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Permutahedral subdivisions and class formulas from Coxeter elements

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Abstract. The Bruhat interval polytope $\mathcal{P}_{u,v}$ is the convex hull of the Bruhat interval $[u, v]$ in S_n, where each permutation z is interpreted as a vector $(z(1), \ldots, z(n)) \in \mathbb{R}^n$. One example is the permutahedron, which is $\mathcal{P}_{e,w_\circ}$. We explore the combinatorics of regular subdivisions of the permutahedron into Bruhat interval polytopes. In particular, we identify 2^{n-2} *finest* such subdivisions, one for each Coxeter element of S_n. For each subdivision, we provide an explicit height vector and determine exactly the constituent Bruhat interval polytopes. We also obtain an algebro-geometric counterpart of these subdivisions: for each Coxeter element $c \in S_n$, we obtain a formula for the class of the permutahedral variety as a sum of Richardson classes, where the terms in the sum exactly correspond to maximal polytopes in the subdivision. We further obtain formulas for the cohomology class of more general subvarieties of G/B which include Hessenberg varieties.

Keywords: Bruhat interval, flag Dressian, permutahedron, Richardson variety

1 Introduction

The permutahedron is the convex hull of the permutation vectors $\{(w(1), \ldots, w(n)) : w \in S_n\}$ in $\mathbb{R}^n$. It is the moment map image of both the complete flag variety Fl_n and the *permutahedral variety* $\overline{Tx}$, the closure of a generic torus orbit in Fl_n. The permutahedron has appeared widely in combinatorics and in combinatorial algebraic geometry. For example, the permutahedral variety has played a key role in the Hodge theory of matroids.

Bruhat interval polytopes are a natural generalization of the permutahedron. Introduced by Kodama–Williams [13], the Bruhat interval polytope $\mathcal{P}_{u,v}$ is the convex hull of the permutations in the Bruhat interval $[u, v]$. The permutahedron is $\mathcal{P}_{e,w_\circ}$. Bruhat interval polytopes are examples of *flag matroid polytopes*, and $\widetilde{\mathcal{P}}_{u,v} := \mathcal{P}_{w_\circ v^{-1}, w_\circ u^{-1}}$ is the moment map image of the Richardson variety $\mathcal{R}_{u,v} \subset \mathrm{Fl}_n$. Their combinatorial properties have been studied in e.g. [8, 10, 14, 18].

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1.1 Main results

The first main result of this extended abstract (Theorem 2.9) concerns the combinatorics of regular subdivisions of the permutahedron $\mathcal{P}_{e,w_\circ}$ into Bruhat interval polytopes. For each Coxeter element $c \in S_n$, we define an explicit height function h^c on the vertices of $\mathcal{P}_{e,w_\circ}$ (Definition 2.6). We show that the regular subdivision induced by h^c is exactly the decomposition

$$\mathcal{P}_{e,w_\circ} = \bigcup_{\substack{w\colon \ell(wc)= \\ \ell(w)+\ell(c)}} \widetilde{\mathcal{P}}_{w,wc}. \tag{1.1}$$

We also show that these subdivisions are *finest*, and discuss the numerology of the maximal cells. We obtain similar results for the type B permutahedron as well.

The second main result (Theorem 3.2) concerns the geometry of the permutahedral variety, and mirrors Equation (1.1). For each Coxeter element $c \in S_n$, we show that the class of the permutahedral variety is a sum of Richardson classes

$$[\overline{Tx}] = \sum_{\substack{w\colon \ell(wc)= \\ \ell(w)+\ell(c)}} [\mathcal{R}_{w,wc}]. \tag{1.2}$$

In fact, this equation is a special case of a formula for a more general class of subvarieties of G/B which includes the Hessenberg varieties, see 3.1.

1.2 Scientific context

The subdivisions of the permutahedron studied here are examples of regular subdivisions of a flag matroid polytope (the permutahedron) into flag positroid polytopes (Bruhat interval polytopes). See Section 2.1.1. Such subdivisions are known to be in correspondence with the cones of a polyhedral fan, the *positive flag Dressian*, which is also the positive tropicalization of Fl_n [7, 12]. Regular matroidal subdivisions of matroid polytopes have been the focus of substantial interest in the past two decades, as have the "positive" and "flag" variants of this setup[1]. Heuristically, the "positive" variants exhibit surprisingly nice behavior compared to the general matroid version [17, 8]. Much of the work on these objects has concerned the structure of the parametrizing polyhedral fans; the combinatorics of e.g. which polytopes appear together in a subdivision remains quite mysterious. The finest subdivisions found here constitute a first investigation of this question in the "positive flag" case. We note that in just the "positive" case, there is a family of finest positroidal subdivisions of the hypersimplex indexed by Catalan objects, whose combinatorics is completely understood. We view our finest Bruhat interval subdivisions of the permutahedron, indexed by Coxeter elements, as roughly analogous.

[1] We direct the reader to [8] for a more comprehensive history and references.

Our results generalize results of [11], who proved (1.1) in the special case of a "standard" Coxeter element $c = s_{n-1} \dots s_1 = n123\dots(n-1)$ via a connection to Gelfand–Tsetlin polytopes. In this special case, (1.2) can be deduced from [4]. When $c = s_{n-1}\dots s_1$, each of the Bruhat interval polytopes $\widetilde{\mathcal{P}}_{w,wc}$ in (1.1) is combinatorially a cube. This cubical subdivision of $\mathcal{P}_{e,w_\circ}$ was utilized in [16] to provide a combinatorial interpretation of a q-analogue of Postnikov's mixed Eulerian numbers. Another perspective on the relationship between (1.1) and (1.2) in the $c = 234\dots n1$ case was given in [15], which showed that there is a degeneration of the permutahedral variety $\overline{Tx}$ to the union $\cup\mathcal{R}_{w,wc}$ which occurs inside the flag variety Fl_n. We speculate that such a degeneration exists for arbitrary c, and in arbitrary type.

2 Subdivisions from Coxeter elements

2.1 Combinatorial background

We use the notation $[n] := \{1, 2, \dots, n\}$. For a permutation $z \in S_n$, both $z(i)$ and z_i are the image of i, and $z[1\colon i] := \{z(1), \dots, z(i)\}$. The simple transposition exchanging i and $i+1$ is s_i. A *reduced word* for $z \in S_n$ is an expression $z = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ where $\ell := \ell(z)$ is minimal. We call $\ell(z)$ the *length* of z. A product uw is *length-additive* if $\ell(uw) = \ell(u) + \ell(w)$. For $u, v \in S_n$, $u \le v$ in the Bruhat order if any reduced expression for v contains a reduced expression for u as a subword. Intervals in Bruhat order are denoted $[u, v] := \{z \in S_n : u \le z \le v\}$. The longest permutation is $w_\circ : i \mapsto n - i + 1$. We define $\widetilde{w} := w_\circ w^{-1}$ and $\widehat{w} := w^{-1}w_\circ$. These operations are involutive anti-automorphisms of the Bruhat order.

Definition 2.1 ([13])**.** Let $u \le v$. The *Bruhat interval polytope* $\mathcal{P}_{u,v}$ is the polytope

$$\mathcal{P}_{u,v} := \mathrm{conv}\{(z(1), z(2), \dots, z(n)) : z \in [u, v]\} \subset \mathbb{R}^n.$$

The *twisted Bruhat interval polytope* $\widetilde{\mathcal{P}}_{u,v}$ is the polytope

$$\widetilde{\mathcal{P}}_{u,v} := \mathrm{conv}\{(\widetilde{z}(1), \widetilde{z}(2), \dots, \widetilde{z}(n)) : z \in [u, v]\} \subset \mathbb{R}^n.$$

The set of Bruhat interval polytopes and the set of twisted Bruhat interval polytopes is the same; the only difference is how each polytope is labeled by an interval. In particular, $\widetilde{\mathcal{P}}_{u,v} = \mathcal{P}_{\widetilde{v},\widetilde{u}}$. The Bruhat interval polytope $\mathcal{P}_{e,w_\circ}$ is the *permutahedron*, which is the convex hull of all permutations. We note that the vertex set of $\mathcal{P}_{u,v}$ is the interval $[u, v]$, and every face of $\mathcal{P}_{u,v}$ is itself a Bruhat interval polytope [18, Theorem 4.1].

Definition 2.2. Let $P \subset \mathbb{R}^d$ be a polytope with vertices $v_1, \dots, v_r$, and let

$$h : \{v_1, \dots, v_r\} \to \mathbb{R}$$

be a *height vector*. We write $v_i^h := (v_i, h(v))$ for the *lifted vertex*, which is a point in $\mathbb{R}^d \times \mathbb{R}$. The lifted polytope is

$$P^h := \mathrm{conv}\{v_i^h : i = 1, \ldots, r\} \subset \mathbb{R}^d \times \mathbb{R}.$$

A *lower face* of P^h is a face which minimizes a linear functional of the form $\langle (x, 1), - \rangle$. The *regular subdivision of P induced by h*, also called the *h-subdivision of P*, is the collection of polytopes

$$\{\mathrm{conv}\{v_i\}_{i \in I} \subset P : \mathrm{conv}\{v_i^h\}_{i \in I} \text{ is a lower face of } P^h\}.$$

Each polytope in the collection is called a *cell* of the subdivision. Cells are partially ordered by inclusion. A regular subdivision of the permutahedron $\mathcal{P}_{e,w_\circ}$ is a regular *Bruhat interval subdivision* if all cells are Bruhat interval polytopes.

2.1.1 Relation to matroidal subdivisions, Dressians, tropical Grassmannians

Before stating our results regarding Bruhat interval subdivisions, we give additional context for the reader familiar with matroid theory. See [8] for additional details.

The twisted Bruhat interval polytope $\widetilde{\mathcal{P}}_{u,v}$ is the *flag matroid polytope* of the flag matroid $M_{u,v} := (M_{u,v}^1, \ldots, M_{u,v}^{n-1})$, where $M_{u,v}^i$ is the matroid with bases $\{z[1\colon i] : z \in [u, v]\}$. The matroid polytope of the constituent matroid $M_{u,v}^i$ is

$$\mathrm{conv}\{e_{z[1\colon i]} : z \in [u, v]\} \subset \mathbb{R}^n$$

where for $I \subset [n]$, $e_I = \sum_{i \in I} e_i$ is the indicator vector of I. We have that $\widetilde{\mathcal{P}}_{u,v}$ is the Minkowski sum of the matroid polytopes of $M_{u,v}^1, \ldots, M_{u,v}^{n-1}$. Each of the matroids $M_{u,v}^i$ is a *positroid*, meaning that $M_{u,v}^i$ can be realized by a $i \times n$ matrix with nonnegative maximal minors. The flag matroids $M_{u,v}$ are exactly the *flag positroids*, meaning they are the only flag matroids which can be realized by an $n \times n$ matrix with nonnegative *flag minors*.

The permutahedron $\mathcal{P}_{e,w_\circ}$ is the flag matroid polytope of the *uniform flag matroid*, meaning the constitutent matroids $M_{e,w_\circ}^i$ are all uniform. Thus, a regular Bruhat interval subdivision of $\mathcal{P}_{e,w_\circ}$ is a regular flag-positroidal subdivision of the uniform flag matroid polytope. The height vectors h giving rise to such subdivisions form a polyhedral fan called the *positive flag Dressian* [12], which is equal to the *positive tropical flag variety* [12, 7]. There is a bijective correspondence between cones of the positive flag Dressian[2] and regular Bruhat interval subdivisions of $\mathcal{P}_{e,w_\circ}$. Containment of cones corresponds to coarsening of subdivisions. We note that, while [7] gives a parametrization of the positive flag Dressian, very little is known about its fan structure—such as the number of rays or maximal cones—or about which Bruhat interval polytopes form regular Bruhat interval subdivisions.

[2] Technically, one can endow the positive flag Dressian with a number of different fan structures. Here, we choose the secondary fan structure.

2.2 Regular Bruhat interval subdivisions from Coxeter elements

In this subsection, we define a height vector h^c for each Coxeter element $c \in S_n$ and state our main result on the combinatorics of the h^c-subdivision of $\mathcal{P}_{e,w_\circ}$.

Definition 2.3. A permutation $c \in S_n$ is a *Coxeter element* if it has a reduced expression in which each simple transposition appears exactly once.

Example 2.4. For S_4, the Coxeter elements are $s_1s_2s_3$, $s_3s_2s_1$, $s_1s_3s_2 = s_3s_1s_2$ and $s_2s_1s_3 = s_2s_3s_1$.

It is well-known that S_n has 2^{n-2} Coxeter elements. We will define a height function on S_n, i.e. on the vertices of $\mathcal{P}_{e,w_\circ}$, using a Coxeter element. We first need the notion of rightmost subexpressions.

Definition 2.5. Let $\mathbf{w} = s_{i_1} \dots s_{i_r}$ be a word (not necessarily reduced) in the simple transpositions of S_n. A *subexpression* for v in $\mathbf{w}$ is an expression for v of the form $v = s^v_{h_1} \dots s^v_{h_r}$ where $s^v_{h_i} \in \{e, s_{h_i}\}$. The set of indices $i \in [r]$ where $s^v_{h_i} \neq e$ is the *support* of the subexpression. The *rightmost subexpression*[3] for v is constructed using a greedy procedure, moving from right to left, as follows: set $v_{(r+1)} = v$. If $v_{(j+1)}$ is already determined, then $v_{(j)}$ is equal to either $v_{(j+1)}$ or $v_{(j+1)}s_{i_j}$, whichever is smaller in the Bruhat order. In the first case, $s^v_{i_j} = e$; in the second, $s^v_{i_j} = s_{i_j}$. See Example 2.7.

For $I \subset [n]$, let v^I be the permutation obtained by putting the elements of I in increasing order, followed by the complement in increasing order. Recall that $\widetilde{w} := w_\circ w^{-1}$ and $\widehat{w} := w^{-1} w_\circ$.

Definition 2.6. Let $c \in S_n$ be a Coxeter element and fix a word $\mathbf{c}$ for c. Let $\mathbf{c}^m = s_{i_1} \cdots s_{i_r}$, where m is large enough that $\mathbf{c}^m$ contains a subexpression for $w_\circ$. Weight each letter of $\mathbf{c}^m$ according to which copy of $\mathbf{c}$ it is in: for $j \in [r]$, $\mathrm{wt}(j) := \#\{k \in [j+1, r] : s_{i_k} = s_{i_j}\}$. The weight of a subexpression of $\mathbf{c}^m$ is the sum of weights of its support. Given $I \subset [n]$, we define $\mathrm{wt}(I)$ to be the weight of the rightmost subexpression for v^I in $\mathbf{c}^m$. Finally, we define

$$h^c(w) := \sum_{i=1}^{n-1} \mathrm{wt}(\widehat{w}[1\colon i]) \quad \text{or equivalently} \quad h^c(\widetilde{w}) := \sum_{i=1}^{n-1} \mathrm{wt}(w[1\colon i]). \tag{2.1}$$

Example 2.7. Let $\mathbf{c} = s_1s_2s_3$, and $\mathbf{c}^m = s_1s_2s_3s_1s_2s_3s_1s_2s_3$. The weights of the letters are 222111000. Consider $w = 3412$ so $\widetilde{w} = 2143$. From the table below, we see $h^c(\widetilde{w}) = 1 + 2 + 0 = 3$.

[3] Also called the *positive distinguished subexpression* for v in the literature.

i	$I = w[1\colon i]$	v^I	rightmost subexp. for v^I in $\mathbf{c}^m$	$\mathrm{wt}(I)$
1	{3}	3124	$s_1s_2s_3s_1\boxed{s_2}s_3\boxed{s_1}s_2s_3$	$1+0=1$
2	{3,4}	3412	$s_1s_2s_3s_1\boxed{s_2}\boxed{s_3}\boxed{s_1}\boxed{s_2}s_3$	$1+1+0+0=2$
3	{1,3,4}	1342	$s_1s_2s_3s_1s_2s_3s_1\boxed{s_2}\boxed{s_3}$	$0+0=0$

Remark 2.8. Recall from Section 2.1.1 that $\mathcal{P}_{e,w_\circ}$ is the Minkowski sum of uniform matroid polytopes, also called hypersimplices, whose vertices are the indicator vectors e_I. In effect, Definition 2.6 defines a height function on vertices of hypersimplices, where the height of e_I is $\mathrm{wt}(I)$. Then h^c is the height function on the vertices of $\mathcal{P}_{e,w_\circ}$ induced by writing each vertex as a sum of hypersimplex vertices and then taking the sum of heights.

The following is the main theorem of this section.

Theorem 2.9. *Let $c \in S_n$ be a Coxeter element and let $c^* := w_\circ c w_\circ$. Then the subdivision of $\mathcal{P}_{e,w_\circ}$ induced by the height function $h^c : S_n \to \mathbb{R}$ is a finest regular Bruhat interval subdivision of $\mathcal{P}_{e,w_\circ}$. The maximal cells of this subdivision are*

$$\{\mathcal{P}_{u,c^*u} : c^*u \text{ length-additive}\} = \{\widetilde{\mathcal{P}}_{w,wc} : wc \text{ length-additive}\}.$$

The dimension i cells are

$$\{\widetilde{\mathcal{P}}_{p,q} : [p,q] \subset [w,wc] \text{ for some } w \text{ with } wc \text{ length-additive},\ \ell(q)-\ell(p)=i\}.$$

To prove Theorem 2.9, we show via careful analysis of h^c that each $\widetilde{\mathcal{P}}_{w,wc}$ lifts to a hyperplane, and that the lifted polytopes satisfy the *local folding condition* of [9, Theorem 2.3.20].

Example 2.10. For $c = s_1s_2s_3 = 2341$, the maximal cells in the h^c-subdivision of the permutahedron $\mathcal{P}_{1234,4321}$ are

$$\widetilde{\mathcal{P}}_{1234,2341},\ \widetilde{\mathcal{P}}_{1324,3241},\ \widetilde{\mathcal{P}}_{1243,2431},\ \widetilde{\mathcal{P}}_{1423,4231},\ \widetilde{\mathcal{P}}_{1342,3421},\ \widetilde{\mathcal{P}}_{1432,4321}.$$

For $c = s_1s_3s_2 = 2413$, the maximal cells in the h^c-subdivision of the permutahedron $\mathcal{P}_{1234,4321}$ are

$$\widetilde{\mathcal{P}}_{1234,2413},\ \widetilde{\mathcal{P}}_{1324,3412},\ \widetilde{\mathcal{P}}_{2314,3421},\ \widetilde{\mathcal{P}}_{1423,4312},\ \widetilde{\mathcal{P}}_{2413,4321}.$$

We obtain the following immediate corollary, utilizing the correspondence between finest regular Bruhat interval subdivisions and the maximal cones of the positive flag Dressian (cf. Section 2.1.1).

Corollary 2.11. *For each Coxeter element $c \in S_n$, the vector h^c lies in a maximal cone in the positive flag Dressian, endowed with secondary fan structure. Thus, there are at least 2^{n-2} maximal cones.*

Our results extend to the type B and type C variants of Bruhat interval polytopes via a folding argument. As this result does not depend on the weight lattice, it suffices to state it for type C. Let $S_n^{\pm}$ be the signed permutation group, acting by reflections on $\mathbb{R}^n$. The group $S_n^{\pm}$ is the type BC Coxeter group, and so is equipped with the (strong) Bruhat order. Given an interval $[u,v]$ in $S_n^{\pm}$, the *type C-Bruhat interval polytope* $\widetilde{\mathcal{P}}^C_{u,v}$ is the polytope

$$\widetilde{\mathcal{P}}^C_{u,v} = \operatorname{conv}\left(w \cdot (n, n-1, \ldots, 1) : w \in [u,v]\right).$$

Theorem 2.12. *Let $c \in S_n^{\pm}$ be a Coxeter element. Then, there is a height function $h_c : S_n^{\pm} \to \mathbb{R}$ such that the corresponding regular subdivision of $\widetilde{\mathcal{P}}^C_{e,w_0}$ is a finest regular subdivision into type C Bruhat interval polytopes with maximal cells*

$$\{\widetilde{\mathcal{P}}^C_{w,wc} : wc \text{ length-additive}\}.$$

2.3 Numerology of maximal cells

In this section, we discuss the number of maximal cells in the h^c-subdivision of $\mathcal{P}_{e,w_\circ}$. As we will see, this number varies depending on the choice of c.

Remark 2.13. Regular Bruhat interval subdivisions of $\mathcal{P}_{e,w_\circ}$ are the "flag" analogue of regular positroidal subdivisions of the hypersimplex $\Delta_{k,n}$. In the latter context, all finest subdivisions have the same f-vector, and in particular have $\binom{n-2}{k}$ maximal cells. In [8], it was shown for $n = 4$ that, in contrast, finest regular Bruhat interval subdivisions of $\mathcal{P}_{e,w_\circ}$ need not have the same number of maximal cells. The results of this section show that this continues for arbitrary n.

We use $\leq_L$ to denote the left weak order on S_n. Recall that $u \leq_L v$ if and only if v has a reduced expression which has a reduced expression for u as a suffix. Equivalently, $u \leq_L v$ if and only if v can be written as $v = xu$ where xu is length-additive. Left weak order can also be rephrased in terms of right inversions.

Definition 2.14. For $u \in S_n$, let $T_R(u) := \{(a,b) : a < b, u_a > u_b\}$ be the set of *right inversions* of u. We define $\succcurlyeq_u$ to be the poset on [n] defined by $a \succcurlyeq_u b$ if $(a,b) \in T_R(u)$ or if $a = b$.

It is straightforward to verify that $\succcurlyeq_u$ is indeed a partial ordering on $[n]$. It follows from [6, Proposition 3.1.3] that for $u \in S_n$

$$[u, w_\circ]_L = \{v \in S_n : v(a) > v(b) \text{ for all } a \neq b \text{ with } a \succcurlyeq_u b\}$$

where the set on the right is exactly the set of linear extensions of $\succcurlyeq_u$.

The following lemma is straightforward.

Lemma 2.15. *Let $c \in S_n$ be a Coxeter element. Then*

$$\{v : c \leq_L v\} \to \{\widetilde{\mathcal{P}}_{w,wc} : wc \text{ length-additive}\}$$
$$v \mapsto \widetilde{\mathcal{P}}_{vc^{-1},v}$$

is a bijection. That is, the maximal cells of the h^c-subdivision of $\mathcal{P}_{e,w_\circ}$ are in bijection with the left weak order interval $[c, w_\circ]_L$ and the linear extensions of $\succcurlyeq_c$.

Using this lemma, we obtain a count for the number of maximal cells in the subdivisions of Theorem 2.9.

Corollary 2.16. *Let $c \in S_n$ be a Coxeter element. The number of maximal cells in the h^c-subdivision of $\mathcal{P}_{e,wo}$ is the number of linear extensions of $\succcurlyeq_c$.*

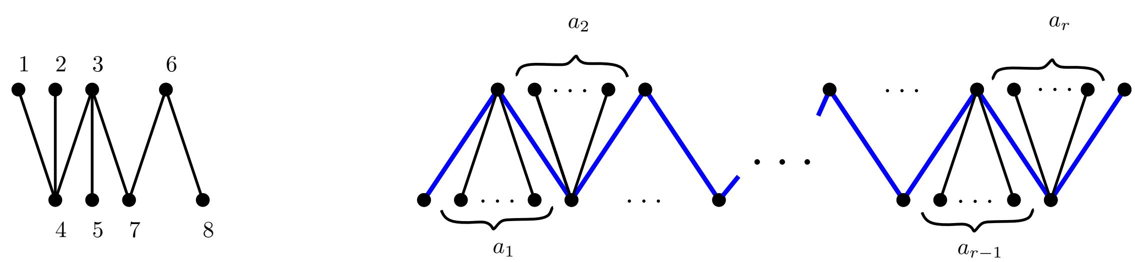

Figure 1: On the left, the Hasse diagram of $\succcurlyeq_c$ when $c = s_5s_7s_6s_1s_2s_4s_3 = 23614857$. On the right, the general shape of the Hasse diagram of $\succcurlyeq_c$, which is a zig-zag (in blue) with claws attached to the peaks and valleys. The numbers $a_1, \ldots, a_r$ are arbitrary nonnegative integers.

See Figure 1 for an example of the poset $([n], \succcurlyeq_c)$, as well as their general "shape". Note that the posets $([n], \succcurlyeq_c)$ are height 2 tree posets on $[n]$. As a result, the number of linear extensions may be computed using a polynomial-time algorithm [5].

Example 2.17. For an h^c-subdivision of $\mathcal{P}_{e,w_\circ}$, the number of maximal cells achieves its maximum when $c = s_1s_2 \ldots s_{n-1} = 234 \cdots n1$ or its inverse. In this case, $([n], \succcurlyeq_c)$ is a *claw*, and has $(n-1)!$ linear extensions. This subdivision was first studied by [11], who showed that each maximal cell is combinatorially a cube.

The number of maximal cells achieves its minimum when $c = \prod_{i \text{ even}} s_i \prod_{i \text{ odd}} s_i$ or its inverse. In this case, $([n], \succcurlyeq_c)$ is a *zig-zag poset*. Linear extensions are in bijection with "up/down" permutations and are counted by Euler numbers (see Table 1).

n	3	4	5	6	7	8	9
$(n-1)!$	2	6	24	120	720	5040	40320
Euler number	2	5	16	61	272	1385	7936

Table 1: Ranging over all Coxeter elements c in S_n, the largest possible number of maximal cells in an h^c-subdivision of $\mathcal{P}_{e,w_\circ}$ is $(n-1)!$. The smallest possible number of maximal cells is the nth Euler number. The two numbers are contrasted above.

3 Formulas for the Permutahedral class from Coxeter elements

In this section, we turn to our second main result on formulas for the permutahedral class.

Let G be a simply connected semisimple algebraic group with Borel subgroup B, opposite Borel B_-, maximal torus T, and Weyl group W. In type A, $G = SL_n$, B is the subgroup of upper triangular matrices, B_- the lower triangular matrices, T is the diagonal matrices, and $W = S_n$. The quotient G/B is the (generalized) flag variety. The flag variety admits two well-known decompositions into B-orbits and B_--orbits:

$$G/B = \bigsqcup_{w \in W} BwB/B = \bigsqcup_{w \in W} B_- wB/B.$$

The *Richardson variety* $\mathcal{R}_{u,v}$ is the closure of the intersection $B_-uB/B \cap BvB/B$.

The (Coxeter) Bruhat interval polytope $\widetilde{\mathcal{P}}_{u,v}$ is the moment map image of the Richardson variety $\mathcal{R}_{u,v} \subset G/B$. The Coxeter permutahedral variety is the torus-orbit closure $\overline{Tx}$ of a generic point $x \in G/B$ and its moment map image is the Coxeter permutahedron.

Given an element $v \in W$ and $X \in T$, we define the subvariety[4] $\mathcal{H}_v(X)$ of G/B by

$$\mathcal{H}_v(X) = \{hB \ : \ h^{-1}Xh \in \overline{BvB}\}.$$

The most general form of our cohomological result calculates the cohomology class of $\mathcal{H}_v(X)$.

In type A_n, when v is a *dominant* permutation, $\mathcal{H}_v(X)$ is a regular semisimple Hessenberg variety, whose definition we now recall. Given a function $h : [n] \to [n]$ with the property $h(i) \geq i$ and a linear operator X, the type A **Hessenberg variety** $\mathrm{Hess}(X, h)$ is the subvariety of the flag variety given by

$$\mathrm{Hess}(X, h) = \{F_\bullet \in \mathrm{Fl}_n \ : \ XF_i \subseteq F_{i+1} \text{ for } i = 1, 2, \ldots, n\}.$$

When h is the function $h(i) = \min\{i+1, n\}$ and X is semisimple and regular, $\mathrm{Hess}(X, h)$ is the permutahedral variety. Hessenberg varieties have appeared in many contexts and

[4] We note that the choice of X does not change the isomorphism class of the variety.

are important, for instance, in the study of the chromatic symmetric function of special graphs. See the survey [2] for more information.

Our proof uses the **double Schubert varieties** $\mathbf{\Omega}_w$ defined by Anderson in [3]. This is the subvariety of $G/B \times G/B$ given by

$$\mathbf{\Omega}_w := \overline{G_\Delta \cdot (wB, B)} = \{(gB, hB) \colon h^{-1}g \in \overline{BwB}\}$$

There are two key ideas in our proof. The first is that $\mathcal{H}_v(X)$ is the projection onto the second factor of $G/B \times G/B$ of the transverse intersection $(X \times 1)\mathbf{\Omega}_e \cap \mathbf{\Omega}_v$. The second is that the double Schubert $\mathbf{\Omega}_w$ degenerates to $\bigcup\{\overline{B_-uB/B} \times \overline{BvB/B} : v^{-1}u \leq w\}$. Combining these ideas together gives our main second result.

Theorem 3.1. *We have*

$$[\mathcal{H}_v(X)] = \sum_{\substack{u \in W \\ uv^{-1} \text{length-additive}}} [\mathcal{R}_{u,uv^{-1}}]$$

in $H^*(G/B)$.

When v is a Coxeter element, $\mathcal{H}_v(X)$ is the permutahedral variety $\overline{Tx}$ which gives as a corollary the geometric counterpart to Theorem 2.9.

Theorem 3.2. *Let c be a Coxeter element in W. Let T be the torus in G and let x be a generic point in G/B. For a pair of permutations $u \leq v$ in W, let $\mathcal{R}_{u,v}$ denote the corresponding Richardson variety.*

Then we have

$$[\overline{Tx}] = \sum_{\substack{w \in W \\ \ell(wc) = \ell(w) + \ell(c)}} [\mathcal{R}_{w,wc}].$$

When $c = s_{n-1}s_{n-2}\dots s_1 = n12\dots(n-1)$ is a "standard" Coxeter element, this recovers the formula for the permutahedral variety in [4].

Remark 3.3. The class of a regular Hessenberg variety $\text{Hess}(X, h)$ does not depend on the choice of regular X [1]. Hence, this also gives formulas for the class of the Peterson variety.

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