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A generalized Lalanne–Kreweras involution for rectangular tableaux

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Abstract. We give a bijective proof of the symmetry of the distribution of the number of descents over standard Young tableaux of any given rectangular shape. Our bijection can be viewed as a generalization of the Lalanne–Kreweras involution on Dyck paths (corresponding to the 2-row case), which proves the symmetry of the Narayana numbers. Using certain arrow encodings, we also describe new statistic-preserving involutions on rectangular tableaux, which give simpler proofs of known results, and allow us to prove a conjecture of Sulanke about the distribution of certain statistics on three-row tableaux. Finally, our construction provides a bijective proof of the symmetry of the number of descents on so-called canon permutations, and suggests a possible notion of rowmotion on standard Young tableaux of rectangular shape.

Keywords: standard Young tableau, descent, bijection, Narayana number, rectangular tableau, canon permutation

1 Introduction

Let $\mathcal{D}_n$ be the set of Dyck paths of semilength n, defined as lattice paths from $(0,0)$ to $(2n,0)$ with steps $\mathtt{u} = (1,1)$ and $\mathtt{d} = (1,-1)$ that never go below the x-axis. A classical result in enumerative combinatorics is that the number of such paths with h valleys (i.e., consecutive pairs $\mathtt{du}$), or equivalently $h+1$ peaks (i.e., consecutive pairs $\mathtt{ud}$), is given by the Narayana numbers, which we denote by $N(2,n,h) = \frac{1}{n}\binom{n}{h}\binom{n}{h+1}$. A consequence of this formula is that

$$N(2,n,h) = N(2,n,n-h-1), \tag{1.1}$$

which says that the number of paths in $\mathcal{D}_n$ with $h+1$ peaks equals the number of those with $n-h$ peaks. A bijective proof of this non-obvious symmetry is given by a beautiful involution that was first considered by Kreweras [8] and later studied by Lalanne [9], often referred to as the Lalanne–Kreweras involution [7, 5].

Dyck paths can be viewed as standard Young tableaux of shape $2 \times n$, and so a natural generalization is to consider tableaux of any rectangular shape. In the rest of the paper, we assume that $k, n \geq 1$. Let $\mathrm{SYT}(n^k)$ denote the set of standard Young tableaux

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of rectangular shape consisting of k rows and n columns. Such tableaux have kn cells, filled with the numbers $1, 2, \dots, kn$ so that entries in rows increase from left to right, and entries in columns increase from top to bottom. For $1 \le i \le kn-1$, we say that i is a descent (resp. ascent) of $T \in \mathrm{SYT}(n^k)$ if i appears in a row higher (resp. lower) than $i+1$ in T. Denote by $\mathrm{Des}(T)$ and $\mathrm{Asc}(T)$ the sets of descents and ascents of T, respectively, and denote the cardinalities of these sets by $\mathrm{des}(T) = |\mathrm{Des}(T)|$ and $\mathrm{asc}(T) = |\mathrm{Asc}(T)|$. For example, for the tableau T in the left of Figure 1, $\mathrm{des}(T) = 13$ and $\mathrm{asc}(T) = 11$.

A straightforward bijection $\Delta : \mathrm{SYT}(n^2) \to \mathcal{D}_n$ is obtained by letting the ith step of the path be $\mathtt{u}$ if i is in the first row of the tableau, and $\mathtt{d}$ otherwise, for each $1 \le i \le 2n$. Under this bijection, ascents and descents of the tableau correspond to valleys and peaks of the Dyck path, respectively. We can thus interpret equation (1.1) as stating that the number of tableaux in $\mathrm{SYT}(n^2)$ with $h+1$ descents equals the number of those with $n-h$ descents.

In general, for an arbitrary number k of rows, the distribution of the number of ascents and descents on standard Young tableaux of rectangular shape is given by the *generalized Narayana* (or *k-Narayana*) *numbers*, which already appear in work of MacMahon [10]. More recently, these numbers were studied by Sulanke [15, 16] in the context of higher-dimensional lattice paths, and they can be defined as

$$N(k,n,h) = |\{T \in \mathrm{SYT}(n^k) : \mathrm{asc}(T) = h\}|. \tag{1.2}$$

As shown in [15], they also have an equivalent definition in terms of descents as

$$N(k,n,h) = |\{T \in \mathrm{SYT}(n^k) : \mathrm{des}(T) = h+k-1\}|. \tag{1.3}$$

Sulanke obtained the following formula for the generalized Narayana numbers from Stanley's theory of P-partitions [13], and he noted that it is implicit in MacMahon's work on plane partitions [10]. Here we state it in a slightly different form.

Theorem 1.1 ([15, Proposition 1])**.** *For* $0 \le h \le (k-1)(n-1)$*, we have*

$$N(k,n,h) = \sum_{\ell=0}^{h} (-1)^{h-\ell} \binom{kn+1}{h-\ell} \prod_{i=0}^{n-1} \prod_{j=0}^{k-1} \frac{i+j+1+\ell}{i+j+1}.$$

Perhaps surprisingly, the generalized Narayana numbers have the following symmetry, which generalizes equation (1.1).

Theorem 1.2 ([15, Corollary 1])**.** *For all* $0 \le h \le (k-1)(n-1)$*, we have*

$$N(k,n,h) = N(k,n,(k-1)(n-1)-h).$$

In [15], Sulanke deduces this symmetry from Theorem 1.1 after some manipulations, but it is in fact a special case of a more general result: the descent polynomial of any graded poset is palindromic. We refer the reader to [13, Section 3.15] for definitions, but let us briefly explain this here. The descent polynomial of a poset enumerates its linear extensions by the number of descents with respect to some natural labeling. It follows from [13, Corollary 3.15.18] that this polynomial is palindromic if and only if the poset is graded, i.e., every maximal chain has the same length. Theorem 1.2 is a special case because the elements of $\mathrm{SYT}(n^k)$ can be viewed as linear extensions of the graded poset $\mathbf{k} \times \mathbf{n}$, defined as the product of a k-element chain by an n-element chain. Ascents of the tableau correspond to descents of the linear extension with respect to the natural labeling where the cell in row r and column c has label $c + (r-1)n$ for all r, c.

However, none of the existing proofs of Theorem 1.2 is bijective. The main result of this paper is a bijective proof of Theorem 1.2.

In Section 2 we describe the main bijection, which relies on certain encodings of standard Young tableaux that use arrows to indicate the placement of the entries. In Section 3, we show that our bijection has the desired properties. In Section 4, we use arrow encodings to introduce other natural involutions on $\mathrm{SYT}(n^k)$, simplifying a result of Sulanke [15] that relates descents with ascents, giving bijective proofs of other symmetries, and describing a map that extends *rowmotion* [14] from Dyck paths to $\mathrm{SYT}(n^k)$. In Section 5, we consider more refined descent statistics, and we prove a conjecture of Sulanke [16] about distributions on 3-row tableaux. We also describe a family of $k!$ statistics on $\mathrm{SYT}(n^k)$ that have a generalized Narayana distribution, proving another conjecture of Sulanke [15]. Our construction gives a bijective proof of the symmetry of the number of descents on certain multiset permutations called canon permutations [5]. Finally, we state some open questions in Section 6. Most proofs are omitted but can be found in [4].

Let us finish this introduction by mentioning some related bijections from the literature. In the case $k = 2$, our main bijection is equivalent to the Lalanne–Kreweras involution on Dyck paths. Generalizations of this classical involution in a different direction have been considered by Hopkins and Joseph [7], who extend it to the piecewise-linear and birational realms, and describe a more general involution, called *rowvacuation*, on the set of order ideals of any graded poset. However, these generalizations do not help when dealing with rectangular tableaux with more than two rows.

The *conjugate* of T is the tableau $T' \in \mathrm{SYT}(k^n)$ obtained by reflecting T along the main diagonal, so that cell (r, c) of T becomes cell (c, r) of T'. For $1 \le i \le kn-1$, we have $i \in \mathrm{Des}(T)$ if and only if $i \notin \mathrm{Des}(T')$, and so $\mathrm{des}(T) + \mathrm{des}(T') = kn - 1$.

The *rotation* of T is the tableau $\theta(T) \in \mathrm{SYT}(n^k)$ obtained by rotating T by 180 degrees, so that cell (r, c) becomes cell $(k+1-r, n+1-c)$, and replacing each entry i with $kn+1-i$. Rotation of rectangular tableaux coincides with *evacuation*, introduced by Schützenberger [11]. For $1 \le i \le kn-1$, we have that $i \in \mathrm{Des}(T)$ if and only if $kn - i \in \mathrm{Des}(\theta(T))$, and similarly $i \in \mathrm{Asc}(T)$ if and only if $kn - i \in \mathrm{Asc}(\theta(T))$. It follows that $\mathrm{des}(T) = \mathrm{des}(\theta(T))$ and $\mathrm{asc}(T) = \mathrm{asc}(\theta(T))$.

Both conjugation and rotation are involutions, meaning that $T'' = T$ and $\theta(\theta(T)) = T$.

2 The bijection

In light of equation (1.3), in order to prove Theorem 1.2 bijectively, it suffices to construct a bijection between tableaux in $\mathrm{SYT}(n^k)$ with $d = h + k - 1$ descents, and tableaux with $(k-1)(n+1) - d = (k-1)(n-1) - h + k - 1$ descents. The following is our main result.

Theorem 2.1. *There exists an involution $\varphi\colon \mathrm{SYT}(n^k) \to \mathrm{SYT}(n^k)$ such that, for all $T \in \mathrm{SYT}(n^k)$,*

$$\mathrm{des}(T) + \mathrm{des}(\varphi(T)) = (k-1)(n+1). \tag{2.1}$$

2.1 Arrow encodings

To describe our construction, we encode tableaux $T \in \mathrm{SYT}(n^k)$ using arrows describing the placement of $1, 2, \ldots, kn$ in T. We assign a (possibly empty) sequence of arrows $\uparrow$ and $\downarrow$ to the right border of each cell, and to the left border of the cells in the first column.

For $1 \le r \le k$ and $1 \le c \le n$, let (r, c) be the cell in row r and column c. We denote by $A_{r,c}$ the sequence of arrows assigned to the right border of the cell (r, c), and by $A_{r,0}$ the sequence of arrows assigned to the left border of the cell $(r, 1)$. For example, in Figure 1, we have $A_{2,3} = \downarrow\uparrow\downarrow$. An array $\{A_{r,c}\}_{1 \le r \le k, 0 \le c \le n}$ is called an *arrow array*.

For $1 \le i \le kn$, denote by $\mathrm{row}_T(i)$ the index of the row where i appears in T. Denote by $T^{\le i}$ be the standard Young tableau consisting of the entries less than or equal to i in T, and let $R_r^{\le i} = A_{r,c}$, where (r, c) is the rightmost filled cell in row r of $T^{\le i}$.

Definition 2.2. The *arrow encoding* of $T \in \mathrm{SYT}(n^k)$ is the arrow array $\{A_{r,c}\}_{1 \le r \le k, 0 \le c \le n}$, where each $A_{r,c}$ is a sequence of arrows in $\{\uparrow, \downarrow\}$ constructed as follows. Initially, set $A_{r,0} = \uparrow$ for $2 \le r \le k$, and $A_{r,c} = \epsilon$ (the empty sequence) for all other cells (r, c). Next, for each i from 1 to $kn - 1$:

- if $\mathrm{row}_T(i) = \mathrm{row}_T(i+1)$, do nothing;
- if $\mathrm{row}_T(i) < \mathrm{row}_T(i+1)$, append a $\downarrow$ to $R_r^{\le i}$ for each $\mathrm{row}_T(i) \le r < \mathrm{row}_T(i+1)$;
- if $\mathrm{row}_T(i) > \mathrm{row}_T(i+1)$, append a $\uparrow$ to $R_r^{\le i}$ for each $\mathrm{row}_T(i) \ge r > \mathrm{row}_T(i+1)$.

One can think of these arrows as describing how to reach the row where to place $i + 1$ from the row where i has been placed. See the left of Figure 1 for an example. Next, we will characterize which arrow arrays are obtained by encoding tableaux in $\mathrm{SYT}(n^k)$.

Definition 2.3. An arrow array $\{A_{r,c}\}_{1 \le r \le k, 0 \le c \le n}$ is *valid* if the following conditions hold:

- (Boundary) We have $A_{1,0} = \epsilon$, $A_{r,0} = \uparrow$ for $2 \le r \le k$, $A_{r,n} = \downarrow$ for $1 \le r \le k - 1$, and $A_{k,n} = \epsilon$.
- (Alternation) For all r, c, the sequence $A_{r,c}$ alternates between $\uparrow$ and $\downarrow$.
- (Matching) For every $1 \le r \le k - 1$, the total number of $\downarrow$ in row r equals the total number of $\uparrow$ in row $r + 1$. There are no $\uparrow$ in row 1 and no $\downarrow$ in row k.
- (Ballot) For every $1 \le r \le k - 1$ and $1 \le c \le n - 1$, the total number of $\downarrow$ in $A_{r,1}, A_{r,2}, \ldots, A_{r,c}$ is no more than the total number of $\uparrow$ in $A_{r+1,1}, A_{r+1,2}, \ldots, A_{r+1,c}$.

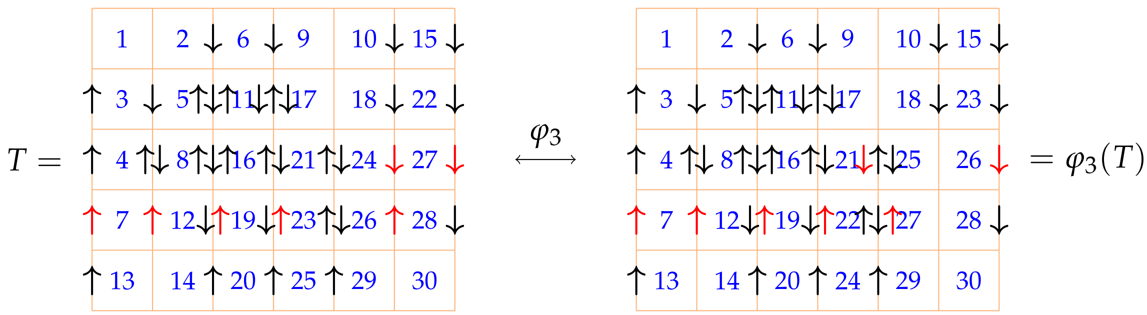

Figure 1: The arrow encoding of a tableau $T \in \mathrm{SYT}(6^5)$, and an example of the involution φ_r for $r = 3$. Each leading $\downarrow$ in row r and each trailing $\uparrow$ in row $r+1$ has been colored in red. The map φ_r removes pairs of red arrows in the same column, and adds such pairs to columns with no red arrows.

Lemma 2.4. *The arrow encoding from Definition 2.2 is a bijection between tableaux in* $\mathrm{SYT}(n^k)$ *and valid arrow arrays* $\{A_{r,c}\}_{1\le r\le k, 0\le c\le n}$.

Next, we define involutions on $\mathrm{SYT}(n^k)$, in terms of the arrow encoding of a tableau.

Definition 2.5. For $1 \le r \le k-1$, let $\varphi_r : \mathrm{SYT}(n^k) \to \mathrm{SYT}(n^k)$ be the map that sends T to the tableau $\varphi_r(T)$ whose arrow encoding is obtained from the arrow encoding $\{A_{r,c}\}$ of T as follows. For every $0 \le c \le n$,

- if $A_{r,c}$ starts with $\downarrow$ and $A_{r+1,c}$ ends with $\uparrow$, remove these two arrows;
- if $A_{r,c}$ does not start with $\downarrow$ and $A_{r+1,c}$ does not end with $\uparrow$, insert $\downarrow$ at the beginning of $A_{r,c}$ and $\uparrow$ at the end of $A_{r+1,c}$.

A $\downarrow$ at the beginning of $A_{r,c}$ will be called a *leading arrow*, and a $\uparrow$ at the end of $A_{r+1,c}$ will be called a *trailing arrow*. An example of φ_r appears in Figure 1.

One can check that φ_r is well defined (i.e., it produces a valid arrow array), and that it is an involution. We are now ready to define the main bijection φ. An example is given in the top of Figure 2. Note that the maps φ_r commute with each other.

Definition 2.6. Let $\varphi : \mathrm{SYT}(n^k) \to \mathrm{SYT}(n^k)$ be the composition $\varphi = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_{k-1}$.

3 Properties of φ

To analyze how φ behaves with respect to descents, we first need to introduce some notation. Let $T \in \mathrm{SYT}(n^k)$ with arrow encoding $\{A_{r,c}\}$. For $1 \le r \le k-1$, define the following subsets of $\{0, 1, \ldots, n\}$:

$$\mathcal{L}_r^{\downarrow}(T) = \{c : A_{r,c} \text{ has a leading } \downarrow\}, \qquad \mathcal{T}_{r+1}^{\uparrow}(T) = \{c : A_{r+1,c} \text{ has a trailing } \uparrow\},$$
$$\mathcal{T}_r^{\downarrow}(T) = \{c : A_{r,c} \text{ has a trailing } \downarrow\}, \qquad \mathcal{L}_{r+1}^{\uparrow}(T) = \{c : A_{r+1,c} \text{ has a leading } \uparrow\}.$$

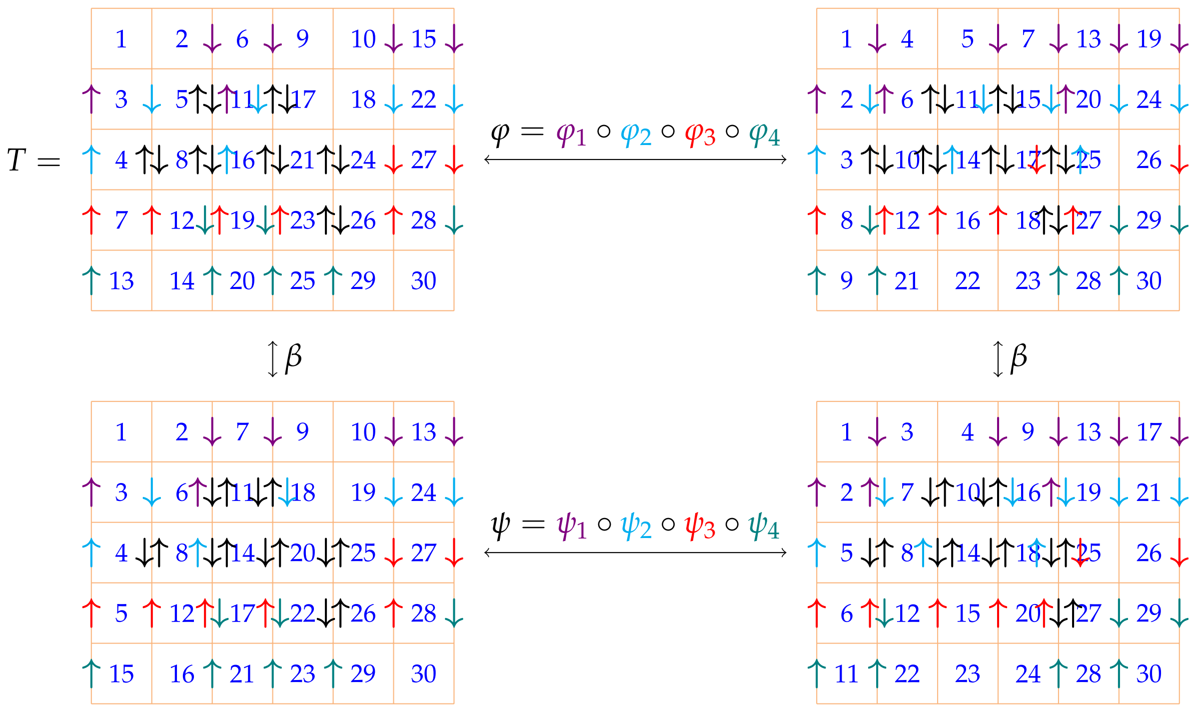

Figure 2: Examples of the involutions φ, β, and ψ. In the top tableaux, all the leading $\downarrow$ in row r and the trailing $\uparrow$ in row $r+1$ have the same color for each fixed r, to help visualize φ. After applying β, these become trailing $\downarrow$ and leading $\uparrow$, respectively.

Note that both $\mathcal{L}_r^{\downarrow}(T)$ and $\mathcal{T}_r^{\downarrow}(T)$ contain n, and both $\mathcal{T}_{r+1}^{\uparrow}(T)$ and $\mathcal{L}_{r+1}^{\uparrow}(T)$ contain 0. If S is any of the above subsets, we denote its complement by $\overline{S} = \{0, 1, \dots, n\} \setminus S$.

Lemma 3.1. *Let* $T \in \mathrm{SYT}(n^k)$ *and* $\widehat{T} = \varphi_r(T)$*, where* $1 \le r \le k-1$*. Then* $\mathcal{L}_r^{\downarrow}(\widehat{T}) = \overline{\mathcal{T}_{r+1}^{\uparrow}(T)}$*,* $\mathcal{T}_{r+1}^{\uparrow}(\widehat{T}) = \overline{\mathcal{L}_r^{\downarrow}(T)}$*, and* $\mathcal{L}_s^{\downarrow}(\widehat{T}) = \mathcal{L}_s^{\downarrow}(T)$*,* $\mathcal{T}_{s+1}^{\uparrow}(\widehat{T}) = \mathcal{T}_{s+1}^{\uparrow}(T)$ *for all* $s \neq r$*.*

The next lemma relates the above sets to the number of descents and ascents.

Lemma 3.2. *For any* $T \in \mathrm{SYT}(n^k)$*, we have*

$$\mathrm{des}(T) = \sum_{r=1}^{k-1} |\mathcal{L}_r^{\downarrow}(T)| = \sum_{r=2}^{k} |\mathcal{T}_r^{\uparrow}(T)|, \quad \mathrm{asc}(T) + k - 1 = \sum_{r=2}^{k} |\mathcal{L}_r^{\uparrow}(T)| = \sum_{r=1}^{k-1} |\mathcal{T}_r^{\downarrow}(T)|.$$

Proof of Theorem 2.1. The bijection $\varphi : \mathrm{SYT}(n^k) \to \mathrm{SYT}(n^k)$ from Definition 2.6 is an involution, since it is a composition of involutions φ_r that commute with each other. Let us show that it satisfies equation (2.1).

By Lemma 3.1 applied to each of the $k-1$ maps in the composition $\varphi = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_{k-1}$, we have $\mathcal{L}_r^{\downarrow}(\varphi(T)) = \overline{\mathcal{T}_{r+1}^{\uparrow}(T)}$ and $\mathcal{T}_{r+1}^{\uparrow}(\varphi(T)) = \overline{\mathcal{L}_r^{\downarrow}(T)}$ for all $1 \le r \le k-1$.

Therefore, using the two equalities in the first part of Lemma 3.2,

$$\begin{aligned}\operatorname{des}(\varphi(T)) &= \sum_{r=1}^{k-1} |\mathcal{L}_r^{\downarrow}(\varphi(T))| = \sum_{r=1}^{k-1} |\overline{\mathcal{T}_{r+1}^{\uparrow}(T)}| = \sum_{r=1}^{k-1} \left(n+1-|\mathcal{T}_{r+1}^{\uparrow}(T)|\right) \\ &= (k-1)(n+1)-\operatorname{des}(T). \end{aligned}$$
□

As an example, if T is the tableau on the top left of Figure 2, then $\operatorname{des}(T) = 13$, $\operatorname{des}(\varphi(T)) = 15$, and $\operatorname{des}(T) + \operatorname{des}(\varphi(T)) = 28 = (5-1)(6+1)$.

In the special case $k = 2$, it can be shown that φ is equivalent to the Lalanne–Kreweras involution for Dyck paths [9, 8], which is often defined in terms of the coordinates of the peaks of the path, see [5]. This equivalence is illustrated in Figure 3.

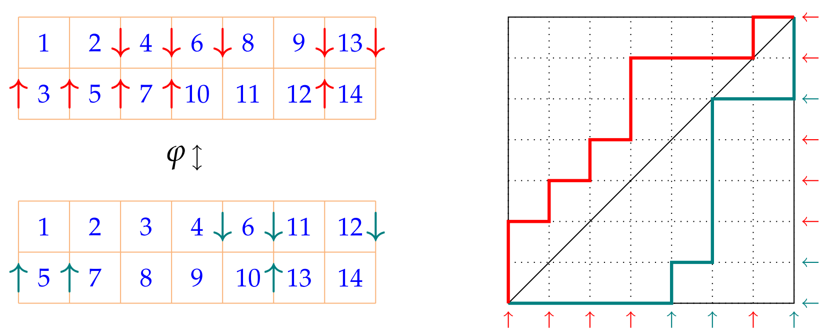

Figure 3: When $k = 2$, our map φ is equivalent to the Lalanne–Kreweras involution.

It is clear from the definition of φ that this bijection commutes with rotation, that is, $\theta(\varphi(T)) = \varphi(\theta(T))$ for all $T \in \mathrm{SYT}(n^k)$.

A much less obvious property is that φ also commutes with conjugation. This is surprising because our definition of φ treats rows and columns very differently.

Theorem 3.3. *For all $T \in \mathrm{SYT}(n^k)$, we have $\varphi(T)' = \varphi(T')$.*

4 Variations

The equivalence of the two definitions (1.2) and (1.3) of the generalized Narayana numbers, in terms of ascents and descents, respectively, is a consequence of a bijection of Sulanke [15, Proposition 2] between the set of tableaux in $\mathrm{SYT}(n^k)$ with h ascents and those with $h+k-1$ descents. His construction is a composition of $k-2$ bijections, and it is not an involution in general. We can use our arrow encodings give a simple description of a similar bijection, which has the advantage of being an involution.

Definition 4.1. Let $\beta : \mathrm{SYT}(n^k) \to \mathrm{SYT}(n^k)$ be the map that sends $T \in \mathrm{SYT}(n^k)$ to the tableau $\beta(T)$ whose arrow encoding is obtained from the arrow encoding $\{A_{r,c}\}$ of T by reversing (i.e., reading from right to left) each sequence $A_{r,c}$.

Examples of β are given in Figure 2. The following property follows from Lemma 3.2.

Theorem 4.2. *The map $\beta : \mathrm{SYT}(n^k) \to \mathrm{SYT}(n^k)$ is an involution such that, for all $T \in \mathrm{SYT}(n^k)$,*

$$\operatorname{des}(T) = \operatorname{asc}(\beta(T)) + k - 1 \quad \textit{and} \quad \operatorname{asc}(T) + k - 1 = \operatorname{des}(\beta(T)).$$

For example, if T is the tableau in the top left of Figure 2, we have $\operatorname{des}(\beta(T)) = 15 = \operatorname{asc}(T) + 4$ and $\operatorname{asc}(\beta(T)) = 9 = \operatorname{des}(T) - 4$.

Using Theorems 2.1 and 4.2, we can show that the composition $\psi = \beta \circ \varphi \circ \beta$ gives a direct bijective proof of Theorem 1.2, in terms of the original interpretation of generalized Narayana numbers as counting tableaux with a given number of ascents.

Theorem 4.3. *The map $\psi\colon \mathrm{SYT}(n^k) \to \mathrm{SYT}(n^k)$ is an involution such that, for all $T \in \mathrm{SYT}(n^k)$,*

$$\operatorname{asc}(T) + \operatorname{asc}(\psi(T)) = (k-1)(n-1).$$

The involution ψ has a direct description that is very similar to the description of φ, where trailing $\downarrow$ play the role of leading $\downarrow$, and leading $\uparrow$ play the role of trailing $\uparrow$. An example of ψ appears at the bottom of Figure 2.

Another symmetry of the generalized Narayana numbers is the following.

Proposition 4.4 ([15])**.** *For $0 \le h \le (k-1)(n-1)$, we have $N(k,n,h) = N(n,k,h)$.*

This symmetry is immediate from the expression in Theorem 1.1, and it is equivalent to [15, Proposition 6]. It also follows from the fact that any two natural labelings of a poset—which in this case are the labelings of $\mathbf{k} \times \mathbf{n}$ obtained by reading the tableaux "by rows" and "by columns"—give rise to the same descent polynomial [13]. But none of these arguments are bijective. Using Theorem 2.1, we can show that the map $T \mapsto \varphi(T)'$ gives a bijective proof of this symmetry.

Entries $i \in \operatorname{Des}(T)$ for which $i+1$ is strictly to the left of i are called *high descents* in [15]. Denoting by $\operatorname{hdes}(T)$ the number of high descents of T, we have $\operatorname{hdes}(T) = \operatorname{asc}(T')$. In the case $k = 2$, the bijection $\Delta : \mathrm{SYT}(n^2) \to \mathcal{D}_n$ described in Section 1 takes high descents of the tableau to *high peaks* of the Dyck path, which are consecutive pairs `ud` that do not touch the x-axis. Proposition 4.4 implies that the statitics hdes and asc are equidistributed on $\mathrm{SYT}(n^k)$. Here is a direct bijection proving this.

Proposition 4.5. *Let $T \in \mathrm{SYT}(n^k)$ and let $\widehat{T} = \beta(\varphi(\beta(T))')'$. Then* $\operatorname{asc}(T) = \operatorname{hdes}(\widehat{T})$.

5 Refined descent statistics and canon permutations

Next we define statistics that keep track not only of the relative position of i and $i+1$, but also of which rows they lie in, and we study how the maps from Sections 2 and 4 behave with respect to these refined statistics. For $T \in \mathrm{SYT}(n^k)$ and $1 \le r, s \le k$, let

$$b_{r,s}(T) = |\{i : \mathrm{row}_T(i) = r \text{ and } \mathrm{row}_T(i+1) = s\}|,$$

where $1 \leq i \leq kn-1$. These statistics record the *bounces* between the rows of T when reading the entries in increasing order. The following is a refinement of Lemma 3.1.

Lemma 5.1. *For $1 \leq r \leq k-1$, the involution φ_r swaps the statistics*

$$b_{r,1}+b_{r,2}+\cdots+b_{r,r} \leftrightarrow b_{1,r+1}+b_{2,r+1}+\cdots+b_{r,r+1}-1,$$
$$b_{r,r+1}+b_{r,r+2}+\cdots+b_{r,k}-1 \leftrightarrow b_{r+1,r+1}+b_{r+2,r+1}+\cdots+b_{k,r+1},$$

and it preserves the statistics $b_{j,\ell}$ for $1 \leq j \leq r-1$ and $1 \leq \ell \leq r$, and the statistics $b_{j,\ell}$ for $r+1 \leq j \leq k$ and $r+2 \leq \ell \leq k$.

In [16, Section 3], Sulanke studies some statistics on $\mathrm{SYT}(n^3)$ that can be described as linear combinations of the $b_{r,s}$. Using certain transformations, he shows that some of them have a 3-Narayana distribution, and he conjectures that the same is true for the others. Sulanke's transformations can be used to reduce these unknown statistics to the following two cases, corresponding to each of columns 3 and 4 of [16, Table 2].

Conjecture 5.2 ([16, Conjecture 1])**.** *On $\mathrm{SYT}(n^3)$, each of the statistics $\mathrm{st}_1 = b_{1,2}+b_{2,2}+b_{2,3}-2$ and $\mathrm{st}_2 = b_{1,1}+b_{1,3}+b_{2,3}-1$ has a 3-Narayana distribution.*

We can prove the statement about st_2 using the biijection φ_1. Indeed, for any $T \in \mathrm{SYT}(n^3)$, Lemma 5.1 implies that

$$\mathrm{des}(T) = b_{1,2}(T)+b_{1,3}(T)+b_{2,3}(T) = 1+b_{1,1}(\varphi_1(T))+b_{1,3}(\varphi_1(T))+b_{2,3}(\varphi_1(T)).$$

It follows that st_2 is equidistributed with $\mathrm{des}-2$, which has a 3-Narayana distribution.

For any permutation $\sigma \in \mathcal{S}_k$, define the statistic

$$\mathrm{des}_\sigma = \sum_{r,s:\sigma(r)>\sigma(s)} b_{r,s} = \sum_{1\leq i<j\leq k} b_{\sigma^{-1}(j),\sigma^{-1}(i)}.$$

These statistics generalize ascents and descents on $\mathrm{SYT}(n^k)$, since

$$\mathrm{des}_{12\ldots k}(T) = \sum_{r>s} b_{r,s}(T) = \mathrm{asc}(T) \quad \text{and} \quad \mathrm{des}_{k\ldots 21}(T) = \sum_{r<s} b_{r,s}(T) = \mathrm{des}(T).$$

Theorem 4.2 implies that $\mathrm{des}_{12\ldots k}$ and $\mathrm{des}_{k\ldots 21}-k+1$ are equidistributed on $\mathrm{SYT}(n^k)$. Our next theorem, whose proof can be found in [6], generalizes this fact, and it proves another conjecture of Sulanke [15, Section 3.1]. Here, $\mathrm{des}(\sigma) = |\{i : \sigma(i) > \sigma(i+1)\}|$ is the number of descents of $\sigma \in \mathcal{S}_k$.

Theorem 5.3. *For every $\sigma \in \mathcal{S}_k$, there exists a bijection $\phi_\sigma : \mathrm{SYT}(n^k) \to \mathrm{SYT}(n^k)$ such that, for all $T \in \mathrm{SYT}(n^k)$,*

$$\mathrm{des}_\sigma(T) = \mathrm{asc}(\phi_\sigma(T)) + \mathrm{des}(\sigma).$$

In particular, the statistic $\mathrm{des}_\sigma - \mathrm{des}(\sigma)$ on $\mathrm{SYT}(n^k)$ has a k-Narayana distribution.

Unlike our description of β, which proves the theorem in the special case $\sigma = k \dots 21$, the construction of ϕ_σ for arbitrary σ is more complicated. The details can be found in [6]. It is done in two steps. First, we describe a sequence of bijections that relates the statistics des_σ and des_λ, where λ is the reverse-layered permutation having the same descent set as σ, and then another sequence of bijections that relates the statistics des_λ and $\mathrm{des}_{12\dots k} = \mathrm{asc}$. The bijection ϕ_σ is obtained by composing these two sequences of bijections.

Part of the motivation for Theorem 5.3 comes from the study of so-called *canon permutations*. These are permutations of the multiset consisting of k copies of each number in $\{1, 2, \dots, n\}$, with the property that the subsequences obtained by taking the jth copy of each entry, for each fixed j, are all identical. For example, 313321214424 is a canon permutation, where the three subsequences are equal to 3124. Canon permutations were introduced in [5] as a variation of *quasi-Stirling* and *Stirling permutations*.

It was shown in [5] that, for $k = 2$, the polynomial that enumerates canon permutations by the number of descents has a nice factorization as a product of an Eulerian polynomial and a Narayana polynomial $\sum_h N(2, n, h)t^h$. In [6], we use the bijections in Theorem 5.3 to generalize this result to arbitrary k. A non-bijective proof using the theory of (P, ω)-partitions has recently been given by Beck and Deligeorgaki [2]. The above factorization, together with the palindromicity of the Eulerian and Narayana polynomials, implies that the distribution of the number of descents on canon permutations is symmetric. Combining our bijections φ and ϕ_σ, we can give a bijective proof of this fact.

6 Further directions

Symmetry of the major index. Another closely related statistic on standard Young tableaux is the *major index*, defined as $\mathrm{maj}(T) = \sum_{i \in \mathrm{Des}(T)} i$. For any partition λ of N, the distribution of this statistic over standard Young tableaux of shape λ is given by Stanley's q-analogue of the hook length formula:

$$\sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = q^{b(\lambda)} \frac{[N]_q!}{\prod_{c \in \lambda} [h_c]_q!}, \tag{6.1}$$

where h_c is the hook length of a cell c in λ, $b(\lambda) = \sum_i (i-1)\lambda_i$, and $[a]_q! = \prod_{i=1}^{a}(1 + q + \cdots + q^{i-1})$. The coefficients of these polynomials are important in algebraic combinatorics, and questions about their unimodality, asymptotic behavior, and internal zeros have been studied in [1, 3]. It follows from equation (6.1) that these polynomials are palindromic. However, no bijective proof of this fact seems to be known.

Problem 6.1. *For any partition λ of N, describe an explicit bijection $\Phi : \mathrm{SYT}(\lambda) \to \mathrm{SYT}(\lambda)$*

such that, for all $T \in \mathrm{SYT}(\lambda)$,

$$\mathrm{maj}(T) + \mathrm{maj}(\Phi(T)) = \binom{N}{2} + b(\lambda) - b(\lambda').$$

Unfortunately, our bijection φ does not have this property for the case of rectangular shapes $\lambda = (n^k)$, unless $k = 2$ or $n = 2$. We point out that, if we restrict to tableaux in $\mathrm{SYT}(\lambda)$ with a fixed number d of descents, then Schützenberger's evacuation proves the symmetry of the distribution of maj on this subset, since $\mathrm{des}(T) = \mathrm{des}(\mathrm{evac}(T))$ and $\mathrm{maj}(T) + \mathrm{maj}(\mathrm{evac}(T)) = dN$. However, this does not solve Problem 6.1.

In the case of rectangular shapes $\lambda = (n^k)$, computational evidence for $k + n \leq 9$ suggests that, in fact, the joint distribution of the number of descents and the major index is jointly symmetric. We wonder if some modification of our bijection φ could be used to prove this.

Problem 6.2. *Describe a bijection* $\Phi : \mathrm{SYT}(n^k) \to \mathrm{SYT}(n^k)$ *such that, for all* $T \in \mathrm{SYT}(n^k)$,

$$\mathrm{des}(T) + \mathrm{des}(\Phi(T)) = (k-1)(n+1) \quad \textit{and} \quad \mathrm{maj}(T) + \mathrm{maj}(\Phi(T)) = \frac{k(k-1)n(n+1)}{2}.$$

A rowmotion map on rectangular tableaux. In dynamical algebraic combinatorics, promotion and rowmotion are important maps on the set of order ideal of a poset [14]. For the type A root poset $\mathbf{A}_{n-1}$, its order ideals are in natural correspondence with the set $\mathcal{D}_n$ of Dyck paths, which in turn are in bijection with $\mathrm{SYT}(n^2)$ via the map Δ from Section 1. Under these bijections, promotion of order ideals of $\mathbf{A}_{n-1}$ translates to promotion on $\mathrm{SYT}(n^2)$, an extensively studied map defined not only on standard Young tableaux of any shape, including $\mathrm{SYT}(n^k)$, but on linear extensions of any poset [12]. On the other hand, no natural rowmotion operation on standard Young tableaux is known.

To define a rowmotion map on $\mathrm{SYT}(n^2)$, one can simply translate rowmotion on order ideals of $\mathbf{A}_{n-1}$ via the above bijections. This map turns ascents of $T \in \mathrm{SYT}(n^2)$ into high descents. Proposition 4.5 describes a more general bijection on $\mathrm{SYT}(n^k)$ that still has this property, providing a good candidate for a rowmotion map on $\mathrm{SYT}(n^k)$. In future work, we will show that this operation can be extended to arbitrary shapes, and that, in fact, it preserves the locations of the cells where ascents and high descents occur.

Generalizations to other posets. Viewing standard Young tableaux of rectangular shape as linear extensions of the poset $\mathbf{k} \times \mathbf{n}$, it is natural to ask whether some of our results can be generalized to other posets. For example, for shapes λ where the corresponding poset is graded, we know from the theory of P-partitions that the distribution of the number of ascents on $\mathrm{SYT}(\lambda)$ is symmetric. It would be interesting to find bijective proofs of these symmetries. For staircase shapes, this is done in [4]. More generally, one could ask for a bijective proof of the palindromicity of the descent polynomial of any graded poset, and of the fact that any natural labeling gives the same descent polynomial.

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