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Domino Tilings and Macdonald Polynomials

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Abstract. There is a general bijection between domino tilings of planar regions in the square lattice and families of non-intersecting Schröder-like paths contained in the region. Motivated by this bijection, we study domino tilings of certain regions R_λ, indexed by partitions λ, weighted according to generalized area and dinv statistics. These statistics arise from the q,t-Catalan combinatorics and Macdonald polynomials. We present a formula for the generating polynomial of these domino tilings in terms of the Bergeron–Garsia nabla operator. When $\lambda = (n^n)$ is a square shape, domino tilings of R_λ are equivalent to those of the Aztec diamond of order n. In this case, we give a new product formula for the resulting polynomials by domino shuffling and its connection with alternating sign matrices. In particular, we obtain a combinatorial proof of the joint symmetry of the generalized area and dinv statistics.

Keywords: Alternating sign matrices; domino tilings; domino shuffling; nabla operator; q,t-Catalan combinatorics; symmetric functions.

1 Introduction

The study of Macdonald polynomials has produced many interesting combinatorial objects, often expressed as weighted sums over sets of lattice paths. Perhaps the most famous and well-studied such objects are the q,t-Catalan numbers introduced by Garsia and Haiman [8], which can be defined combinatorially as the sum over Dyck paths weighted by the area and dinv statistics (see the book [10] and the references therein). Among the many generalizations of the q,t-Catalan numbers are the extension to Schröder paths defined by Egge, Haglund, Killpatrick, and Kremer [4], and to nested families of Dyck paths due to Loehr and Warrington [12]. All of these objects have natural interpretations in terms of the *nabla operator* ∇, introduced by Bergeron and Garsia [1], on the ring of symmetric functions Λ in the countably infinite set of variables $\{x_i\}_{i\geq 1}$ with coefficients in the field of rational functions $\mathbb{Q}(q,t)$.

The nabla operator ∇ is the unique $\mathbb{Q}(q,t)$-linear map on Λ such that $\nabla(\tilde{H}_\mu) = q^{n(\mu')}t^{n(\mu)}\tilde{H}_\mu$ for all μ, where $\tilde{H}_\mu$ is the modified Macdonald polynomial studied by Macdonald [14] and appearing in Garsia and Haiman's work on diagonal harmonics [8], μ'

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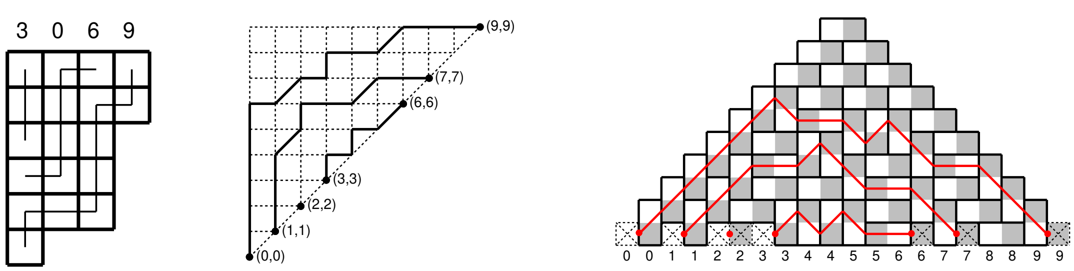
is the conjugate of a partition μ, and $n(\mu) = \sum(i-1)\mu_i$. One can obtain interesting combinatorial objects by applying ∇ to elements of some basis of Λ and expanding the result in terms of another basis or, equivalently, taking the Hall inner product $\langle \cdot, \cdot \rangle$ with an element of the dual basis. Some of the classical bases of Λ are m_μ (monomial), e_μ (elementary), h_μ (complete homogeneous) and s_μ (Schur), each are indexed by partitions μ. Note that $s_{(n)} = h_n$ and $s_{(1^n)} = e_n$.

We list some of these combinatorial objects in Table 1. For other formulas of this kind, see [12].

Algebraic Object	Combinatorial Model	Special Case of Theorem 1.1
$\langle \nabla(e_n), e_n \rangle$	the q,t-Catalan numbers [8, 7]	$\lambda = (1^n)$ and $d = 0$
$\langle \nabla(e_n), h_d e_{n-d} \rangle$	the q,t–Schröder numbers [4, 9]	$\lambda = (1^n)$
$\mathsf{sgn}(\lambda)\langle \nabla(s_\lambda), e_n \rangle$	the λ-family of Dyck paths [12]	$d = 0$

Table 1: Some combinatorial objects of the nabla operator.

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, Loehr and Warrington [12] consider strictly nested collections of Dyck paths with starting points $(0,0), (1,1), (2,2), \ldots$, and ending points determined by λ in the following way. Decompose the Ferrer's diagram of λ into border strips and label each column with the length of the strip whose rightmost box is in that column (and with label 0 if there is no such strip) as shown in Figure 1(a) for the partition $\lambda = (4,4,3,3,3,1)$. Let $n_0, \ldots, n_k$ denote these labels read from right to left where $k = \lambda_1 - 1$. A *λ-family of Dyck paths* is a $(k+1)$-tuple $\pi = (\pi_0, \ldots, \pi_k)$ of pairwise non-intersecting lattice paths, where each π_i is a lattice path consisting of unit north $(0,1)$ and east $(1,0)$ steps that lies weakly above the diagonal $y = x$, starts at (i,i), and ends at $(i+n_i, i+n_i)$.



(a) (b) (c)

Figure 1: (a) The decomposition of $\lambda = (4,4,3,3,3,1)$ into border strips of lengths $(n_0, n_1, n_2, n_3) = (9,6,0,3)$. (b) A λ-family of Schröder paths for $\lambda = (4,4,3,3,3,1)$ with 5 diagonal steps. (c) The region R_λ for $\lambda = (4,4,3,3,3,1)$ with the domino tiling that corresponds to the λ-family in Figure 1(b).

In this extended abstract, we present the Schröder generalization of these nested families of paths. Using the same notation as for the Dyck paths, a *λ-family of Schröder paths* is a $(k+1)$-tuple $\pi = (\pi_0, \ldots, \pi_k)$ of pairwise non-intersecting lattice paths, where each π_i is a lattice path consisting of unit north $(0,1)$, east $(1,0)$, and diagonal $(1,1)$ steps that lies weakly above the diagonal $y = x$, starts at (i,i), and ends at $(i+n_i, i+n_i)$. Figure 1(b) shows a λ-family of paths for the partition in Figure 1(a). Let $\mathcal{S}_\lambda$ denote the set of all λ-families of Schröder paths and $\mathcal{S}_{\lambda,d}$ the set of all λ-families of Schröder paths with d total diagonal steps among all the paths.

We define area and dinv statistics for $\mathcal{S}_{\lambda,d}$ (see Section 2.1) that specialize to the corresponding statistics in the cases of λ-families of Dyck paths, (single) Schröder paths, and (single) Dyck paths. Our first result expresses λ-families of Schröder paths weighted according to these statistics in the language of Macdonald polynomials. In the following formula $\mathsf{sgn}(\lambda) = \pm 1$ is a sign factor introduced by Loehr and Warrington whose definition is recalled in Section 2.1.

Theorem 1.1. *For any partition $\lambda \vdash n$ and $0 \le d \le n$,*

$$\sum_{\pi \in \mathcal{S}_{\lambda,d}} q^{\mathsf{area}(\pi)} t^{\mathsf{dinv}(\pi)} = \mathsf{sgn}(\lambda) \langle \nabla(s_\lambda), h_d \, e_{n-d} \rangle. \tag{1.1}$$

Theorem 1.1 is the common generalization of several formulas relating q,t-Catalan combinatorics to Macdonald polynomials through the nabla operator (Table 1).

Our present interest in these polynomials is due to a general bijection [13] between domino tilings of regions R on the square lattice and families of non-intersecting Schröder-like paths contained in R (see Section 2.2). A *domino tiling* of R is a covering of R using dominoes (union of two unit squares sharing an edge) without gaps or overlaps. Let $\mathcal{T}(R)$ be the set of domino tilings of R. Motivated by this bijection, we define below a region R_λ whose domino tilings correspond to $\mathcal{S}_\lambda$. This region is related to the Aztec diamond introduced by Elkies, Kuperberg, Larsen, and Propp [5, 6] in 1992. One goal of this paper is to establish a bridge between the rich combinatorics surrounding domino tilings of the Aztec diamond (and similar regions) and q,t-Catalan combinatorics.

The *Aztec diamond* of order n, denoted by AD_n, is the union of all unit squares inside the diamond-shaped region $\{(x,y) \in \mathbb{R}^2 : |x| + |y| \le n+1\}$. Given a partition λ decomposed into border strips of lengths $(n_0, \ldots, n_k)$ as above, R_λ is defined as a certain subset of the upper-half, $y \ge 0$, of AD_{n_0+1}. Equip the Aztec diamond with the checkerboard coloring so that the unit squares along the top right side are colored black. Label the boxes in the bottom row of the top half of the Aztec diamond $0, 0, 1, 1, 2, 2, \ldots, n_0, n_0$ from left to right. The region R_λ is defined by removing from this region the white boxes in the bottom row labeled $0, 1, \ldots, k$ and the black boxes in the bottom row labeled $i + n_i$ for each $i = 0, \ldots, k$. Figure 1(c) shows the domino tiling of R_λ corresponding to the λ-family in Figure 1(b).

Passing through this bijection [13], we define statistics area, dinv, and diags directly on domino tilings of R_λ that agree with the corresponding statistics on $\mathcal{S}_\lambda$ (see Section 2.1). We then study the following weighted sums over domino tilings in the set $\mathcal{T}(R_\lambda)$ motivated by Theorem 1.1:

$$P_\lambda(z;q,t) := \sum_{T\in\mathcal{T}(R_\lambda)} z^{\mathsf{diags}(T)} q^{\mathsf{area}(T)} t^{\mathsf{dinv}(T)} = \mathsf{sgn}(\lambda) \sum_{d=0}^{n} z^d \langle \nabla(s_\lambda), h_d\, e_{n-d}\rangle. \tag{1.2}$$

An immediate consequence of Theorem 1.1 is the symmetry of $P_\lambda(z;q,t)$ in the variables q and t.

Corollary 1.2. *For all partitions λ, we have $P_\lambda(z;q,t) = P_\lambda(z;t,q)$.*

As is often the case in q,t-Catalan combinatorics, the joint symmetry of the area and dinv statistics remains mysterious combinatorially, but we give a direct combinatorial proof of Corollary 1.2 for the special cases when $\lambda = (n^n)$.

A special role is played by the partitions of square shape, $\lambda = (n^n)$, in which case the region R_λ reduces to AD_n. For convenience, we write

$$\mathrm{AD}_n(z;q,t) = P_{(n^n)}(z;q,t). \tag{1.3}$$

Surprisingly, the combinatorics of alternating sign matrices and domino shuffling introduced in [5, 6] interact well with the area, dinv and diags statistics coming from q,t-Catalan combinatorics. We use this machinery to prove the following product formula for $\mathrm{AD}_n(z;q,t)$.

Theorem 1.3. *When $\lambda = (n^n)$ is a partition of square shape, we have the product formula*

$$\mathrm{AD}_n(z;q,t) = (qt)^{n^2(n-1)/2} \prod_{i,j\geq 0 \text{ and } i+j<n} (z + q^i t^j). \tag{1.4}$$

Specializing all variables to 1, we recover the well-known fact that the number of domino tilings of AD_n is $2^{n(n+1)/2}$. This product formula is similar to the formula given in [5, 6] weighted according to different statistics. In fact, our statistic $\mathsf{diags}(T)$ is the same as their statistic $v(T)$ counting vertical dominoes up to a simple change of variables. Our $\mathsf{area}(T)$ statistic is similar to their rank statistic $r(T)$, although they are not the same, and our $\mathsf{dinv}(T)$ statistic appears to be completely new. Since our proof of Theorem 1.3 is entirely combinatorial and (1.4) is evidently symmetric in q and t, we obtain a direct proof (not using Theorem 1.1) of the q,t-symmetry in Corollary 1.2 for $\lambda = (n^n)$.

This extended abstract is organized as follows. In Section 2, we define statistics on λ-families of Schröder paths and domino tilings, and state some background information on related combinatorics of alternating sign matrices and domino shuffling. In Section 3, we indicate how Theorem 1.1 can be obtained from the Loehr–Warrington formula [12, 2, 11]. In Section 4, we discuss the special case when $\lambda = (n^n)$.

2 Preliminaries

2.1 Statistics on Families of Nested Schröder Paths

We define area, dinv, and diags statistic statistics for λ-families of Schröder paths building upon the statistics for Schröder paths and λ-families of Dyck paths, see [4] and [12] for their detailed original definitions.

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition and $\pi = (\pi_0, \ldots, \pi_k)$ a λ-family of Schröder paths $\mathcal{S}_\lambda$, where $k = \lambda_1 - 1$. For each path π_j in π, we define $\mathsf{diags}(\pi_j)$ to be the number of diagonal $(1,1)$ steps in the path π_j, and set $\mathsf{diags}(\pi) = \sum_j \mathsf{diags}(\pi_j)$. The λ-family π in Figure 1(b) has $\mathsf{diags}(\pi) = 2 + 2 + 1 = 5$.

A triangle with vertices (c, d), $(c + 1, d)$ and $(c + 1, d + 1)$ for some integers c, d is called an *area triangle*. The *area* of a path π_j, $\mathsf{area}(\pi_j)$, is the total number of area triangles below π_j and above the diagonal $y = x$, and set $\mathsf{area}(\pi) = \sum_j \mathsf{area}(\pi_j)$. The λ-family in Figure 1(b) has area $27 + 11 + 0 = 38$.

To define the dinv statistic, we designate certain pairs of steps in the paths π to be *dinv pairs* and record the total number of such pairs. As shown in Figure 2, dinv pairs are the four types of pairs of north or diagonal steps in some, possibly different, paths in π. As in [12], the *dinv adjustment* of λ is $\mathsf{adj}(\lambda) = \sum_{j:n_j>0}(\lambda_1 - 1 - j)$, and the *sign* of λ is $\mathsf{sgn}(\lambda) = (-1)^{\mathsf{adj}(\lambda)}$. The dinv of a λ-family π is defined to be the quantity

$$\mathsf{dinv}(\pi) = \mathsf{adj}(\lambda) + \#\{\text{dinv pairs in } \pi\}. \tag{2.1}$$

Our running example in Figure 1(b) has $\mathsf{sgn}(\lambda) = -1$ and $\mathsf{dinv}(\pi) = 5 + 29 = 34$.

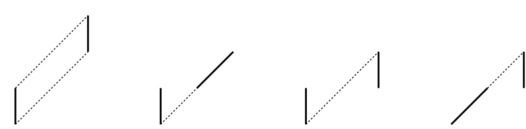


Figure 2: The four types of dinv pairs. The dotted lines indicate the alignment of the pairs of edges relative to some translation of the diagonal $y = x$.

2.2 Bijection with Domino Tilings and Statistics

In general, there is a bijection [13] between domino tilings of a region R on the square lattice and families of non-intersecting paths contained in R. In particular, domino tilings of R_λ are in bijection with λ-families of Schröder paths. This bijection is described below.

Given a domino tiling of R_λ, the corresponding family of paths is given by decorating each of the four types of dominoes that can appear relative to the checkerboard coloring with a (possibly empty) path step as shown in Figure 3. Regarding each of these path steps as oriented from left to right, each step enters a domino in a black square and exits a domino in a white square. The union of all of these individual path steps therefore forms a collection of non-intersecting paths in R_λ. The starting points of these paths are the black squares in R_λ such that the white square immediately to its left is not in R_λ and the ending points of these paths are the white squares in R_λ such that the black square immediately to its right is not in R_λ.

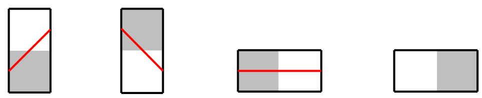


Figure 3: The four types of dominoes and their corresponding path steps.

One can view the resulting family of paths as λ-family of Schröder paths by a change of coordinates that amounts to rotating 45° counterclockwise and rescaling so that the longest path begins at $(0,0)$ and ends at (n_0, n_0); see Figures 1(b) and 1(c). Conversely, given a λ-family of Schröder paths, we can undo the affine transformation above to obtain a family of paths in R_λ. We then cover R_λ with dominoes based on the four configurations in Figure 3.

Let T be a domino tiling of R_λ and $U(T)$ (resp., $H(T)$) be the collection of all vertical (resp., horizontal) dominoes in which the bottom (resp., left) unit square is colored black. These two particular types of domino correspond to the up and diagonal steps of its λ-family of Schröder paths. The diags statistic is given by $\mathsf{diags}(T) = |H(T)|$.

Recall that the area statistic counts the total number of area triangles of each path. This can be expressed as a sum over the "area" of each vertical domino in $U(T)$ and horizontal domino in $H(T)$, where each area contribution depends on the y-coordinate of that domino. To be more precise, for a vertical domino $d_u \in H(T)$ and a horizontal domino $d_h \in U(T)$ whose bottom edge lies on $y = i$, define $\mathsf{area}(d_u) = \mathsf{area}(d_h) = i$. Then the area statistic can be reinterpreted as

$$\mathsf{area}(T) = \sum_{d_u \in U(T)} \mathsf{area}(d_u) + \sum_{d_h \in H(T)} \mathsf{area}(d_h). \tag{2.2}$$

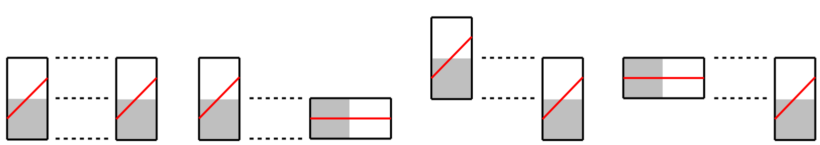


Figure 4: The four types of domino pairs. The dotted lines indicate the alignment of dominoes.

The dinv pairs in Figure 2 are equivalent to the domino pairs shown in Figure 4. So, the dinv statistic is given by

$$\mathsf{dinv}(T) = \mathsf{adj}(\lambda) + \#\{\text{domino pairs in } T\}. \tag{2.3}$$

2.3 Alternating Sign Matrices and Aztec Diamonds

An alternating sign matrix (ASM) of order n is an $n \times n$ matrix with entries $0, 1$ or -1 such that all row and column sums are equal to 1 and the non-zero entries alternate in sign in each row and column. Mills, Robbins, and Rumsey introduced them [15] in the early 1980s. The connection between ASMs and domino tilings of the Aztec diamond was initially discovered in [5, 6]. They showed that the number of domino tilings of the Aztec diamond can be expressed as a weighted enumeration of ASMs. This connection was made explicit later by Ciucu [3, Section 2], the idea is stated as follows.

For two integers x and y, we say a (lattice) point (x, y) in AD_n is *even* (resp., *odd*) if $x + y$ and n have the same (resp., opposite) parity. For each domino tiling T of AD_n, we first add four unit segments connecting points $(0, n)$ with $(0, n+1)$, $(-n, 0)$ with $(-(n+1), 0)$, $(0, -n)$ with $(0, -(n+1))$, and $(n, 0)$ with $(n+1, 0)$. Second, we define the maps M_e and M_o below.

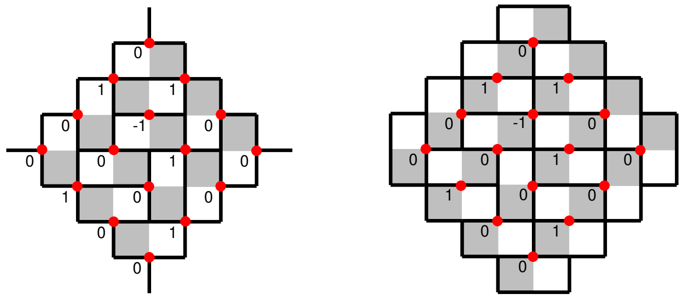
$$M_e(v_e) = \begin{cases} 1 & \text{if } \deg(v_o) = 4, \\ 0 & \text{if } \deg(v_o) = 3, \\ -1 & \text{if } \deg(v_o) = 2, \end{cases} \quad \text{and} \quad M_o(v_o) = \begin{cases} 1 & \text{if } \deg(v_e) = 2, \\ 0 & \text{if } \deg(v_e) = 3, \\ -1 & \text{if } \deg(v_e) = 4, \end{cases} \tag{2.4}$$

where $\deg(\cdot)$ is the degree of an even point v_e or odd point v_o by viewing a domino tiling as a graph. Finally, the image of all even (resp., odd) points under the map M_e (resp., M_o) forms a square array (which is an ASM) of size n (resp., $n+1$); see Figures 5(a) and 5(b). We write $M_e(T)$ and $M_o(T)$ for such ASMs that correspond to T.

2.4 Domino Shuffling

Recall that we equip AD_n the checkerboard coloring so that the unit squares along the top right side are black. Domino shuffling [5, 6] is a process that involves moving dominoes in a specific way, we follow closely their idea and summarize it below. Given a domino tiling T of AD_n, if we can find a 2×2 block in T (consisting of two horizontal or two vertical dominoes), then we say this 2×2 block is *even* (resp., *odd*) if the top right unit square in this block is black (resp., white). An *odd-deficient* (resp., *even-deficient*) tiling of T is obtained from T by removing all odd (resp., even) blocks of T.

Domino shuffling is a map S which sends an odd-deficient tiling T_o of order n to an even-deficient tiling $T_e = S(T_o)$ of order $n+1$ by the following actions: (1) slide a vertical domino in which the bottom square is black of T_o one unit to its left, (2) slide



(a) (b)

Figure 5: (a) An example of $M_e(T)$ where $T \in \mathcal{T}(\mathrm{AD}_3)$, even points are marked in red. (b) An example of $M_o(T')$ where $T' \in \mathcal{T}(\mathrm{AD}_4)$, odd points are marked in red.

a vertical domino in which the top square is black of T_o one unit to its right, (3) slide a horizontal domino in which the left square is black of T_o one unit above it, and (4) slide a horizontal domino in which the right square is black of T_o one unit below it. We should emphasize that the colors are moved along with dominoes in our case (in [5, 6], the colors are swapped after shuffling, so our terminologies are slightly different from theirs). See Figures 6(a) and 6(b) for an example.

Suppose that a domino tiling T of AD_n contains m odd blocks. Let T_o be the odd-deficient tiling of T and $T_e = S(T_o)$. The key facts are listed below: (1) T_e has $m+n+1$ even blocks, (2) the center of an odd (resp., even) block is assigned -1 (resp., 1) under the map M_e (resp., M_o), and (3) the ASMs $M_e(T_o)$ and $M_o(T_e)$ are identical.

3 On the Proof of Theorem 1.1

The Loehr–Warrington formula gives the monomial expansion of $\nabla(s_\lambda)$ for any Schur function s_λ as a sum over labeled, weakly-nested λ-families of Dyck paths. We describe these objects informally below, referring to [12] for the rigorous definitions. The formula involves nested families of Dyck paths $(\pi_0, \ldots, \pi_k)$ with the same starting and ending points as our λ-families of Schröder paths that are allowed to touch and share vertical steps but may not cross, may not share horizontal steps, and no path can meet the starting point of any other path. A labeling is an assignment of nonnegative integers to the vertical steps of these paths such that the labels on consecutive vertical steps in each path increase from bottom to top, and whenever labels $r < r'$, on paths π_i and $\pi_{i'}$ respectively, appear in the same column with r' one row above r we have $i < i'$. Let LNDP_λ denote the set of all pairs (π, w) where π is a weakly-nested λ-family of Dyck paths and w is a labeling as described above.

Loehr and Warrington [12] define area and dinv statistics on pairs $(\pi, w) \in \mathrm{LNDP}_\lambda$. The Loehr–Warrington formula, recently proved by Blasiak, Haiman, Morse, Pun, and Seelinger [2] and by Kim and Oh [11], states that for any partition λ,

$$\nabla(s_\lambda) = \mathsf{sgn}(\lambda) \sum_{(\pi,w)\in \mathrm{LNDP}_\lambda} q^{\mathsf{area}(\pi,w)} t^{\mathsf{dinv}(\pi,w)} x^w, \tag{3.1}$$

where $x^w = \prod_i x_i^{\#i\text{'s in } w}$. We briefly sketch how Theorem 1.1 can be obtained from (3.1), following Haglund's argument in the single Schröder path case ([10, Chapter 6]).

Since $\{h_\mu\}$ and $\{m_\mu\}$ are dual bases, the coefficient $\langle \nabla(s_\lambda), h_\mu \rangle$ can be computed as the area and dinv weighted sum over those $(\pi, w) \in \mathrm{LNDP}_\lambda$ such that there are μ_i i's in the labeling w, in which case we write $\mathsf{wt}(w) = \mu$. A labeling is called *standard* if $\mathsf{wt}(w) = (1, \ldots, 1)$, and we write SLNDP_λ for the set of all standard labeled λ-families of weakly nested Dyck paths. Following [10, Chapter 6], one can define a *standardization* operation on labelings $w \mapsto w'$ which gives an area and dinv-preserving bijection between $\{(\pi, w) \in \mathrm{LNDP}_\lambda \mid \mathsf{wt}(w) = \mu\}$ and the set of $(\pi, w') \in \mathrm{SLNDP}_\lambda$ such that the *reading word* of w' with respect to a certain reading order is a μ-shuffle. As in the single path case, the reading order for these labelings is from top to bottom along diagonals starting from the furthest diagonal from $y = x$, and for labels on coinciding vertical edges, we break ties by reading labels from the weakly innermost path to the weakly outermost path.

By [10, Theorem 6.10], $\langle \nabla(s_\lambda), h_d e_{|\lambda|-d} \rangle$ is the area and dinv-weighted sum over (π, w) in SLNDP_λ such that the reading word of w is a shuffle of the decreasing sequence $|\lambda| - d, \ldots, 1$ and the increasing sequence $|\lambda| - d + 1, \ldots, |\lambda|$. Such labeled paths can be interpreted as λ-families of Schröder paths by replacing pairs of consecutive vertical and horizontal steps with diagonal steps whenever the label on the vertical step is from the increasing sequence $|\lambda| - d + 1, \ldots, |\lambda|$. One checks that this defines an area and dinv preserving bijection between such labeled λ-families of (weakly-)nested Dyck paths and λ-families of (strictly-)nested Schröder paths with d diagonals, which gives Theorem 1.1.

4 Partitions of Square Shape

When $\lambda = (n^n)$, domino tilings of R_λ reduce to domino tilings of *extended* AD_n where the region outside AD_n is forced to be covered by vertical dominoes. In this case, the dinv adjustment simplifies to $\mathsf{adj}(\lambda) = \binom{n}{2}$, so the dinv of a domino tiling T of the extended AD_n reduces to

$$\mathsf{dinv}(T) = \binom{n}{2} + \#\{\text{domino pairs in } T\}. \tag{4.1}$$

Example 4.1. Table 2 shows 8 weighted domino tilings of the extended AD_2 where the original AD_2 is enclosed by bold edges. One can check that their total weight factors into $\mathrm{AD}_2(z; q, t) = (qt)^2(z + 1)(z + q)(z + t)$, which agrees with (1.4).

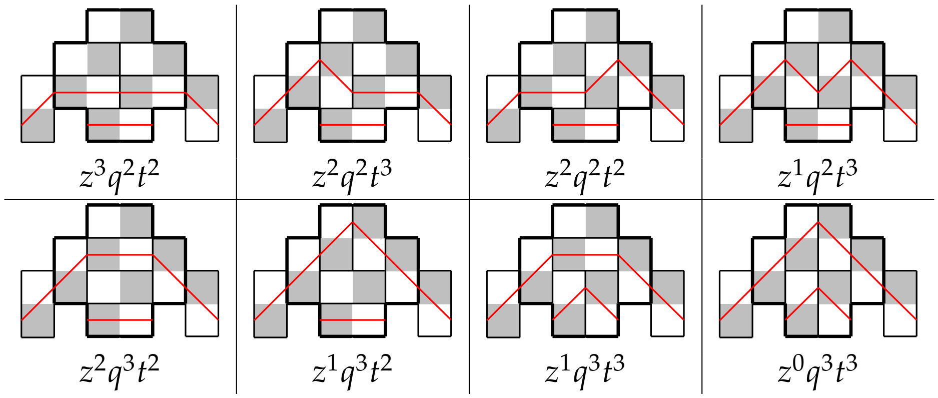


Table 2: The weight $z^{\mathsf{diags}(T)}q^{\mathsf{area}(T)}t^{\mathsf{dinv}(T)}$ of each domino tiling T of the extended AD_2.

For a domino tiling T of the extended AD_n, its corresponding *extended* ASM is obtained from the ASM $M_e(T|_{\mathrm{AD}_n})$ (when restricting T to AD_n; see Section 2.3) by adding a triangular array of 0's of size n on the southwestern and southeastern sides, respectively. The resulting extended ASM is a triangular array of size $2n+1$ (see Figure 6(c)).

Given a domino tiling T of the extended AD_n with m odd blocks. Let T_o be the odd-deficient tiling of T and $\hat{T}_o$ be the tiling obtained from T_o by completing all odd blocks with two *horizontal* dominoes. Define diags and dinv of T_o as usual, and define area of T_o by $\mathsf{area}(T_o) = \mathsf{area}(\hat{T}_o)$.

To apply the domino shuffling (Section 2.4), we restrict T_o to AD_n and apply the domino shuffling to obtain the even-deficient tiling of AD_{n+1}; note that there are $m+n+1$ even blocks. Let T_e be such even-deficient tiling of the extended AD_{n+1} where the region outside AD_{n+1} is covered by vertical dominoes (see Figures 6(a) and 6(b)). We write $\hat{T}_e$ for the tiling obtained from T_e by completing all even blocks with two *horizontal* dominoes. Similarly, define diags and dinv of T_e as usual, and define area of T_e by $\mathsf{area}(T_e) = \mathsf{area}(\hat{T}_e)$. We can prove how the area and dinv statistics interact with the domino shuffling in the following proposition.

Proposition 4.2. *Let T be a domino tiling of the extended AD_n with m odd blocks, T_o the odd-deficient tiling of T, and T_e the even-deficient tiling under the domino shuffling of T_o. Then*

$$\mathsf{area}(T_e) = \mathsf{area}(T_o) - \mathsf{diags}(T_o) + n(2n+1) - m, \quad \textit{and} \tag{4.2}$$

$$\mathsf{dinv}(T_e) = \mathsf{dinv}(T_o) + n(3n+1)/2. \tag{4.3}$$

The key idea of proving Proposition 4.2 is based on interpreting the area and dinv statistics on T as some statistics on its extended ASM $M_e(T)$. We label each row from bottom to top of $M_e(T)$ by 1 to $2n+1$ and define two statistics on $M_e(T)$ as follows. Suppose x is an entry of $M_e(T)$ at row i, then the *level* of x is defined to be $\mathsf{level}(x) = i$. The *top partial sum* of x, denoted $\mathsf{tp}(x)$, is defined to be the sum of all entries of $M_e(T)$

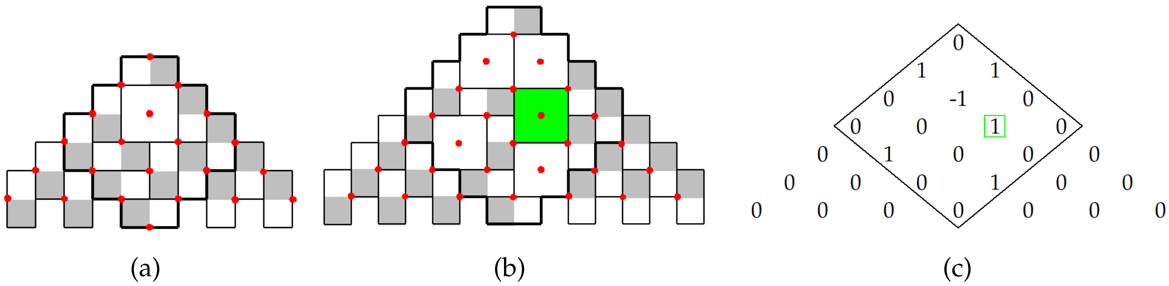


Figure 6: (a) An odd-deficient tiling T_o of the domino tiling given in Figure 5(a), even points are marked in red. (b) The even-deficient tiling T_e under the domino shuffling of T_o, odd points are marked in red. (c) The extended ASM $M_e(T_o) = M_o(T_e)$.

located from rows $i+1$ to $2n+1$ and to the right of x on the same row. Considering the tilings $\hat{T}_o$ and $\hat{T}_e$, we can show that the area contribution from each odd and even block is given by the level statistic.

Lemma 4.3. *Let $d_h \in H(\hat{T}_o)$ be a horizontal domino contained in an odd block of $\hat{T}_o$ and $d'_h \in H(\hat{T}_e)$ be a horizontal domino contained in an even block of $\hat{T}_e$. Let x be an entry of $M_e(T_o)$ corresponding to the center of an odd or even block. Then*

$$\mathsf{level}(x) - 1 = \begin{cases} \mathsf{area}(d_h) & \text{if } x = -1, \\ \mathsf{area}(d'_h) & \text{if } x = 1. \end{cases} \tag{4.4}$$

In Figures 6(b) and 6(c), the even block shown in green corresponds to the entry $x = 1$ in the green box. The area of the horizontal domino in that even block is given by $\mathsf{level}(x) - 1 = 4 - 1 = 3$.

Let T be a domino tiling of the extended AD_n. We say an entry x of $M_e(T)$ (or $M_o(T)$) is *associated* with a domino of T if $x = M_e(v)$ (or $M_o(v)$) and v is the midpoint of the longer edge of that domino. Let $A(T_o)$ (resp., $A(T_e)$) be the collection of entries x of $M_e(T_o)$ (resp., $M_o(T_e)$) such that x is associated with a vertical domino of $U(T_o)$ (resp., $U(T_e)$) except for those touch the line $y = 0$ and is not in the Aztec diamond region. We can show that the number of domino pairs of T_o or T_e is given by the tp statistic.

Lemma 4.4. *Let T, T_o, and T_e be as in Proposition 4.2. Then*

$$\#\{\textit{domino pairs in } T_o\} = \sum_{x \in A(T_o)} (\mathsf{tp}(x) - 1) + (n-1)^2, \textit{ and} \tag{4.5}$$

$$\#\{\textit{domino pairs in } T_e\} = \sum_{x \in A(T_e)} (\mathsf{tp}(x) - 1) + n^2. \tag{4.6}$$

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