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Identifying Orbit Lengths for Promotion

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Abstract. In this work we study Schützenberger's promotion operator on standard Young tableaux via a corresponding graphical construction known as $m-$diagrams. In particular, we prove that certain internal structures of SYT are preserved under promotion and correspond to distinct components of $m-$diagrams. By treating these structures as atomic parts of the $m-$diagram, we provide a simple algorithm for computing the promotion orbit length of rectangular SYT.

Keywords: Young tableaux, jeu de taquin, promotion, web

1 Introduction

Young tableaux are fundamental combinatorial objects. With various choices of fillings and rules for manipulations, Young tableaux encode representations and algebraic operations on representations [1, 5, 7, 11], Schubert varieties and the cohomology ring structure induced by Schubert classes [19, 20], and other constructions in algebra and geometry [9].

We study Schützenberger's *promotion* operator on *standard Young tableaux*, which is important in part because of its role in the cyclic-sieving phenomenon [13, 14]. A seminal — and very surprising — result of Haiman showed that the promotion orbit of an $m \times n$ rectangular standard Young tableaux has order at most mn [5]. This has been explored and extended in different contexts, using related operators [8, 17, 18] or equivariant maps to other combinatorial objects admitting a group action [2, 10, 15, 21].

Yet, very little has been done to analyze or characterize the sizes of specific promotion orbits, despite analogous classification and stability questions in representation theory. A result by Purbhoo and Rhee does identify all minimal orbits for rectangular tableaux [12]. In this paper, we do the following:

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1. define a notion of uniformly equivalent tableaux and investigate promotion via a graphical construction known as $m-$diagrams, which are essentially a union of noncrossing matchings;

2. prove that, while promotion of rectangular tableaux is not equivariant with rotation of $m-$diagrams, the *rotational symmetry* of the $m-$diagram coincides with the length of promotion orbits. In particular, this allows us to analyze orbit lengths when the $m-$diagrams are disconnected (see Theorem 3.1).

2 Preliminaries

2.1 Tableaux

We begin by fixing notations and conventions for tableaux.

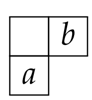
Definition 2.1 (Standard Young Tableau)**.** Suppose $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ is a partition of n. A *Young diagram* of shape λ is a grid of left- and top-aligned boxes with λ_i boxes in the i^{th} row for each $1 \leq i \leq k$. If all parts of the partition λ are the same then we call λ a rectangular partition and its Young diagram a *rectangular Young diagram*.

A *standard Young tableau (SYT)* is a filling of the Young diagram for λ with the values $1, \ldots, n$ without repetition such that rows strictly increase left-to-right and columns strictly increase top-to-bottom.

Given an SYT T, we use $|T|$ to denote the number of boxes in T and T_{ij} to denote the entry in the i^{th} row and j^{th} column of T. Next, we review Schützenberger's jeu de taquin promotion on SYT. Promotion is a function from SYT of fixed shape λ to SYT of the same shape [16].

Definition 2.2 (Promotion)**.** Given an SYT T, the *promotion* of T is the SYT $P(T)$ created as follows:

1. Erase 1 in the top left corner of T and leave an empty box.

2. Given the configuration 

and $b < a$ then slide b left; else if $a < b$ slide a up.

3. Repeat the above process until there are no nonempty boxes below or to the right of the empty box.

4. Decrement all entries by 1 and insert the largest entry of T into the empty box.

Example 2.3. Figure 1 gives an examples of an SYT T and its promotion $P(T)$. This tableau will serve as our running example.

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 6 & 7 & 14 & 19 \\ \hline 3 & 8 & 9 & 15 & 18 & 21 \\ \hline 4 & 10 & 11 & 16 & 20 & 23 \\ \hline 5 & 12 & 13 & 17 & 22 & 24 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|} \hline 1 & 5 & 6 & 13 & 17 & 18 \\ \hline 2 & 7 & 8 & 14 & 19 & 20 \\ \hline 3 & 9 & 10 & 15 & 21 & 22 \\ \hline 4 & 11 & 12 & 16 & 23 & 24 \\ \hline \end{array} = P(T)$$

Figure 1: The tableau T and its promotion $P(T)$

Note that while we define the promotion function on SYT, the same operation can be performed on any filling of a Young diagram such that the entries are non-repeating with strictly increasing rows and columns.

We are particularly interested in the orbits of tableaux under promotion. For a given SYT T, we denote the promotion orbit of T by $\mathcal{O}(T)$. The key observation in our approach is that certain internal structures of SYT are preserved during promotion. To this end, we introduce the following definition.

Definition 2.4 (Equivalence)**.** Suppose S and T are skew tableaux with $|S| = |T| = n$ and contents $\{s_1, s_2, \ldots, s_n\}$ and $\{t_1, t_2, \ldots, t_n\}$ respectively. We say that S is *equivalent* to T, or $S \equiv T$, if S and T are the same after each tableau is left justified with the entries relabeled smallest to largest so that the content becomes $\{1, 2, \ldots, n\}$. Moreover, we say S is *uniformly equivalent* to T if the content of S is $\{k+1, k+2, \ldots, k+n\}$ and the content of T is $\{\ell+1, \ell+2, \ldots, \ell+n\}$ for some k, ℓ and $S \equiv T$.

Uniform equivalence is a powerful condition that allows us to treat uniformly proper subtableaux as atomic parts of the corresponding m-diagrams and webs (see the next definition for more). However, there are contexts in which more generality is useful.

Definition 2.5 (Uniformly Proper Subtableau)**.** Suppose T is a tableau with n rows and $S \subsetneq T$ is a subtableau. We call S *uniformly proper* if:

1. S has n rows;

2. S is uniformly equivalent to a standard Young tableau.

We call S a uniformly proper *rectangular* subtableau if in addition all of its rows have the same length. If T contains no uniformly proper subtableau, then we say that T is *minimal.*

When an SYT T is not minimal, and thus has some uniformly proper subtableau S, we can write the tableau T as the horizontal concatentation $T = T_1 S T_2$, where T_1 and T_2 are also subtableaux of T, see Figure 2. More generally, given two skew tableaux T and S, we write TS to refer to the tableau resulting from the horizontal concatenation of T and S when such an operation makes sense.

1	2	6	7	14	19
3	8	9	15	18	21
4	10	11	16	20	23
5	12	13	17	22	24

T

=

1	2
3	
4	
5	

T_1

	6	7	14
8	9	15	
10	11	16	
12	13	17	

S

	19
18	21
20	23
22	24

T_2

Figure 2: Decomposition of tableau T with uniformly proper subtableau S

We will see in Section 3 that the problem of determining the orbit length $|\mathcal{O}(T)|$ of any non-minimal tableau T can be reduced to determining the orbit lengths of certain smaller minimal tableaux.

2.2 $m-$Diagrams

Webs are planar graph morphisms in diagrammatic categories for certain representations of the quantum groups $U_q(\mathfrak{sl}_n)$ [3, 6]. There is a rich literature on how promotion of tableaux corresponds to rotation of webs in the cases $n = 2$ and $n = 3$ [10, 15, 21]; but this is constrained for larger n by our limited understanding of webs in that case (though see [4] for a rotation-invariant basis when $n = 4$).

We instead focus on an intermediary between webs and tableaux: m-diagrams, which are collections of arcs that satisfy certain noncrossing conditions.

Definition 2.6 ($m-$Diagram)**.** A *matching* $\mathcal{M}$ on the set $\{1, 2, \ldots, n\}$ is a collection of disjoint pairs

$$\mathcal{M} = \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\} \subseteq \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}.$$

We often refer to the pair (i, j) as an *arc* and assume in our notation that $i \leq j$. We say that an integer i' is *on the arc* (i, j) if $i' \in \{i, j\}$ and *below the arc* (i, j) if $i < i' < j$.

A matching $\mathcal{M}$ is:

- *perfect* if every number $1, 2, \ldots, n$ is used on an arc;
- *crossing* if it contains two arcs $(i, j), (i', j') \in \mathcal{M}$ with $i < i' < j < j'$;
- *noncrossing* if it is not crossing;
- *with repetition* if it contains at least one arc of the form (i, i);
- *standard* if every integer i' below an arc (i, j) is itself on an arc.

An m-diagram on the set $\{1, 2, \ldots, n\}$ is the union $\bigcup_{i=1}^{r} \mathcal{M}_i$ where each $\mathcal{M}_i$ is a noncrossing matching.

Note that an m-diagram itself may not be a noncrossing matching since arcs from one matching are allowed to cross arcs from another matching. An m-diagram may have arcs of the form (i, i) but is itself not a multiset, so each arc appears at most once in an m-diagram regardless of how many different matchings contain it.

We represent matchings and m-diagrams by drawing semicircular arcs in the upper half plane between the points i and j on the x-axis. The next lemma demonstrates a simple bijection between SYT and $m-$diagrams, the proof of which can be found in [15, Lemma 1], [21, Proposition 2.4].

Lemma 2.7. *Let T be an SYT with r rows and content $\{1, \ldots, n\}$. For each $i \leq r-1$ the following recursive process constructs a noncrossing matching $\mathcal{M}_i$ on the integers filling rows i and $i+1$ of T:*

1. *If row i of T has λ_i boxes then denote them by $t_i(1), t_i(2), \ldots, t_i(\lambda_i)$ and similarly for row $i+1$.*
2. *Create an arc $(t_i(j), t_{i+1}(1))$ where*
$$j = \max\{1 \leq s \leq \lambda_i : t_i(s) \leq t_{i+1}(1)\}.$$
3. *If $t_{i+1}(1), \ldots, t_{i+1}(\ell)$ are all on arcs then create an arc $(t_i(j), t_{i+1}(\ell+1))$ where*
$$j = \max\{1 \leq s \leq \lambda_i : t_i(s) \leq t_{i+1}(\ell+1) \\ \text{and } (t_i(s), t_{i+1}(j')) \text{ is not an arc for any } j' \leq \ell\}.$$

We rephrase the previous result in terms of m-diagrams as follows. All of the proofs are immediate from the previous lemma together with the definitions.

Corollary 2.8. *The algorithm in Lemma 2.7 defines a function $\varphi(T) = M_T$ from r-row SYT to m-diagrams built from $r-1$ matchings. In addition, all of the following hold:*

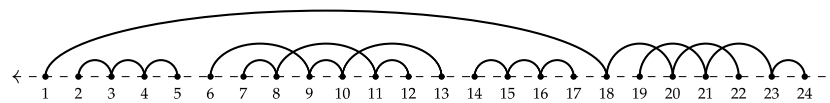
- *The number of arcs in the matching $\mathcal{M}_i$ is equal to the number of boxes in row $i+1$ of T.*
- *Each matching $\mathcal{M}_i$ is without repetition.*
- *The matching $\mathcal{M}_i$ is standard with respect to the entries in rows i and $i+1$ of T.*
- *If T is a standard Young tableau on a rectangular Young diagram then $\varphi(T) = M_T$ is the union of $r-1$ matchings $\mathcal{M}_i$ of the same cardinality. Moreover every integer $1, 2, \ldots, |T|$ is one of three types: the end of an arc in $\mathcal{M}_i$ and the start of an arc in $\mathcal{M}_{i+1}$, or the start of an arc on $\mathcal{M}_1$, or the end of an arc in $\mathcal{M}_{r-1}$.*

An example of an $m-$diagram corresponding to an SYT is given in Figure 3. We can define a rotation operation on m-diagrams algebraically as follows. Intuitively, this consists of connecting the endpoints of the boundary to form a circle, rotating the circle one step clockwise, and then disconnecting the circle back to a boundary line again.

1	2	6	7	14	19
3	8	9	15	18	21
4	10	11	16	20	23
5	12	13	17	22	24

T

$\longrightarrow$



$\varphi(T)$

Figure 3: Tableau T and its corresponding $m-$diagram $\varphi(T)$

Definition 2.9 (Rotation)**.** Denote by $\rho : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ the cyclic permutation given by $\rho(i) = i-1 \mod n$. Then ρ induces a map from the set of matchings on $\{1,2,\ldots,n\}$ to itself by sending each arc $(a,b) \mapsto (\rho(a),\rho(b))$. We call this map *rotation* and denote the image of a matching $\mathcal{M}$ under rotation by $\rho(\mathcal{M})$.

The following two observations about rotation are almost immediate from the definitions.

Lemma 2.10. *If $\mathcal{M}$ is a perfect matching then so is the rotation $\rho(\mathcal{M})$. If $\mathcal{M}$ is a noncrossing matching then so is the rotation $\rho(\mathcal{M})$.*

We would like for rotation to respect promotion in the sense that rotating the m-diagram for SYT T produces the m-diagram for the promotion $P(T)$, but usually it does not. For example, it is not true that the $m-$diagram $\varphi(P(T))$ is equal to $\rho(\varphi(T))$ for the tableau T in our running example. However, the next section describes a different way to connect rotations to promotions. The arguments in Section 3 utilize the idea of *sub-diagrams* and *components* of an $m-$diagram, which we introduce here.

Definition 2.11 (Sub-diagrams and Components)**.** Consider an $m-$diagram $M = \cup_{i=1}^{r}(a_i,b_i)$ where each (a_i,b_i) is an arc. The proper subcollection $C = \{(a_{i_k},b_{i_k})\}_{k=1}^{s}$ forms a *sub-diagram* of M if whenever a is an endpoint of some arc in C, all arcs with a as an endpoint are in C and no (a_j,b_j) for $j \notin \{i_1,\ldots,i_s\}$ crosses any arc in C. The sub-diagram is *uniform* if its endpoints are all adjacent and a *component* if it has no proper subcollection that forms a sub-diagram. Two sub-diagrams C, C' are equivalent if $\rho^N(C) = C'$ for some N.

An example of components and sub-diagrams of an $m-$diagram is given in Figure 4.

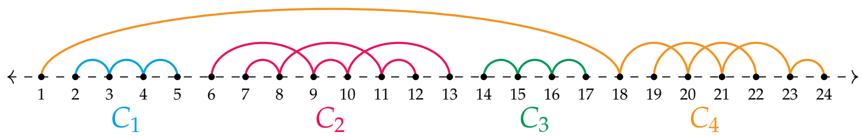


Figure 4: The components of the $m-$diagram $\varphi(T)$ are C_1, C_2, C_3, and C_4. The components C_1, C_2, C_3 are uniform, whereas C_4 is nonuniform. Note that the components C_1 and C_3 are equivalent since $\rho^{12}(C_3) = C_1$. Any union of components forms a sub-diagram of $\varphi(T)$. For example, $C_1 \cup C_2$ forms a uniform sub-diagram, whereas $C_3 \cup C_4$ forms a non-uniform sub-diagram.

3 Determining Orbit Lengths

Our main result is an identification between the orbit lengths of rectangular SYT and the rotatational symmetry of their corresponding $m-$diagrams. This identification allows us to explicitly compute the orbit length of a non-minimal tableau by only considering a subset of its, typically much smaller, subtableaux. To this end, we provide a simple algorithm for the computation of orbit lengths.

Theorem 3.1. *Fix a rectangular SYT T and its corresponding $m-$diagram $M = \varphi(T)$. Let N be the smallest positive integer such that $\rho^N(M) = M$. Then, the equality $|\mathcal{O}(T)| = N$ holds.*

The proof of Theorem 3.1 involves a combination of three lemmas that relate the internal structure of T and $P(T)$ to the geometric structures of M and $\rho(M)$. We begin by identifying all uniformly proper subtableaux of T with uniform sub-diagrams of M.

Lemma 3.2. *For a fixed rectangular SYT T and its corresponding $m-$diagram $M = \varphi(T)$, we have the following correspondences:*

1. *The set of uniformly proper rectangular subtableaux of T is in bijection with the uniform sub-diagrams of M.*

2. *The set of minimal uniformly proper rectangular subtableaux of T is in bijection with the uniform components of M.*

Then, we show that the action of promotion on T preserves the set of uniformly proper subtableaux that do not contain the minimal element of the tableau.

Lemma 3.3. *If $T = T_1 S T_2$, then $P(T) = T_1' S' T_2'$ where $T_1' T_2' \equiv P(T_1 T_2)$ and S' is a uniformly proper rectangular subtableau of $P(T)$ that is uniformly equivalent to S.*

The key insight in the proof of Lemma 3.3 is that the sliding path of the empty box is entirely horizontal as it passes through a uniformly proper subtableau. This

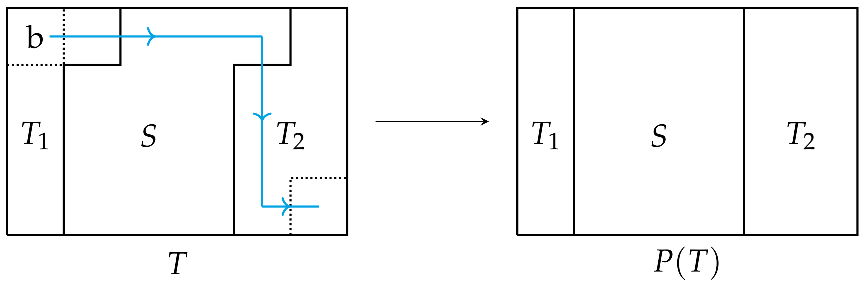


Figure 5: The sliding path of b through T and structure of $P(T)$

ensures that the internal structure of each uniformly proper subtableau is preserved during promotion. This is illustrated in Figure 5.

The combination of Lemmas 3.2 and 3.3 shows that the sub-diagrams of M are simply rotated to form the sub-diagrams of the $m-$diagram $\varphi(P(T))$.

Lemma 3.4. *Let $M = \varphi(T)$ and choose a sub-diagram C. Suppose that $E_C = \{(a_{i_k}, b_{i_k})\}_{k=1}^{s}$ is the set of endpoints of C and T_C is the tableau corresponding to C.*

1. *The arcs with endpoints $\rho(E_C)$ in $\varphi(P(T))$ form a sub-diagram of $\varphi(P(T))$.*
2. *If E_C does not contain 1, the arcs with endpoints $\rho(E_C)$ in $\varphi(P(T))$ are $\rho(C)$.*
3. *If E_C does contain 1, the arcs with endpoints $\rho(E_C)$ in $\varphi(P(T))$ form the $m-$diagram $\varphi(P(T_C))$ when endpoints are appropriately relabelled.*

The proof of Theorem 3.1 is then a straightforward combination of Lemmas 3.2 to 3.4. Now, since the orbit length $|\mathcal{O}(T)|$ is identified with the rotational symmetry of the $m-$diagram $\varphi(T)$, we can calculate the order of promotion of T by calculating the order of rotational symmetry of its corresponding $m-$diagram. We outline a simple process for this when 1 and n are not in the same component of $\varphi(T)$ in Theorem 3.5.

Theorem 3.5. *Let M be an $m-$diagram on n vertices such that the vertices 1 and n are not in the same component. The order of rotational symmetry of M can be determined as follows:*

1. *Let k be the number of non-equivalent components of M. Color each vertex of the $m-$diagram a color from $[k]$ such that two vertices are the same color if and only if they belong to equivalent components.*
2. *Define N to be the smallest positive integer such that for any vertex i, $i - N \mod n$ has the same color as i and for any component C, the rotation $\rho^N(C)$ is an equivalent component.*
3. *For each component C of M with a vertex in $[N]$, let N_C be the number of vertices of C that are less than or equal to N and T_C the SYT corresponding to C.*

4. *Let ℓ be the smallest positive integer such that $|\mathcal{O}(T_C)| | \ell N_C$ for each C.*

Then, M has rotational symmetry of order ℓN.

If the vertices 1 and n are in the same component of $\varphi(T)$, we can still apply the algorithm of Theorem 3.5 by first pre-processing $\varphi(T)$ in the following way:

1. Let C_1 be the component of $\varphi(T)$ containing both 1 and n. Suppose the first k vertices of $\varphi(T)$ are contained in C_1 and let T_1 be the Young tableau to C_1.

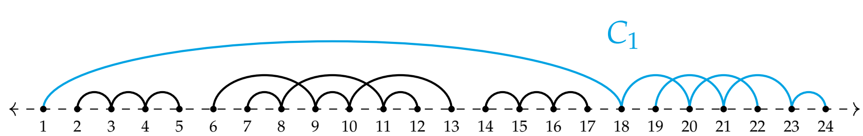
2. Determine the tableau $P^k(T_1)$ and its corresponding $m-$diagram $\varphi(P^k(T_1))$. Note that while T_1 is not an SYT, the promotion operation can still be performed since the entries of T_1 strictly increase both row and column-wise.

3. Define M to be the $m-$diagram $\rho^k(\varphi(T))$ with the rightmost component replaced by $\varphi(P^k(T_1))$.

Note that by Lemma 3.3, the $m-$diagram M defined in the above process exactly corresponds to $\varphi(P^k(T))$ and so the rotational symmetry of M corresponds to the orbit length $|\mathcal{O}(T)|$. Moreover, our approach of identifying uniform components of $\varphi(T)$ with the uniformly proper subtableaux of T allows us to calculate $\varphi(P^k(T))$, and thus $P^k(T)$, without requiring the $k-$fold evaluation of promotion on T itself.

Following the algorithm in Theorem 3.5, we see that the orbit length $|\mathcal{O}(T)|$ is determined by examining orbit lengths of a set of minimal standard Young tableaux. While there is no known method for identifying the orbit lengths of minimal tableaux, typically these tableaux will be much smaller than T and thus directly computing their orbit lengths will be much less computationally expensive than computing the entire promotion orbit of T. Furthermore, while there may be many components of $\varphi(T)$ that have at least one vertex in $[N]$, only those that contain vertices outside of $[N]$ need be considered in the calculation of k : if all vertices of the component C are contained in $[N]$, then $|\mathcal{O}(T)| | N_C$ by [5].

Example 3.6. We can now calculate the promotion orbit length of our running example T. Since 1 and 24 are in the same component in the $m-$diagram $\varphi(T)$, we first pre-process the $m-$diagram following the above steps. We begin by identifying C_1, the minimal component of $\varphi(T)$ that contains both 1 and 24 as vertices.



$\varphi(T)$

We then form the tableau T_1 corresponding to T_1. Since the first vertex not contained in C_1 is vertex 2, we need only promote T_1 once.

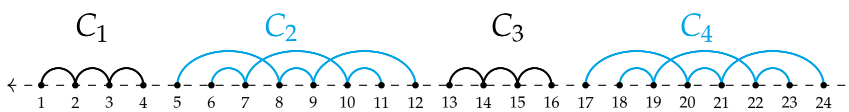
1	19
18	21
20	23
22	24

$\longrightarrow$

17	18
19	20
21	22
23	24

T_1 $\quad$ $P(T_1)$

After computing the promotion $P(T_1)$, we form the new $m-$diagram M by computing the rotation $\rho(\varphi(T))$ and replacing the rightmost component with $\varphi(P(T_1))$. We then use the steps described in Theorem 3.5 to determine the rotational symmetry of M.



M

Using the same notation as in Theorem 3.5, we see that $N = 12$ since the components C_1 and C_3 are equivalent and the components C_2 and C_4 are equivalent. Since the components C_1 and C_2 are entirely contained in the first 12 vertices of M, we conclude that M has rotational symmetry of order 12. It follows that $|\mathcal{O}(T)| = 12$.

4 Extending Results to Semistandard Young Tableaux

Thus far, we have only considered rectangular SYT. However, our results generalize to rectangular (column) semistandard Young tableaux in a natural way.

Definition 4.1. A *(column) semistandard Young tableau (SSYT)* is a filling of the Young diagram for the partition λ with the numbers $1, 2, \ldots, n$ so that each number appears at least once, rows strictly increase left-to-right, and columns weakly increase top-to-bottom.

The *content* of an SSYT T is the collection of integers filling its boxes with repetition. If each number i appears e_i times in T then we denote the content $\{1^{e_1}, 2^{e_2}, \ldots, n^{e_n}\}$. Note that the content may be a multiset.

Remark 4.2. Usually *semistandard* refers to tableaux in which columns strictly increase and rows are weakly increasing. For this reason, we usually stress *column* in our terminology. Note that each column semistandard tableau is the transpose of a (row) semistandard tableau. Since the promotion orbit length is preserved under the transposition of tableaux, it is sufficient to characterize the orbits of column SSYT.

The definition of promotion given in Definition 2.2 is easily generalizable to a column SSYT T by repeating steps (1) to (3) until there are no copies of 1 remaining in T and then inserting the maximum entry of T into every empty box. Moreover, $m-$diagrams can be constructed from SSYT using the same process as in Lemma 2.7. Our Theorem 3.1 can also be used to determine the promotion orbit lengths of an SSYT T.

Corollary 4.3. *Let T be an SSYT and $M = \varphi(T)$ be its corresponding $m-$diagram. If the vertices 1 and n are not in the same component, the orbit length $|\mathcal{O}(T)|$ can be determined using the algorithm in Theorem 3.5.*

There are several key observations that allow us to extend our result to SSYT. Firstly, there is a natural map ψ from the set of SSYT to SYT of the same shape. In essence, the map ψ expands the content of T so that any entry appears in the tableau only once while ensuring that the relative ordering of entries is preserved. If the number 1 appears k times in the SSYT T, then the $k-$fold promotion of the SYT $\psi(T)$ corresponds to 1 promotion of T and so the equality $\psi(P(T)) = P^k(\psi(T))$ holds. Moreover, there is a well defined "contraction" map κ on the $m-$diagram $\varphi(\psi(T))$ so that the equality $\kappa(\varphi(\psi(T))) = \varphi(T)$ holds. The contraction map also respects rotation in the following sense: if the number 1 appears k times in the SSYT T, then k rotations of the $m-$diagram $\varphi(\psi(T))$ followed by contraction corresponds to 1 rotation of the $m-$diagram $\varphi(T)$, i.e. the equality $\kappa(\rho^k(\varphi(\psi(T)))) = \rho(\varphi(T))$ holds.

Combining these observations with the arguments arising from the study of SYT completes the proof of Corollary 4.3. As with SYT, if the vertices 1 and n are in the same component of $\varphi(T)$, the $m-$diagram can be pre-processed as outlined above and then Corollary 4.3 can be applied accordingly.

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