*Séminaire Lotharingien de Combinatoire* **93B** (2025)
Article #81, 12 pp.

*Proceedings of the 37$^{th}$ Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)*

# Web bases for two-column tableaux from hourglass plabic graphs

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**Abstract.** Web bases give a diagrammatic calculus for spaces of $U_q(\mathfrak{sl}_r)$-tensor invariants, but are known to exist only in certain cases. Recently, we introduced *hourglass plabic graphs* to give the first rotation-invariant basis in the case $r = 4$, corresponding to 4-row rectangular tableaux. Separately, Fraser introduced a rotation-invariant web basis for the case of 2-column rectangular tableaux. Here, we show that Fraser's basis agrees with that predicted by the hourglass plabic graph framework. Together with our earlier results, this implies that hourglass plabic graphs give a uniform description of all known rotation-invariant $U_q(\mathfrak{sl}_r)$-web bases. Moreover, this provides a single combinatorial model simultaneously generalizing the Tamari lattice, the alternating sign matrix lattice, and the lattice of plane partitions.

**Keywords:** web basis, plabic graph, promotion, standard Young tableau

# 1 Introduction

*Webs* in type *A* give a diagrammatic way to represent morphisms between $\mathrm{SL}_r(\mathbb{C})$-modules. This allows very complicated algebraic manipulations to be expressed compactly and understandably in a graphical calculus. Moreover, webs exist for the quantum deformation $U_q(\mathfrak{sl}_r)$, allowing for applications to quantum link invariants.

A major challenge over the past few decades has been to find "nice" *web bases*. Kuperberg [8] gave a remarkable web basis for the tensor invariants of $U_q(\mathfrak{sl}_3)$, which consists

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of certain planar, trivalent graphs embedded in the disk with a proper white/black coloring of the vertices and a marked initial boundary vertex. Such a web corresponds to a tensor invariant in $\mathsf{Inv}_{U_q(\mathfrak{sl}_3)}(V_q^{\otimes n})$, where there are $n$ black boundary vertices. A key property of Kuperberg's basis is that it is *rotation-invariant*, i.e. marking a different boundary vertex as initial yields another such basis element. Finding rotation-invariant web bases for $r \geq 4$ has been a significant open problem for nearly 30 years. See [1, 3] for further discussion.

In [4], we introduced the framework of *hourglass plabic graphs* and used it to construct rotation-invariant web bases for spaces of $U_q(\mathfrak{sl}_r)$-invariants for $r \leq 4$, i.e. rank at most 3. For $r = 3$, the framework recovers Kuperberg's basis. In this abstract, we describe how the framework also yields web bases in arbitrarily high rank with *Plücker degree at most* 2, corresponding to tensor invariants in $\mathsf{Inv}_{U_q(\mathfrak{sl}_r)}(V_q^{\otimes 2r})$. Full details are available in [6], which builds on Fraser's work in [2].

Hourglass plabic graphs are a variation on Postnikov's theory of *plabic graphs*. See Section 2.2 for the precise definition. One key difference is that we allow *hourglass edges*, which are certain half-twist multi-edges. An $r$-hourglass plabic graph has a total of $r$ strands around any interior vertex, including hourglass multiplicity. See Figure 3 (lower right) for an example with $r = 7$.

A plabic graph is in many ways governed by its *trip permutation*, which is obtained by traveling through the graph following the *rules of the road*: taking a left at white vertices and a right at black vertices. An $r$-hourglass plabic graph $G$ instead has a tuple $\mathsf{trip}_\bullet(G) = (\mathsf{trip}_1(G), \ldots, \mathsf{trip}_{r-1}(G))$ of *trip permutations*, where $\mathsf{trip}_i$ takes the $i$th left at white vertices and the $i$th right at black vertices. The half-twists in hourglass edges become essential when computing $\mathsf{trip}_\bullet$. See Figure 3 (lower right) for a sample trip calculation. One may check that the square move in Figure 1 preserves all trip permutations. The $r$-valence condition immediately implies $\mathsf{trip}_i(G)^{-1} = \mathsf{trip}_{r-i}(G)$.

Hopkins–Rubey [7] observed that Kuperberg's basis could be viewed as certain *reduced* plabic graphs and interpreted the trip permutation. Building on their work, for each $r$-row rectangular tableau $T \in \mathsf{SYT}(r \times d)$, in [5] we associate a tuple $\mathsf{prom}_\bullet(T) = (\mathsf{prom}_1(T), \ldots, \mathsf{prom}_{r-1}(T))$ of *promotion permutations* with the property $\mathsf{prom}_i(T)^{-1} = \mathsf{prom}_{r-i}(T)$. See Section 2.1 for details. In [4], we give a bijection between 4-row rectangular standard tableaux $T$ and certain 4-hourglass plabic graph move-classes with the property that $\mathsf{trip}_\bullet(G) = \mathsf{prom}_\bullet(T)$ for any $G$ in the move-class. Under this bijection, web rotation corresponds to tableau *promotion*.

Fraser [2] gave a web basis for arbitrary $\mathrm{SL}_r$ in Plücker degree 2 (which in fact coincides with Lusztig's *dual canonical basis*, though this is rarely true in general). Fraser's construction gives an explicit bijection $\mathcal{F}$ associating to each 2-column tableau $T \in \mathsf{SYT}(r \times 2)$ a certain generalized-square-move-equivalence class $\mathcal{F}(T)$ of webs. See Section 2.3 for details. We augment Fraser's map $\mathcal{F}$ by interpreting the resulting webs as hourglass plabic graphs. Our first main result is:

**Theorem 1.1.** *For any $T \in \mathsf{SYT}(r \times 2)$ and any r-hourglass plabic graph $G \in \mathcal{F}(T)$, we have* $\mathsf{trip}_\bullet(G) = \mathsf{prom}_\bullet(T)$. *Promotion of $T$ intertwines with rotation of $\mathcal{F}(T)$.*

Our second main result provides an intrinsic characterization of graphs $G$ appearing in the image of Fraser's construction. Plabic graphs have a notion of being *reduced*, which for instance forbids certain double crossings of trips. In [4], we introduced a corresponding notion of *fully reduced* 4-hourglass plabic graphs. Here we provide a notion suitable for general $r$. See Definition 2.6. We show:

**Theorem 1.2.** *An r-hourglass plabic graph with* $2r$ *black boundary vertices is in the image of $\mathcal{F}$ if and only if it is fully reduced.*

A consequence of these results is a strengthening of Fraser's main result in [2] of a rotation-invariant web basis for $\mathsf{Inv}_{U_q(\mathfrak{sl}_r)}(V_q^{\otimes 2r})$ by giving an intrinsic graph-theoretic characterization of the basis webs. This stands in contrast to [2], where the basis webs are characterized only as the image of $\mathcal{F}$.

*Remark* 1.3. Our web basis consists of *square move* equivalence classes of certain $r$-hourglass plabic graphs. However, square moves do not change the tensor invariant. The smallest case where such equivalence classes are absolutely necessary is in $\mathsf{Inv}_{\mathrm{SL}_4}(V^{\otimes 8})$, which is the basic $r = 4$ square move living in Plücker degree 2 (see Figure 1). Here the corresponding SYT $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline 7 & 8 \\ \hline \end{array}$ has promotion order 2, while either of these graphs has rotation order 4. But the entire equivalence class has rotation order 2. As we shall see, move equivalence classes have rich connections to statistical mechanics and lattices. In our experience, what may be initially perceived as a defect is in fact a feature of the framework.

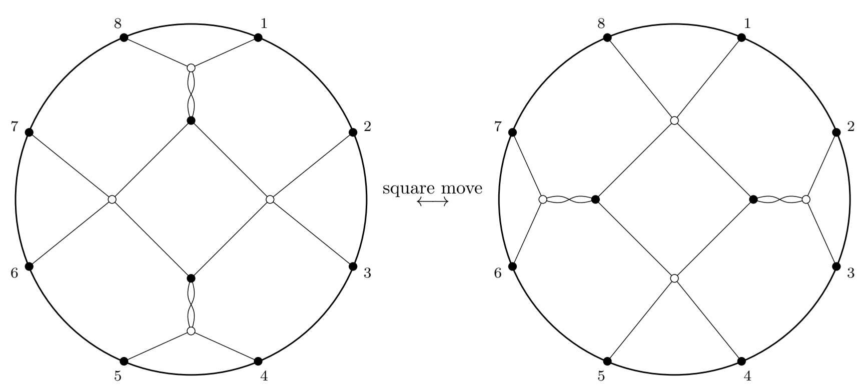


**Figure 1:** The square move equivalence class $\mathcal{F}(T)$ for $T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline 7 & 8 \\ \hline \end{array}$.

Furthermore, the hourglass plabic graph framework unifies the following three classical constructions: Alternating sign matrices, plane partitions in a box, and the Tamari lattice. See [4, Section 8] and Proposition 4.4 for details.

The remainder of this extended abstract is organized as follows. Section 2 collects background and preliminaries on tableaux and promotion, on hourglass plabic graphs, and on Fraser's map $\mathcal{F}$. In Section 3, we sketch the proof of the first part of Theorem 1.1. Section 4 characterizes when square faces are fully reduced, shows that square moves preserve trip permutations, and discusses the connection between square moves and the Tamari lattice. Section 5 sketches the core of the proof of Theorem 1.2.

This is an extended abstract of the full paper [6].

# 2 Background and preliminaries

## 2.1 Promotion permutations for standard Young tableaux

A *partition* with $r$ rows is a tuple $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{N}^r_{>0}$ where $\lambda_1 \geq \cdots \geq \lambda_r$. The *diagram* of $\lambda$ is a collection of upper-left justified boxes where the $i$th row from the top has $\lambda_i$ boxes. A *standard Young tableau (SYT) of shape* $\lambda$ is a bijective filling of the diagram of $\lambda$ with the numbers $\{1, \ldots, n\}$, where $n = \sum_i \lambda_i$. When $\lambda$ is a rectangle with $r$ rows and $d$ columns, we denote this set as $\mathsf{SYT}(r \times d)$.

Given $T \in \mathsf{SYT}(\lambda)$, define the *(jeu de taquin) promotion* of $T$, denoted $\mathcal{P}(T)$, as follows. Delete the 1 from $T$, leaving an empty box. Move the smallest of the numbers to the right or below the empty box into the empty box. Continue this process with the new empty box, until the empty box is at an outer corner. Fill the empty box with $n+1$, then subtract 1 from all entries. The *promotion path* consists of the numbers that move in this process, and is denoted by arrows in Example 2.2.

In [5, Definition 6.1], we defined *promotion functions* for fluctuating tableaux, a class containing SYT. When the shape of the tableaux $T$ is rectangular with $r$ rows, we showed these functions are permutations, and in fact, $\mathsf{prom}_i(T)^{-1} = \mathsf{prom}_{r-i}(T)$ [5, Theorem 6.7]. In our setting, we do not need the definition in full generality, but give the following, which was [5, Proposition 6.9].

**Definition 2.1.** Let $T \in \mathsf{SYT}(r \times d)$ and $1 \leq i \leq r-1$. Then the $i$th *promotion permutation* is constructed as $\mathsf{prom}_i(T)(b) \equiv a + b - 1 \pmod{r \cdot d}$ if and only if $a$ is the unique value that moves from row $i+1$ to $i$ in the application of jeu de taquin promotion to $\mathcal{P}^{b-1}(T)$, where $1 \leq b \leq r \cdot n$.

*Example* 2.2. Consider the promotion orbit of $T \in \mathsf{SYT}(7 \times 2)$ shown in Figure 2. One

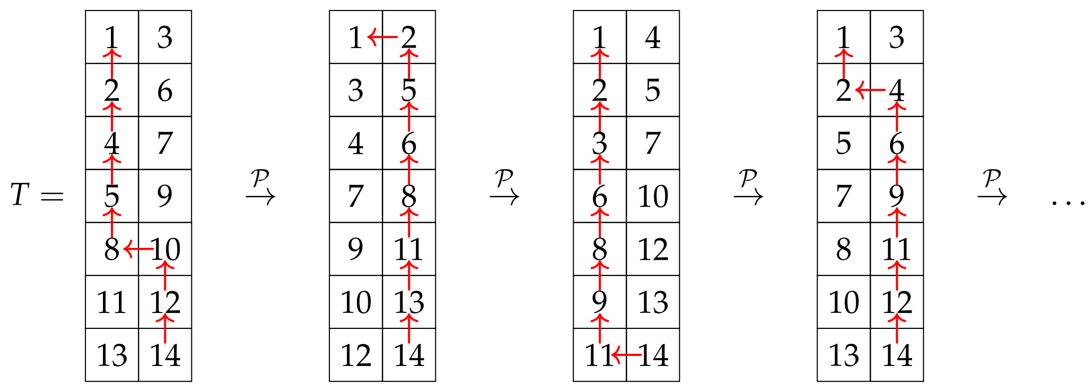


**Figure 2:** The promotion orbit discussed in Example 2.2.

computes that

$$\begin{aligned}
\mathsf{prom}_1(T) &= 2\ 6\ 4\ 5\ 9\ 7\ 8\ 12\ 10\ 11\ 14\ 13\ 3\ 1 = \mathsf{prom}_6^{-1}(T),\\
\mathsf{prom}_2(T) &= 4\ 7\ 5\ 9\ 12\ 8\ 10\ 14\ 11\ 13\ 3\ 1\ 6\ 2 = \mathsf{prom}_5^{-1}(T), \text{ and}\\
\mathsf{prom}_3(T) &= 5\ 9\ 8\ 12\ 14\ 10\ 11\ 1\ 13\ 3\ 6\ 2\ 7\ 4 = \mathsf{prom}_4^{-1}(T).
\end{aligned}$$

Note that the first entries in these permutations are given by the entries siding up in the promotion path of $T$, the second entries are given by the entries sliding up in the promotion path of $\mathcal{P}(T)$ plus one.

## 2.2 Hourglass plabic graphs

Following [4], we define hourglass plabic graphs. An *hourglass graph* $G$ is an underlying planar embedded graph $\widehat{G}$, together with a positive integer multiplicity $m(e)$ for each edge $e$. The hourglass graph $G$ is drawn in the plane by replacing each edge $e$ of $\widehat{G}$ with $m(e) > 1$ with $m(e)$ *strands*, twisted so that the clockwise orders of these strands around the two incident vertices are the same. We call this twisted edge an $m(e)$*-hourglass*, and call an edge with $m(e) = 1$ a *simple edge*. The *degree* $\deg(v)$ of a vertex $v \in G$ is the number of edges incident to $v$, counted with multiplicity, while its *simple degree* $\widehat{\deg}(v)$ is its degree in the underlying graph $\widehat{G}$.

**Definition 2.3.** An *r-hourglass plabic graph* is a bipartite hourglass graph $G$, with a fixed proper black-white vertex coloring, embedded in a disk, with all internal vertices of degree $r$, and all boundary vertices of simple degree one, labeled clockwise as $b_1, b_2, \ldots, b_n$. We consider $G$ up to planar isotopy fixing the boundary circle.

We say that $G$ is of *standard type* if all boundary vertices are colored black and of degree one, and in this case we say that $G$ has *Plücker degree* $\frac{n}{r}$. In the remainder of the paper, all $r$-hourglass plabic graphs are assumed to be standard type, unless otherwise noted. We say $G$ is *contracted* if all internal vertices have simple degree at least 3.

**Definition 2.4.** Let $G$ be an $r$-hourglass plabic graph with boundary vertices $b_1, \dots, b_n$. For $1 \le a \le r-1$, the *a-th trip permutation* $\mathsf{trip}_a(G)$ is the permutation of $[n]$ obtained as follows: for each $i$, begin at $b_i$ and walk along the edges of $G$, taking the $a$-th leftmost turn at each white vertex, and $a$-th rightmost turn at each black vertex, until arriving at a boundary vertex $b_j$. Then $\mathsf{trip}_a(G)(i) = j$. The walk taken is called a $\mathsf{trip}_a$*-segment*. Note that $\mathsf{trip}_a(G)^{-1} = \mathsf{trip}_{r-a}(G)$. We write $\mathsf{trip}_\bullet(G)$ for the tuple of these trip permutations. See Figure 3 for an example of a trip segment.

**Definition 2.5.** A trip segment has a *self-intersection* if it passes through a vertex more than once. In particular, all trip segments not reaching the boundary have a self-intersection.

Now suppose $G$ has no self-intersections. We define what it means for two different trip segments $\ell, \ell'$ of $G$ to intersect. First suppose the four endpoints of $\ell, \ell'$ are distinct. Draw the segments $\ell, \ell'$ on the underlying graph $\widehat{G}$. Contract any edges that $\ell, \ell'$ both pass through to a point. The result is a network of $\times$'s which we call *intersections*. Each intersection has four distinct directions at which the two segments $\ell, \ell'$ enter and leave. An intersection is *essential* if $\ell$ and $\ell'$ cross and *inessential* if $\ell$ and $\ell'$ bounce off of each other. If instead segments $\ell$ and $\ell'$ share a boundary vertex $b_k$, we consider them to have an essential intersection at the unique internal vertex incident to $b_k$. We say $\ell$ and $\ell'$ have an *oriented double crossing* if they have an essential intersection followed in the forwards direction along both segments by another essential intersection.

The following definition is central to the paper. Here an *isolated component* of an hourglass graph $G$ is a connected component which does not contain a boundary vertex.

**Definition 2.6.** An hourglass plabic graph $G$ is *fully reduced* if it has no isolated components, it has no self-intersections, no two $\mathsf{trip}_i$-segments have an oriented double crossing, and no pair of $\mathsf{trip}_i$- and $\mathsf{trip}_{i+1}$-segments have an oriented double crossing.

*Remark* 2.7. In [4], we called the condition of Definition 2.6 "monotonic", but showed that for $r \le 4$ it was equivalent to another definition of "fully reduced".

## 2.3 Fraser's map

We give a brief description of the map of Fraser [2, Section 1.1-1.3] from 2-column standard Young tableaux to certain webs.

Define a map $\mathcal{F}$ on $\mathsf{SYT}(r \times 2)$ as the composition of four maps, an example to follow the construction is given in Figure 3. First, draw the matching given by the standard Catalan bijection of the transposed tableau. That is, given $T \in \mathsf{SYT}(r \times 2)$, let $\mathfrak{M}(T)$ be the unique noncrossing matching on $2r$ points in which each number in the first column is the smallest in its pair.

Given a noncrossing matching $M$ of $2r$ points with $s$ short arcs (arcs between adjacent boundary vertices), let $i_1, i_2, \ldots, i_s$ be the sequence of numbers such that $i_j$ and $i_j + 1 \bmod 2r$ are joined by an arc. That is, $i_1, i_2, \ldots$ are the cyclically left endpoints of all the short arcs. The *claw sets* are the cyclic intervals $C_j := (i_j, i_{j+1}]$. (These were called color sets in [2].) We use the claw sets to obtain a *weighted dissection* $\mathfrak{d}(M)$ of the $s$-gon by merging the boundary vertices in each claw set and replacing multiple edges by a single edge with the corresponding weight. Note that the total weight of $\mathfrak{d}(M)$ is the number of arcs in $M$, which is $r$. Similarly, the weight of $\mathfrak{d}(M)$ at boundary vertex $j$ in the $s$-gon equals the cardinality of the claw set $C_j$ of $M$.

Given such a weighted dissection $\mathfrak{d}$, we then add in diagonals of weight zero to create a weighted triangulation $\mathfrak{t}(\mathfrak{d})$. (This step is not unique; any choice of triangulation is related to any other by a sequence of flip moves on weight zero diagonals.)

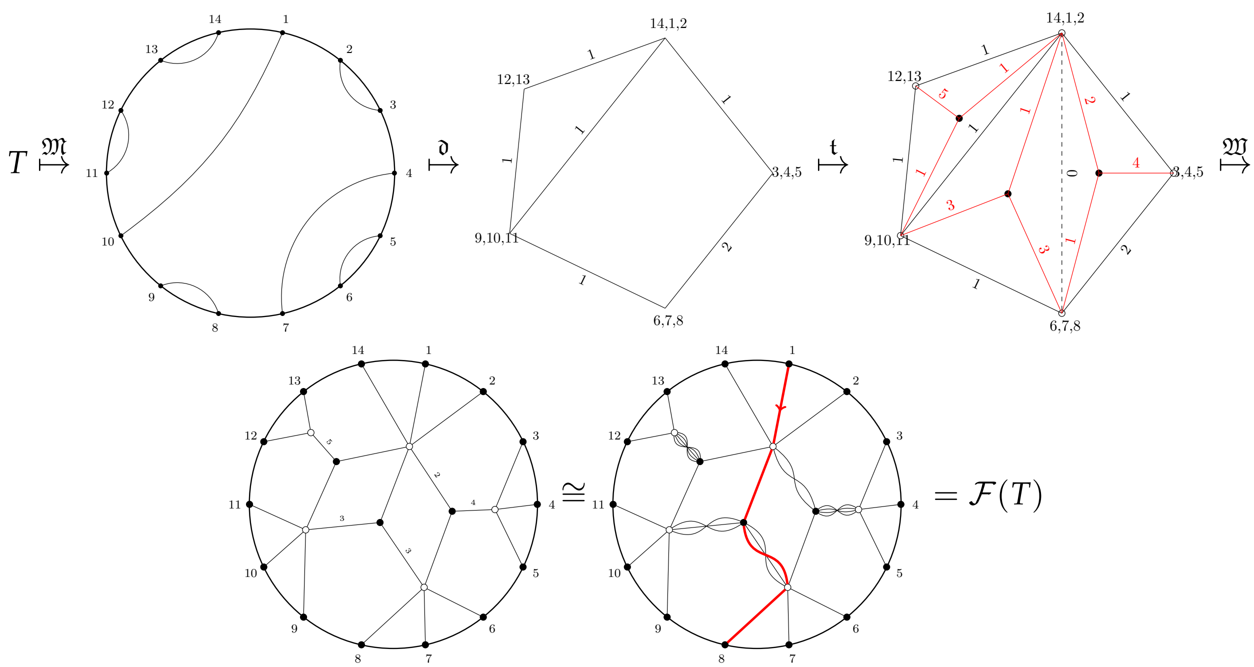

**Figure 3:** In illustration of the construction discussed in Section 2.3. Top left: The noncrossing matching corresponding to the transpose of the first SYT of Example 2.2. Top middle and right: Its corresponding weighted dissection $\mathfrak{d}$ and weighted triangulation $\mathfrak{t}$. Bottom left: The web produced from $\mathfrak{t}$ via the construction of Fraser [2]. Bottom right: The hourglass plabic graph constructed by replacing edges of weight $> 1$ with hourglasses. It is fully reduced. The red path is $\mathsf{trip}_4(1)$. It ends at 8 since the number 8 slides into the 4th row in the first step of Example 2.2.

Given such a weighted triangulation $\mathfrak{t}$, construct a web $\mathfrak{W}(\mathfrak{t})$ as follows. $\mathfrak{W}(\mathfrak{t})$ has $2r$ black boundary vertices and $s$ internal white vertices (one for each vertex of the $s$-gon). The $j$th white vertex in $\mathfrak{W}(\mathfrak{t})$ is adjacent to the entire claw set $C_j$ of boundary vertices by weight 1 edges. $\mathfrak{W}(\mathfrak{t})$ also has an internal trivalent black vertex $b(\Delta)$ for each triangle $\Delta$ in $\mathfrak{t}$. Let $\delta_1, \delta_2, \delta_3$ denote the vertices of $\Delta$ in $\mathfrak{t}$ and $w(\delta_1), w(\delta_2), w(\delta_3)$ denote the corresponding white vertices in $\mathfrak{W}(\mathfrak{t})$; these are the three vertices $b(\Delta)$ is adjacent to in $\mathfrak{W}(\mathfrak{t})$. The weights of these three edges is given as follows. Let $e_{\delta_i}$ denote the edge of $\Delta$ opposite to $\delta_i$. The edge $e_{\delta_i}$ cuts the triangulation $\mathfrak{t}$ into two parts; call (*) the one that does not contain $\Delta$. Then the weight of the edge between $b(\Delta)$ and $w(\delta_i)$ is the sum of the weights of the edges in (*). In the case where $e_{\delta_i}$ is on the boundary (*) only consists of the edge $e_{\delta_i}$.

We add a final step to the construction of [2]. Given the web $\mathfrak{W}$, replace each edge of weight $m > 1$ with an hourglass of multiplicity $m$ to form an $r$-hourglass plabic graph $\mathcal{F}(T)$. We say that an hourglass plabic graph is *Fraser* if it can be produced from this construction. Fraser showed that the result of this construction is a web, and that the web invariant does not depend on the choice of triangulation [2, Proposition 1.13].

# 3 The $\mathsf{trip}_\bullet = \mathsf{prom}_\bullet$ property

In this section we sketch the proof of the first statement in Theorem 1.1, i.e. if $G \in \mathcal{F}(T)$, then $\mathsf{trip}_\bullet(G) = \mathsf{prom}_\bullet(T)$. Our approach first describes promotion permutations directly in terms of the matching of $T$. We then identify "ears" in Fraser graphs which may be removed inductively. We show that the equality $\mathsf{trip}_\bullet = \mathsf{prom}_\bullet$ is appropriately preserved under ear removal.

## 3.1 Promotion permutations for two column rectangles

We regard non-crossing perfect matchings as permutations $[2r] \to [2r]$. For such a matching $M$ we say that $i$ is an *opener* if $i < M(i)$ and $i$ is a *closer* otherwise.

Let $T \in \mathsf{SYT}(r \times 2)$ and $\mathfrak{M}(T) : [2r] \to [2r]$ be the corresponding non-crossing perfect matching (from the first step of the map $\mathcal{F}$). We call $\{1, \mathfrak{M}(T)(1)\}$ the *barrier*. Let $o_1 < \cdots < o_k$ be the openers of $\mathfrak{M}(T)$ strictly between 1 and $\mathfrak{M}(T)(1)$ and let $c_{k+1} < \cdots < c_{r-1}$ be the closers of $\mathfrak{M}(T)$ strictly larger than $\mathfrak{M}(T)(1)$. Note that every matching in $\mathfrak{M}(T)$ except the barrier has exactly one element in $\{o_1, \ldots, o_k, c_{k+1}, \ldots, c_{r-1}\}$. In Example 2.2 these openers are $\{2, 4, 5, 8\}$ and the closers are $\{12, 14\}$.

**Proposition 3.1.** *With the above notation, we have*

$$\mathsf{prom}_i(T)(1) = \begin{cases} o_i & \text{if } 1 \leq i \leq k, \\ c_i & \text{if } k+1 \leq i \leq r-1. \end{cases}$$

## 3.2 Claws and ears

Our analysis of Fraser graphs is inductive and involves the following structures. See [6] for full details.

**Definition 3.2.** Let $G$ be an hourglass plabic graph. A *claw* $C$ of $G$ is a maximal collection of boundary vertices connected to the same white internal vertex by edges with hourglass multiplicity 1. The *size* of a claw is the number of its boundary vertices. Their common internal vertex is the *center* of the claw. An *ear* $(A, B, C)$ of type $(p, q)$ of $G$ is a particular configuration of claws $A, B, C$ as in the lower left of Figure 4. Call an ear of a Fraser web *proper* if neither vertex of the barrier $\{1, \mathfrak{M}(T)(1)\}$ appears as a boundary vertex of the claw $B$.

**Definition 3.3.** To *remove an ear* $(A, B, C)$ of type $(p, q)$ in a web $G$, do the following; First delete the claw $B$ as well as the edges and vertex connecting the centers of $A, B, C$. Add $q$ clockwise-last boundary edges to $A$, resulting in a new claw $A'$, and add $p$ clockwise-first boundary edges to $C$, resulting in a new claw $C'$.

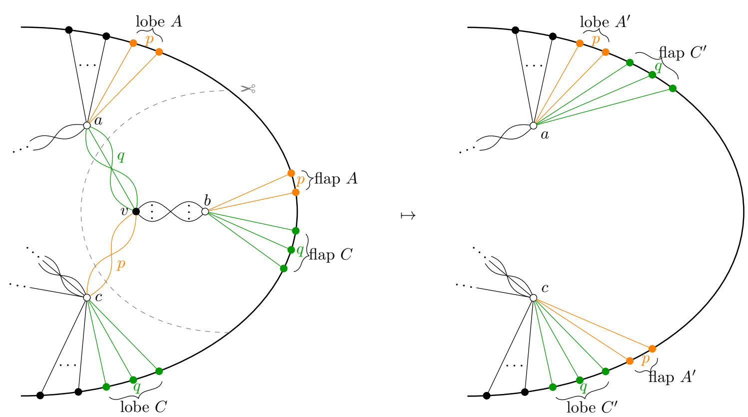


**Figure 4:** Removal of an ear $(A, B, C)$ of type $(p, q)$ resulting in new claws $(A', C')$.

Suppose $G$ is Fraser with triangulation $\mathfrak{t}$. Ears correspond precisely to triangles in $\mathfrak{t}$ with at least two edges on the boundary. Removing an ear corresponds precisely to collapsing such a triangle. In particular, removing an ear from a Fraser web results in a Fraser web with fewer vertices. This idea now allows an induction to prove Theorem 1.1.

*Proof of Theorem 1.1 (sketch).* In the following we denote with $T(\cdot)$ the SYT corresponding to a Fraser graph. We induct on the number of vertices of $G$. In the base case, $G$ is the disjoint union of two claws, and the result is straightforward to verify. Otherwise, one may show $G$ has a proper ear. Let $H$ be the result of removing this ear. As noted above, $H$ is Fraser with fewer vertices, so by induction $\mathsf{trip}_\bullet(H) = \mathsf{prom}_\bullet(T(H))$.

By analyzing closers and openers and using Proposition 3.1, we show $\mathsf{trip}_i(G)(1) = \mathsf{prom}_i(T(G))(1)$. Using the fact that $\mathcal{F}$ intertwines promotion of the tableau with rotation of the web, we obtain $\mathsf{prom}_\bullet(T(G)) = \mathsf{trip}_\bullet(G)$. □

# 4 Square faces

In this section, we define *square moves* (see Definition 4.2 and Figure 5) on *square faces*, which play a special role in the theory of hourglass plabic graphs. In the setup of Fraser's map, the square move corresponds to flips of diagonals, which yields a connection between square moves and the Tamari lattice in Proposition 4.4. We also characterize when square faces are fully reduced (Theorem 4.5) and show that square moves preserve trip permutations (Theorem 4.6).

**Definition 4.1.** The edges of a plabic graph decompose the embedding disk into regions. If such a region is not adjacent to the boundary of the disk, we call it a *face*. A *face* of an hourglass plabic graph is a face of the underlying plabic graph. We call a face a *square* if this cycle has exactly 4 edges. For a face $F$ of an hourglass plabic graph $G$ bounded by edges $e_1, \ldots, e_{2f}$, define $m(F) := \sum_{i=1}^{2f} m(e_i)$.

Note that the lacunae between the strands of an hourglass edge are not faces.

**Definition 4.2.** Let $F$ be a square in an $r$-hourglass plabic graph $G$ with $m(F) = r$ and with vertices $v_1, v_2, v_3, v_4$. We say $G'$ is obtained from $G$ by applying a *square move* at the square $F$ if $G'$ is obtained by the following procedure: Remove the edges between the vertices $v_1, v_2, v_3, v_4$, insert a new square $v'_1, v'_2, v'_3, v'_4$ and hourglasses such that $v_i$ and $v'_i$, for $1 \leq i \leq 4$, are adjacent and $m(v_i v_{i+1}) = m(v'_{i+2} v'_{i+3})$, the result is bipartite and all vertices have degree $r$. If by this process a vertex with simple degree 2 is generated, remove it and its incident edges and contract its neighbors into a single vertex. (A visual aid for this definition is in Figure 5.)

As shown in the proof of [2, Proposition 1.13] and depicted in Figure 6, square moves correspond to flips of weight 0 diagonals in the triangulation produced in the construction of $\mathcal{F}(T)$.

**Proposition 4.3** ([2, Proposition 1.13])**.** *For a fixed $T \in \mathsf{SYT}(r \times 2)$, the hourglass plabic graph $\mathcal{F}(T)$ is determined uniquely, up to square moves.*

Given a partition shape $\lambda$, let the *superstandard* SYT of shape $\lambda$ be constructed by filling $\lambda$ in order left to right, then top to bottom. For the superstandard SYT of shape $r \times 2$, $\mathcal{F}$ gives diagonals all of weight zero. Thus the move equivalence class is counted by the Catalan numbers and is indeed equivalent to the Tamari lattice; see Proposition 4.4 below. In [4, Proposition 8.2], we showed the square move equivalence class of the

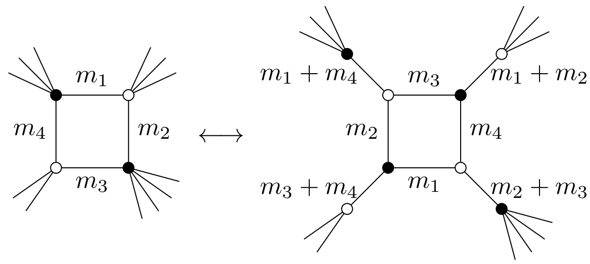


**Figure 5:** The square move. Here we assume $m_1 + m_2 + m_3 + m_4 = r$. Edge multiplicities on the boundary of the picture are not shown, and are preserved by the move.

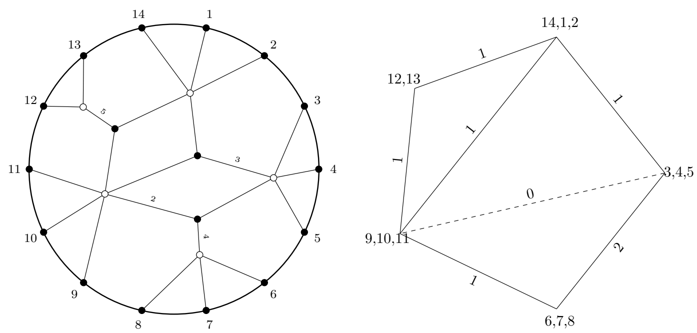


**Figure 6:** A square move applied to our running example. A square move on the web corresponds to a flip of a diagonal with weight 0 in the underlying triangulation.

hourglass plabic graph corresponding to the superstandard $4 \times n$ SYT is in bijection with the set of $n \times n$ alternating sign matrices.

**Proposition 4.4.** *Let $T$ be the $r \times 2$ superstandard SYT. The square move equivalence class of $\mathcal{F}(T)$ is in bijection with triangulations of an $r$-gon connected by diagonal flips, i.e. the Tamari lattice.*

The following results hold for arbitrary hourglass plabic graphs and thus are important results for the general framework.

**Theorem 4.5.** *If a square $F$ in an $r$-hourglass plabic graph $G$ satisfies $m(F) > r$, then $G$ is not fully reduced.*

The following theorem can be obtained as a consequence of Theorem 1.1.

**Theorem 4.6.** *Applying a square move to an hourglass plabic graph preserves the trip permutations* $\mathsf{trip}_\bullet$.

*Remark* 4.7. If $W'$ is an hourglass plabic graph that differs from $W$ by a sequence of square moves, then the associated tensor invariants are equal.

# 5 Characterization of fully reduced graphs

We now quickly sketch the intrinsic characterization of Fraser graphs.

*Proof (sketch) of Theorem 1.2.* A careful analysis of the argument mentioned in Section 3 (cutting away ears) shows that Fraser graphs are fully reduced. For the converse, a key observation is that if $G$ is a fully reduced hourglass plabic graph, then the underlying plabic graph $\widehat{G}$ is reduced. This fact is then combined with the observation that, for $H$ a contracted, bipartite, reduced plabic graph, whose trip permutation has two antiexcedances, all internal black vertices have degree 3 and all white vertices are adjacent to the boundary. The final ingredient is that, for $G$ be a connected, contracted, fully reduced hourglass plabic graph of Plücker degree two, all internal faces of $G$ are squares. The proof is completed by constructing a triangulated polygon $\mathfrak{t}$ whose vertices correspond to the white vertices of $G$ and whose diagonals correspond to the square faces in $G$. □

## Acknowledgements

We thank Chris Fraser for his helpful comments. Work on this project was carried out during visits to NDSU (partially supported by NSF DMS-2247089) and to ICERM. We are grateful for the excellent working environments provided by these institutions.

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