ldots w_{n-1} w_n$, if

$$\begin{cases} \overline{1} \in \mathrm{BAsc}(w) & \text{and} \quad u = \overline{w_1} w_2 \ldots w_{n-1} w_n; \text{ or} \\ i \in \mathrm{BAsc}(w) \cap [n-1] & \text{and} \quad u = w_1 \ldots w_{i-1} w_{i+1} w_i w_{i+2} \ldots w_n; \text{ or} \\ n \in \mathrm{BAsc}(w) & \text{and} \quad u = w_1 w_2 \ldots w_{n-1} \overline{w_n}. \end{cases} \tag{3.1}$$

Proposition 3.3. *The relation $\rightharpoonup$ is the covering relation of a partial order $\leq$ on X_n.*

Figure 1 shows the Hasse diagram of $(X_n, \leq)$ for $n = 2, 3$. Proposition 3.3 is a corollary of a stronger result: Theorem 3.8 shows that, in fact, $(X_n, \leq)$ is isomorphic to $\mathcal{I}_{C_n}$ by constructing an explicit poset isomorphism.

3.2 Poset isomorphism

In order to describe an explicit isomorphism $X_n \to \mathcal{I}_{C_n}$, we first need to understand the circular descent statistic of Lam and Postnikov Equation (2.1) on signed permutations for our particular choice of simple roots at the end of Section 2.

Definition 3.4. We say that a signed permutation $w \in \mathfrak{B}_n$ has a (type C) **circular descent** at position $i \in [\![n]\!]$ if $w_i \gg w_{i^+}$. Let $\mathrm{CDes}(w)$ denote the set of circular descents of w.

One can easily verify that the definition above is compatible with the statistic of Lam and Postnikov, in the sense that $\mathrm{cdes}(w) = |\,\mathrm{CDes}(w)|$ for all signed permutations w.

Example 3.5. Consider the signed permutation with complete notation $w = \overline{5}\,\overline{1}\underset{\circ}{4}\underset{\circ}{2}\overline{3}\underset{\circ}{3}\underset{\circ}{\overline{2}}\overline{4}1\underset{\circ}{5}$. The marked positions represent the circular descents of w, thus $\mathrm{CDes}(w) = \{\overline{3}, \overline{2}, 1, 2, 5\}$.

Given a signed permutation $w \in X_n$, construct points $v^1(w), \ldots, v^{n+1}(w) \in \mathbb{R}^n$ recursively as follows. First, $v^{n+1}(w)$ is the point with integer coordinates

$$v_i^{n+1}(w) := \left| [\overline{i}, \overline{1}] \cap \mathrm{CDes}(w^{-1}) \right|, \qquad \text{for } i = 1, 2, \ldots, n.$$

Then, for $k \in [n]$, let

$$v^k(w) := v^{k+1}(w) + \tfrac{1}{2} e_{w_k} = v^{n+1}(w) + \tfrac{1}{2}\left(e_{w_k} + \cdots + e_{w_n}\right).$$

Notice that, since $w^{-1}(1) \in [n]$ for all $w \in X_n$, the first coordinate of $v^{n+1}(w)$ is always 0.

Remark 3.6. We can explicitly describe the set $\mathrm{CDes}(w^{-1})$ using the complete notation of w: $i \in \mathrm{CDes}(w^{-1})$ if and only if i^+ appears to the left of i in $w_{\overline{n}} \cdots w_{\overline{1}} w_1 \cdots w_n$.

Example 3.7. Consider the signed permutation with complete notation $w = \overline{5}3\overline{1}2\overline{4}4\overline{2}1\overline{3}5$. It is the inverse of the permutation in Example 3.5. One can directly verify that, for instance, $\overline{2} \in \mathrm{CDes}(w^{-1})$ by observing that $\overline{1} = \overline{2}^+$ appears before $\overline{2}$ in complete notation. Since $\mathrm{CDes}(w^{-1}) \cap [\overline{5}, \overline{1}] = \{\overline{2}, \overline{3}\}$, then $v^6(w) = (0, 1, 2, 2, 2)$. The other $v^i = v^i(w)$ are

$$v^5 = (0,1,2,2,\tfrac{5}{2}),\ \ v^4 = (0,1,\tfrac{3}{2},2,\tfrac{5}{2}),\ \ v^3 = (\tfrac{1}{2},1,\tfrac{3}{2},2,\tfrac{5}{2}),\ \ v^2 = (\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2},2,\tfrac{5}{2}),\ \text{ and }\ v^1 = (\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2},\tfrac{5}{2}).$$

Theorem 3.8. *For $w \in X_n$, let $A(w)$ be the convex hull of the points $v^k(w)$ for $k \in [n+1]$. Then, $A(w)$ is an alcove of $\mathcal{H}_{C_n}$ contained in Π_{C_n}. Moreover,*

$$A : X_n \to \mathcal{I}_{C_n} : w \mapsto A(w)$$

is a poset isomorphism.

Proof sketch. For any w, the vectors $\{\frac{1}{2}e_{w_i}\}_{i\in[n]}$ form a basis of $\mathbb{R}^n$. Hence, the polytope $A(w)$ is a full-dimensional simplex. Moreover, the vertices of $A(w)$ completely determine w since they can be ordered by its number of non-integer coordinates. Hence, if well-defined, the map A is necessarily injective.

The poset isomorphism is shown by a case by case analysis. We verify that alcoves $A' \in \mathcal{I}_{C_n}$ covering $A(w)$ uniquely correspond to relations $w \rightharpoonup w'$ in X_n that moreover satisfy $A' = A(w')$. □

In fact, one can be more explicit about the location of the alcove $A(w)$. Observe that

$$\langle v^{n+1}(w), \theta\rangle = 2|\operatorname{CDes}(w^{-1}) \cap [\overline{n}, \overline{1}]| = \begin{cases} \operatorname{cdes}(w^{-1}) - 1 & \text{if } w^{-1}(n) > 0, \\ \operatorname{cdes}(w^{-1}) & \text{if } w^{-1}(n) < 0. \end{cases}$$

Since $\langle \frac{1}{2}e_n, \theta\rangle = 1$, we deduce the following.

Proposition 3.9. *For every $w \in X_n$, the alcove $A(w)$ lies inside the hypersimplex $\Delta_{C_n,\operatorname{cdes}(w^{-1})}$.*

3.3 First formula for the h^*-polynomials

Let P be a polytope with a unimodular triangulation Γ. Suppose that we have a linear ordering[2] $T_1 \prec T_2 \prec \ldots$ of the maximal simplices in Γ such that the intersection of T_k with $T_1 \cup \cdots \cup T_{k-1}$ is a union of facets of T_k; let $c(T_k)$ be the number of facets in this intersection. Then, a result of Stanley [10] shows that

$$h^*_P(t) = \sum_i t^{c(T_i)}. \tag{3.2}$$

Proof of Theorem 1.1. For $n, k \geq 1$, let $P_{n,k} = \{x \in \Pi_{C_n} \mid 2x_1 \leq k\}$. The polytope $P_{n,k}$ is the disjoint union of $\Delta'_{C_n,1}, \ldots, \Delta'_{C_n,k}$, and so $L_{P_{n,k}}(r) = L_{\Delta'_{C_n,1}}(r) + \cdots + L_{\Delta'_{C_n,k}}(r)$. Moreover, since all these polytopes have the same dimension, $h^*_{P_{n,k}}(t) = h^*_{\Delta'_{C_n,1}}(t) + \cdots + h^*_{\Delta'_{C_n,k}}(t)$.

The collection $\mathcal{P}_{n,k}$ of alcoves contained in $P_{n,k}$ forms an order ideal of $\mathcal{I}_{C_n}$. Fix any linear extension $\prec$ of $\mathcal{I}_{C_n}$ such that alcoves in $\mathcal{P}_{n,k}$ appear before those in $\mathcal{P}_{n,k+1} \setminus \mathcal{P}_{n,k}$. Since every linear extension of the weak order on a hyperplane arrangement [3] gives a shelling of the complex of regions (see for instance [9, Proposition 3.4]), $\prec$ defines a shelling on the alcove triangulation of each $P_{n,k}$. Moreover, the number of facets $c(A)$ of an alcove A that appear in the intersection with the preceding alcoves is precisely the number of elements covered by A in the weak order. Finally, using Theorem 3.8, Proposition 3.9, and Stanley's formula Equation (3.2), we deduce

$$h^*_{P_{n,k}}(t) = \sum_{\substack{w\in X_n: \\ \operatorname{cdes}(w^{-1})\leq k}} t^{\operatorname{basc}(w)}, \quad \text{and} \quad h^*_{\Delta'_{C_n},k}(t) = h^*_{P_{n,k}}(t) - h^*_{P_{n,k-1}}(t) = \sum_{\substack{w\in X_n: \\ \operatorname{cdes}(w^{-1})=k}} t^{\operatorname{basc}(w)}. \quad \square$$

[2]Such an ordering is called a **shelling** of Γ, and it does not necessarily exist.

[3]Considering the cone over the finite arrangement of hyperplanes intersecting Π_{C_n}.

4 Relation with flag statistics

So-called *flag-statistics* are now classical in the study of signed permutations; we recall the definition of two that will play an important roll in what follows. The **flag-descent** and **flag-excedance** statistics on a signed permutation $w \in \mathfrak{B}_n$ are, respectively,

$$\begin{aligned} \operatorname{fdes}(w) &:= 2|\{i \in [n-1] \mid w_i > w_{i+1}\}| + \varepsilon_1(w), \\ \operatorname{fexc}(w) &:= 2|\{i \in [n-1] \mid w_i > i\}| + |\{i \in [n] \mid w_i < 0\}|. \end{aligned} \tag{4.1}$$

where $\varepsilon_1(w) \in \{0,1\}$ is 1 if and only if $w_1 < 0$. For example, if $w = \bar{3}52\bar{1}\bar{4}$, then $\operatorname{fdes}(w) = 2 \cdot 3 + 1 = 7$ and $\operatorname{fexc}(w) = 2 \cdot 1 + 3 = 5$.

The statistic fdes was introduced at the turn of the millennium by Adin, Brenti, and Roichman in [2], and it is related to the **Coxeter descent** statistic des_W through the simple formula $\operatorname{des}_W(w) = \lfloor \frac{\operatorname{fdes}(w)+1}{2} \rfloor$. Nearly a decade later, Foata and Han introduced the fexc statistic in [4], where they also show that both fexc and fdes have the same distribution. Their proof involved computing a generating function for the pair of statistics $(\operatorname{fexc}, \operatorname{fdes})$ over all hyperoctahedral groups $\{\mathfrak{B}_n\}_n$, and observing that the function is symmetric in the two parameters. By manipulating their formula [4, Equation (9.3)], we derive the following generating function for $(\operatorname{fexc}, \operatorname{des}_W)$ restricted to the subsets $\{X_n\}_n$.

Proposition 4.1. *The following identity holds.*

$$\sum_{n \geq 1} \sum_{w \in X_n} s^{\operatorname{fexc}(w)} t^{\operatorname{des}_W(w)} \frac{u^n}{(1-t)^{n+1}} = \frac{1}{(1+s)} \sum_{r \geq 0} \frac{(1-us^2)^{r+1} - (1-u)^{r+1}}{(1-u)^{r+1} - s(1-u)(1-us^2)^r} t^r. \tag{4.2}$$

The next formula is derived by counting integer points inside the dilations $r\Delta'_{C_n,k+1}$ and using similar techniques to those in the work of Li for type A [8].

Proposition 4.2. *For every $r \geq 0$,*

$$\sum_{n \geq 1} \sum_{k \geq 0} L_{\Delta'_{C_n,k+1}}(r) s^k u^n = \frac{(1-us^2)^{r+1} - (1-u)^{r+1}}{(1+s)\big((1-u)^{r+1} - s(1-u)(1-us^2)^r\big)}. \tag{4.3}$$

Proof of Theorem 1.2. Multiply both sides of Equation (4.3) by t^r and sum over all $r \geq 0$. The resulting right-hand side is exactly the right-hand side of Equation (4.2). Therefore,

$$\sum_{n \geq 1} \sum_{k \geq 0} \sum_{r \geq 0} L_{\Delta'_{C_n,k+1}}(r) t^r s^k u^n = \sum_{n \geq 1} \sum_{w \in X_n} s^{\operatorname{fexc}(w)} t^{\operatorname{des}_W(w)} \frac{u^n}{(1-t)^{n+1}}.$$

Finally, extracting the coefficient of $s^{k-1}u^n$ on both sides of this equation, we obtain

$$\sum_{r \geq 0} L_{\Delta'_{C_n,k}}(r) t^r = (1-t)^{-(n+1)} \sum_{\substack{w \in X_n: \\ \operatorname{fexc}(w)=k-1}} t^{\operatorname{des}_W(w)}. \qquad \square$$

5 Further results

5.1 Relation with type B Eulerian numbers

Recall from the Introduction the well-known relation between the volumes of (type A) hypersimplices and Eulerian numbers: $A_{n-1,k-1} = \mathrm{Vol}(\Delta_{n,k})$. Since $W_{\Phi_{C_n}} = W_{\Phi_{B_n}} = \mathfrak{B}_n$, it is customary to denote by $B_{n,k}$ the **type BC Eulerian numbers**, defined by

$$B_{n,k} := |\{w \in \mathfrak{B}_n \mid \mathrm{des}_W(w) = k\}|.$$

In contrast to the type A case, $\mathrm{des}_W(w)$ only takes $n+1$ possible values, whereas there are $2n-1$ hypersimplices $\Delta_{C_n,k}$. Thus, a direct connection between the type BC Eulerian numbers and the volume of the type C hypersimplices–similar to the one in type A–is not expected. Nonetheless, we find the following surprising relation.

Proposition 5.1. *For all $n \geq 1$ and $k \geq 0$,*

$$B_{n,k} = \mathrm{Vol}(\Delta_{C_n,2k-1}) + 2\,\mathrm{Vol}(\Delta_{C_n,2k}) + \mathrm{Vol}(\Delta_{C_n,2k+1}),$$

where $\mathrm{Vol}(\Delta_{C_n,j}) = 0$ whenever $j < 1$ or $j > 2n-1$.

Proof. Consider the bijection $\phi : X_n \to \mathfrak{B} \setminus X_n : w \mapsto \phi(w)$ which negates the 1 in the window notation of w. It follows from the definition of fexc in Equation (4.1) that $\mathrm{fexc}(\phi(w)) = \mathrm{fexc}(w) + 1$ for all $w \in X_n$. Moreover, since fexc and fdes have the same distribution and $\mathrm{des}_W(w) = \lfloor \frac{\mathrm{fdes}(w)+1}{2} \rfloor$ for all $w \in \mathfrak{B}_n$ (see Section 4), we have

$$\begin{aligned} B_{n,k} &= |\{w \in \mathfrak{B}_n \setminus X_n \mid \mathrm{fexc}(w) \in \{2k-1, 2k\}\}| + |\{w \in X_n \mid \mathrm{fexc}(w) \in \{2k-1, 2k\}\}| \\ &= |\{w \in X_n \mid \mathrm{fexc}(w) \in \{2k-2, 2k-1\}\}| + |\{w \in X_n \mid \mathrm{fexc}(w) \in \{2k-1, 2k\}\}| \\ &= \mathrm{Vol}(\Delta_{C_n,2k-1}) + 2\,\mathrm{Vol}(\Delta_{C_n,2k}) + \mathrm{Vol}(\Delta_{C_n,2k+1}), \end{aligned}$$

where the last equality follows from Corollary 1.3. □

5.2 Limiting poset

Theorem 1.4 provides a simple description of the exponential generating function for the distribution of cover relations in $\mathcal{I}_{\Phi_{C_n}} \cong (X_n, \leq)$. This prompts natural questions about the asymptotic behavior of these partial orders. In type A, Laget-Chapelier, Reutenauer, and the first author [1] proved that $\mathcal{I}_{\Phi_{A_n}}$, the weak order on the alcoves contained in Π_{A_n}, converges to Young's lattice of partitions as n goes to infinity. We establish an analogous result for $(X_n, \leq)$.

Consider the order ideal $Y_n := \{w \in X_n \mid \mathrm{cdes}(w^{-1}) \leq 2\}$ of X_n. It consists of those signed permutations $w \in X_n$ whose associated alcoves $A(w)$ lie inside $\Delta_{C_n,1} \cup \Delta_{C_n,2}$. Let $\tau_n : Y_n \to X_{n+1}$ be the function $\tau(w_1 w_2 \ldots w_n) = 1 w'_1 w'_2 \cdots w'_n$ where $w'_i \in [\![n+1]\!]$ has the same sign as w_i and satisfies $|w'_i| = |w_i| + 1$. Let $\mathcal{SP}$ be the lattice of strict partitions.

Theorem 5.2. *For all $n \geq 1$, $\tau_n(Y_n) \subset Y_{n+1}$ and τ_n is actually a poset embedding. Moreover, the colimit in the category of posets of the diagram $\left(Y_1 \xrightarrow{\tau_1} Y_2 \xrightarrow{\tau_2} Y_3 \xrightarrow{\tau_3} \dots\right)$ is $\mathcal{SP}$.*

Proof sketch. The cover relations in Y_n are only of the second (with $w_i > 0 > w_{i+1}$) and third type in Equation (3.1). Therefore, the window notation of elements $w \in Y_n$ are precisely the shuffles of the words $12\dots k$ and $\overline{n}\dots\overline{k+1}$ for some $1 \leq k \leq n$. Define a map $\lambda_n : Y_n \to \mathcal{SP}$ by $\lambda_n(w) := (n+1-i \mid i \in [n], w_i < 0)$. It follows form the form of the cover relations in Y_n that λ_n sends cover relations in Y_n to cover relations in $\mathcal{SP}$.

Since τ_n only adds a positive letter in the first position, $\lambda_{n+1} \circ \tau_n = \lambda_n$. This shows that $\mathcal{SP}$ (together with the maps λ_n) is a *cocone* over the diagram $\left(Y_1 \xrightarrow{\tau_1} Y_2 \xrightarrow{\tau_2} \dots\right)$. To prove that $\mathcal{SP}$ is in fact the colimit of this diagram, it suffices to observe that any strict partition μ with $\mu_1 \leq n$ determines a unique shuffle w of $1,\dots,k$ and $\overline{n}\dots\overline{k+1}$ such that $\lambda_n(w) = \mu$ and $k = n - \ell(\mu)$: the shuffle for which the negative letters appear in positions $\mu_{\ell(\mu)},\dots,\mu_2,\mu_1$. $\square$

6 Open questions and closing remarks

6.1 What about type B and D?

Let Φ_{B_n} be the root system of type B of rank $n \geq 3$. Unlike in type C, the lattice spanned by the vertices of the fundamental alcove is not stable under the action of $\widetilde{W}_{\Phi_{B_n}}$; and therefore it does not contain all the vertices of the alcoves of $\mathcal{H}_{\Phi_{B_n}}$. Thus, the alcove triangulation of the parallelepiped Π_{B_n} is not unimodular with respect to any lattice containing the vertices of all the alcoves. The same occurs in type D, starting at rank 4.

In particular, we cannot apply the ideas in Section 3.3 to interpret the cover-counting polynomials $\Psi_{B_n}(t)$ (resp. $\Psi_{D_n}(t)$) of the interval $\mathcal{I}_{\Phi_{B_n}}$ (resp. $\mathcal{I}_{\Phi_{D_n}}$) as the h^*-polynomial of the fundamental parallelepiped. However, computational evidence suggests the following intriguing relation between $\Psi_{B_n}(t)$ and the type D Eulerian polynomials $\mathrm{D}_n(t)$.

Conjecture 6.1. *For $n \geq 3$, $\Psi_{B_n}(t) + t^n\Psi_{B_n}(t^{-1}) = 2D_n(t)$.*

Recall that, in type C, Theorem 3.8 shows that $\Psi_{C_n}(t) = h^*_{\Pi_{C_n}}(t)$. Thus, the type C analog of the conjecture above follows almost directly from Equation (1.4).

Proposition 6.2. *For $n \geq 2$, $\Psi_{C_n}(t) + t^n\Psi_{C_n}(t^{-1}) = B_n(t)$.*

6.2 Equidistribution of pairs of statistics

Theorems 1.1 and 1.2 provide formulas for the h^*-polynomial of type C hypersimplices, each using a different pair of statistics on X_n. The first formula involves the pair $(\mathrm{cdes}, \mathrm{basc})$, while the second utilizes $(\mathrm{fexc}, \mathrm{des}_W)$. This prompts a natural question.

Open problem 6.3. Give a bijective proof of the fact that the pairs

$$(\mathrm{cdes}(w^{-1}), \mathrm{basc}(w)) \qquad \text{and} \qquad (\mathrm{fexc}(w)+1, \mathrm{des}_W(w))$$

have the same joint-distribution over X_n.

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