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A new shifted Littlewood–Richardson rule and related developments

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Abstract. As Littlewood–Richardson rules compute linear representation theory of symmetric groups and cohomology of ordinary Grassmannians, shifted Littlewood–Richardson rules compute analogous *projective* representation theory of symmetric groups and cohomology of *orthogonal* Grassmannians. The first shifted Littlewood–Richardson rule is due to Stembridge (1989). We give a new shifted Littlewood–Richardson rule that is provably more efficient in some cases and is more convenient for hand calculations. Our rule builds on ideas of Lascoux–Schützenberger (1981), Haiman (1989), and Serrano (2010).

In addition, we obtain the first algebraic proof of Serrano's shifted Littlewood–Richardson rule (2010) and a new proof of the Hiller–Boe shifted Pieri rule (1986). We also give a new characterization of Serrano's shifted plactic monoid as the largest monoid satisfying a short list of natural axioms and propose a solution to an open problem of Cho (2013) asking for a satisfactory definition of plactic skew Schur P-functions.

Keywords: shifted tableau, rectification, mixed insertion, shifted plactic monoid

1 Introduction

The Littlewood–Richardson (LR) coefficients $c^{\nu}_{\lambda,\mu}$ arise in the linear representation theory of the symmetric group and the cohomology of complex Grassmannians. The classical Littlewood–Richardson rules compute the number $c^{\nu}_{\lambda,\mu}$ in a manifestly integral and non-negative fashion. Key ingredients of these rules are semistandard Young tableaux, Schur functions, insertion algorithms, jeu de taquin, and plactic structure.

The *shifted Littlewood–Richardson coefficients* $b^{\nu}_{\lambda,\mu}$ analogously arise in the projective representation theory of the symmetric group and the cohomology of orthogonal Grassmannians. There are analogous *shifted Littlewood–Richardson rules* for computing $b^{\nu}_{\lambda,\mu}$, the first being due to Stembridge [7]; however, all are significantly more complicated than

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their classical counterparts. The key combinatorial ingredients of these rules are shifted Young tableaux, Schur P- and Q-functions, shifted insertions, and shifted analogs of jeu de taquin and the plactic monoid.

Our main result is a new shifted LR rule for the coefficients $b^{\nu}_{\lambda,\mu}$. In some cases, our rule is provably more efficient than the original rule of [7]. Moreover, we find it significantly more convenient for hand calculations. At the center of our approach is the identification of a shifted plactic class that we call *barely Yamanouchi words*, which has many desirable algebraic and combinatorial properties. The straight-shaped tableaux of these classes (*barely Yamanouchi tableaux*; see Figure 1) are a useful substitute for the traditional "Yamanouchi" tableaux, although they appear very different. This newly identified shifted plactic class is characterized by its decreasing subsequences in parallel with a characterization for classical Yamanouchi words.

1	1	1	$3'$	$4'$	$6'$	$7'$
	2	2	$4'$	$5'$		
		3				

Figure 1: An example of a barely Yamanouchi tableau of shifted shape $(7,4,1)$.

Constructed tableaux are the shifted tableaux that can become barely Yamanouchi after a sequence of Haiman's *mixed insertions* [2]. Constructed tableaux can be built directly (without consideration of mixed insertion) by laying down sequences of (generalized) rimhooks satisfying a couple of conditions that are easy to use but slightly technical to state. We defer the details of these definitions until Section 2. For an example of a constructed tableau, see Figure 2.

Theorem 1.1. *Let λ, μ, and ν be strict partitions with $|\lambda| + |\mu| = |\nu|$ and $\mu < \nu$. Then, $b^{\nu}_{\lambda,\mu}$ equals the number of shifted tableaux of shape λ constructed from $\mu < \nu$.*

1	$2'$	$3'$	3	$4'$	$5'$	$8'$	$9'$
	4	$5'$	5	$6'$	$7'$	$10'$	$11'$
		6	$7'$	$8'$	$9'$		

Figure 2: A tableau of shape $\lambda = (8,7,4)$ constructed from $\mu = (4,2) < \nu = (11,9,5)$. Here, the coloring is redundant but identifies the different generalized rimhooks.

For an illustration of the use of Theorem 1.1 to compute shifted LR coefficients $b^{\nu}_{\lambda,\mu}$, see Example 5.9. Our proof of Theorem 1.1 makes heavy use of Serrano's *shifted plactic monoid* [6]. This monoid is an analog of the plactic monoid of Lascoux–Schützenberger

[4]; however, it is more complicated than its classical analog and less directly connected to jeu de taquin. Serrano [6] used his shifted plactic monoid to give another shifted LR rule for $b^{\nu}_{\lambda,\mu}$; this rule is rather difficult to apply in practice due to being formulated in terms of equivalence classes. A consequence of our analysis is the first algebraic proof of Serrano's result (Serrano's original proof being fundamentally combinatorial). We also obtain a new proof of the first Pieri rule for Schur P-functions, originally due to Hiller–Boe [3] (and equivalent to a special case of Stembridge's formula [7]).

We also further analyze the shifted plactic monoid. Unlike the classical plactic monoid whose relations are easily derived from the jeu de taquin algorithm on tableaux, the relations defining the shifted plactic monoid are less obvious to discover and rely on Haiman's mixed insertion algorithm instead of shifted jeu de taquin.

Inspired by a universal characterization of the plactic monoid by Lascoux and Schützenberger [4], we give a new characterization of the shifted plactic monoid as the greatest quotient of the free monoid satisfying a short list of natural axioms. Equivalently, this identifies the shifted plactic monoid as the initial object of a category of monoids that we define. The only combinatorial structure going into these axioms is a definition of free Schur P-functions for strict partitions of the integers 1 and 3. In particular, this characterization does not employ mixed insertion, jeu de taquin, or Young tableaux.

Finally, we propose a solution to [1, Open Problem 7.12(1)], asking for a new definition of *plactic skew Schur P-functions* $\mathcal{P}_{\nu/\lambda}$ satisfying certain desirable properties. This open problem arises from an earlier definition of Serrano [6], which he conjectured to satisfy these properties, but which was shown by Cho [1] to fail them in some cases. Our definition involves a mixture of two different insertion algorithms; more specifically, we propose to take a formal sum over skew tableaux with a given Sagan–Worley rectification [5, 8] of the rectification's mixed reading word.

2 Background and definitions

Throughout this paper, we use the English tableau orientation convention. A *strict partition* is a partition with distinct parts. We draw the Young diagram of a strict partition as a shifted shape, indenting the ith row $i-1$ boxes. For example, the tableau of Figure 1 has shifted shape equal to the strict partition $(7,4,1)$. We consider the alphabet $\mathcal{N} = \{1 < 2 < 3 < \dots\}$ and the doubled alphabet $\mathcal{D} = \{1' < 1 < 2' < 2 < \dots\}$. A *shifted semistandard tableau* of *shape* λ is a filling of the the boxes of the shifted Young diagram of λ with elements of $\mathcal{D}$ such that the box labels weakly increase from left to right along rows and from top to bottom down columns, each k' appears at most once in any row, each k appears at most once in any column, and the leftmost box of each row contains an unprimed number. Figures 1 and 2 both show examples of such tableaux.

We write $\mathrm{ShSSYT}(\lambda)$ for the set of all shifted semistandard tableaux of strict partition shape λ. The *content* of a shifted semistandard tableau T is the integer vector $c(T) = (c_1, c_2, \dots)$ where c_i denotes the number of boxes that are filled with either the value i' or i. A tableau $T \in \mathrm{ShSSYT}(\lambda)$ is *standard* if it contains no primed entries and its content is $(c_1, c_2, \dots)$, where $c_i = 1$ for $i \leq |\lambda|$ and $c_i = 0$ for $i > |\lambda|$. We write $\mathrm{ShSYT}(\lambda)$ for the set of all shifted standard tableaux of strict partition shape λ. The *Schur P-function* P_λ is the symmetric function

$$P_\lambda := \sum_{T \in \mathrm{ShSSYT}(\lambda)} \mathbf{x}^{c(T)} = \sum_{T \in \mathrm{ShSSYT}(\lambda)} \prod_i x_i^{c_i}.$$

We will also need the *Schur Q-function* $Q_\lambda := 2^{\ell(\lambda)} P_\lambda$. Note that we can also think of Q_λ as the partition function for a generalization of shifted semistandard tableaux where we allow primed entries in any box; we call these *Q-tableaux*.

Let $w = w_1 \dots w_n \in \mathcal{N}^\star$. Then, the *mixed insertion* [2] of w is the shifted tableau constructed as follows:

- Place w_1 in the first row of the diagram and set $i = 2$.
- Compare w_i with the entries of the first row. If it is largest, insert it at the end of the row. If not, find the least x in that row such that $x > w_i$. Insert w_i in the position of x and save x along with its previous position. If x is unprimed and not on the main diagonal, insert x in the next row in the same way. If x is unprimed and on the main diagonal, insert x' in the next column. If x is primed, insert it into the next column. For column insertions of z, we mean insert it at the bottom of the column if it is greater than the entries of that column; and otherwise insert it at the position of the least y in the column with $y > z$. In the latter case, save the value of y as well as its previous position and repeat this step with $z = y$.
- After the last insertion has been performed, set $i = i + 1$ and repeat step 2 if $i \leq n$; otherwise end and return the shifted tableau.

We also need the *Sagan–Worley insertion* [8, 5] of $w = w_1 \dots w_n \in \mathcal{D}^\star$.

- Place w_1 in the first row of the diagram and set $i = 2$.
- Let y be the smallest letter in the first row such that $\boxed{y}\boxed{w_i}$ is not a legal Q-tableau. If y is not on the diagonal, replace y with w_i and insert y into the next row. In the other case, if $w_i = y$ or y', then unprime w_i and insert it into the next column; otherwise, replace y with w_i and insert y into the next column.
- We iterate this process until no letters are bumped with the caveat that once we start to displace by columns, all subsequent bumps must also be column-bumps.
- After the last insertion has been performed, set $i = i + 1$ and repeat step 2 if $i \leq n$; otherwise end and return the shifted tableau.

3 A universal property for the shifted plactic monoid

Let $\mathcal{A}$ be a totally ordered alphabet. A *hook subword* of a word w is a subword $d \cdot i$ such that d is strictly decreasing and i is weakly increasing. If w is a hook subword of itself, we say w is a *hook word*. Let λ be a strict partition. Denote by $\mathsf{hook}(\lambda)$, the set of words $w = w_1 \ldots w_k$, where each w_i is a hook word of length $\lambda_{\ell(\lambda)-i+1}$ and such that, for all $i > 1$, w_i is a longest hook subword of $w_{i-1}w_i$. Then the *shifted free Schur function* of shape λ is

$$\widehat{\mathcal{P}}_\lambda := \sum_{w\,:\,w \in \mathsf{hook}(\lambda)} w \in \mathbb{Z}\langle\langle x_1, x_2, \ldots \rangle\rangle.$$

(Serrano [6] gives an equivalent definition of $\mathsf{hook}(\lambda)$ as the set of mixed reading words for certain tableaux.)

The *shifted plactic monoid* $\mathbf{S}$ [6] is the quotient of the free monoid on $\mathcal{N}$ by 8 homogeneous degree-4 relations (the *shifted Knuth relations*). Serrano shows that two words are equivalent in $\mathbf{S}$ if and only if they have the same mixed insertion tableau. We assume the reader is familiar with the ordinary plactic monoid $\mathbf{P}$ [4]; let κ be the quotient map from the free monoid to $\mathbf{P}$. We write ρ_I for the restriction map on words given by deleting all letters outside the interval I.

Consider the category $\underline{\mathsf{SPlac}}(\mathcal{A})$ of monoids $\mathbf{M}$ equipped with homomorphisms

$$\phi : \mathbf{F}(\mathcal{A}) \twoheadrightarrow \mathbf{M} \quad \text{and} \quad \psi : \mathbf{M} \to \mathbf{P}(\mathcal{A}) \quad \text{such that}$$

(SPlac.1) $\psi \circ \phi = \kappa$;

(SPlac.2) the images $\phi(\widehat{\mathcal{P}}_{\square}), \phi(\widehat{\mathcal{P}}_{\square\square})\in \mathbb{Z}\mathbf{M}$ of the free Schur P-functions $\{\widehat{\mathcal{P}}_{\square}, \widehat{\mathcal{P}}_{\square\square}\} \subset \mathbb{Z}\mathbf{F}(\mathcal{A})$ commute;

(SPlac.3) for any ordered endomorphism $\omega : \mathbf{F}(\mathcal{A}) \to \mathbf{F}(\mathcal{A})$ and any $w_1, w_2 \in \mathbf{F}(\mathcal{A})$, if $\phi(w_1) = \phi(w_2)$, then $(\phi \circ \omega)(w_1) = (\phi \circ \omega)(w_2)$; and

(SPlac.4) for any interval $I \subseteq \mathcal{A}$ and any $w_1, w_2 \in \mathbf{F}(\mathcal{A})$, if $\phi(w_1) = \phi(w_2)$, then

$$(\kappa \circ \rho_I)(w_1) = (\kappa \circ \rho_I)(w_2).$$

The defining properties of $\underline{\mathsf{SPlac}}(\mathcal{A})$ are mostly given by directly adapting those for a category of plactic monoids that we build based on ideas of Lascoux–Schützenberger [4]. The main surprise is the appearance of κ in (SPlac.4), where one might naturally expect ϕ by analogy. A morphism $\theta : (\mathbf{M}', \phi', \psi') \to (\mathbf{M}, \phi, \psi)$ in $\underline{\mathsf{SPlac}}(\mathcal{A})$ is a monoid homomorphism such that the diagram

$$\begin{tikzcd}
& \mathbf{M}' \arrow[dd, "\theta"] \arrow[dr, "\psi'"] & \\
\mathbf{F}(\mathcal{A}) \arrow[ur, "\phi'"] \arrow[dr, "\phi"'] & & \mathbf{P}(\mathcal{A}) \\
& \mathbf{M} \arrow[ur, "\psi"'] &
\end{tikzcd} \tag{3.1}$$

commutes.

Let $\sigma : \mathbf{F}(\mathcal{A}) \to \mathbf{S}(\mathcal{A})$ be the quotient map and let $\pi : \mathbf{S}(\mathcal{A}) \to \mathbf{P}(\mathcal{A})$ be the projection map given by quotienting by the ordinary Knuth relations. It is straightforward to see that $\kappa = \pi \circ \sigma$.

We give the following intrinsic characterization of $\mathbf{S}(\mathcal{A})$ without reference to the shifted Knuth relations. Our approach also avoids consideration of Haiman's *mixed insertion* and is analogous to the Lascoux–Schützenberger [4] characterization of $\mathbf{P}(\mathcal{A})$.

Theorem 3.1. *The shifted plactic monoid $(\mathbf{S}(\mathcal{A}), \sigma, \pi)$ is the initial object of the category $\underline{\mathsf{SPlac}}(\mathcal{A})$.*

Note that the final object of $\underline{\mathsf{SPlac}}(\mathcal{A})$ is just $(\mathbf{P}(\mathcal{A}), \kappa, \mathrm{id})$. We also show that the analog of Theorem 3.1 holds if $\widehat{\mathcal{P}}_{\ydiagram{2,1+1}}$ is replaced by $\widehat{\mathcal{P}}_{\ydiagram{3}}$ in (SPlac.2). If instead $\widehat{\mathcal{P}}_{\ydiagram{2,1+1}}$ is replaced by $\widehat{\mathcal{P}}_{\ydiagram{2}}$, we obtain the free monoid as initial object, so that category is less interesting.

4 A proposed solution to Cho's open problem

First we recall the *plactic Schur P-functions* $\mathcal{P}_\lambda \in \mathbb{Z}\mathbf{S}(\mathcal{N})$ [6]. They are the lifting of the ordinary Schur P-functions P_λ given by $\sigma(\widehat{\mathcal{P}}_\lambda)$. Consider the set of all Q-tableaux of shape ν/λ. For the sake of convenience we denote them instead as $\mathrm{ShSSYT}(\nu/\lambda)'$.

Let $\mu < \lambda$ be strict partitions. There has long been desired an appropriate definition of "skew plactic Schur P-functions" $\mathcal{P}_{\nu/\mu}$ as lifts of $P_{\nu/\mu}$ to $\mathbb{Z}\mathbf{S}(\mathcal{N})$ with good properties. Serrano [6] proposed a definition, but Cho [1] disproved his conjecture that it had the desired properties; Cho [1, Open Problem 7.12(1)] asked for a new definition of $\mathcal{P}_{\nu/\mu}$ that makes it a member of $\mathbb{Q}[\mathcal{P}_\lambda]_\lambda$ (in agreement with Serrano's original conjecture); and such that the expansion of $\mathcal{P}_{\nu/\mu}$ in the basis $(\mathcal{P}_\lambda)_\lambda$ can be described in a nice way.

We propose such a definition of $\mathcal{P}_{\nu/\mu}$ by lifting instead $Q_{\nu/\mu}$ and adjusting by the appropriate constant.

Definition 4.1. *The* skew plactic Schur P-function *is*

$$\mathcal{P}_{\nu/\mu} := \frac{1}{2^{\mathrm{diag}(\nu/\mu)}} \sum_{T \in \mathrm{ShSSYT}(\nu/\mu)'} [F \circ \mathrm{rect}(T)]_{\mathbf{S}} \in \mathbf{S}(\mathcal{A})$$

where $\mathrm{ShSSYT}(\nu/\mu)'$ *stands for the set of Q-tableaux of shape* λ*,* F *is the map that forgets primes in diagonal positions,* $\mathrm{rect}(\bullet)$ *signifies Sagan–Worley rectification [8, 5], and* $\mathrm{diag}(\nu/\mu)$ *denotes the number of diagonal positions in the skew shape* ν/μ*.*

Note that Sagan–Worley's algorithm is employed to rectify T, but after it has been rectified its shifted plactic class (codifying equivalence under Haiman's insertion) is taken. The following shows that the skew plactic Schur P-functions, as defined in Definition 4.1, live in the correct ring and have the desired expansion into plactic Schur P-functions.

Theorem 4.2. *Let* $\mu < \nu$ *be strict partitions. Then* $\mathcal{P}_{\nu/\mu} \in \mathbb{Q}[\mathcal{P}_\lambda]_\lambda$ *and*

$$\mathcal{P}_{\nu/\mu} = \sum_{\lambda} \frac{2^{\ell(\lambda)}}{2^{\mathrm{diag}(\nu/\mu)}} b^{\nu}_{\lambda,\mu} \mathcal{P}_\lambda.$$

In particular, the expansion coefficients of $\mathcal{P}_{\nu/\mu}$ *in the plactic Schur P-basis are equal to the expansion coefficients of* $P_{\nu/\mu}$ *in the ordinary Schur P-basis.*

Similar ideas give us the first algebraic proof of Serrano's [6] shifted LR rule. That is, following a strategy of Lascoux–Schützenberger [4] for the unshifted case, we obtain a shifted plactic expression for the multiplication $\widehat{\mathcal{P}}_\lambda \cdot \widehat{\mathcal{P}}_\mu$ based on the duality between mixed insertion and Sagan–Worley insertion. Serrano's rule then follows:

Corollary 4.3 (Serrano [6])**.** *Fix a tableau* $T \in \mathrm{ShSSYT}(\nu)$*. Then the shifted Littlewood–Richardson coefficient* $b^{\nu}_{\lambda,\mu}$ *can be determined as follows:*

$$b^{\nu}_{\lambda,\mu} = |\{([U],[V]) \,:\, [U]\cdot[V] = [T] \in \mathbf{S},\ U \in \mathrm{ShSSYT}(\lambda),\ \text{and}\ V \in \mathrm{ShSSYT}(\mu)\}|.$$

5 A new shifted Littlewood–Richardson rule

We will now consider a new analog of Yamanouchi tableaux for the shifted context.

Definition 5.1. *Let* ν *be a strict partition of* n *with* $\ell(\nu) = \ell$*. The* canonical word for ν *is the sequence* $\hat{y}_\nu \in \mathcal{N}^*$ *given by*

$$\nu_\ell \quad \nu_\ell - 1 \quad \ldots \quad 1 \quad \nu_{\ell-1} \quad \nu_{\ell-1} - 1 \quad \ldots \quad 1 \quad \ldots \quad \nu_1 \quad \nu_1 - 1 \quad \ldots \quad 1.$$

A word is barely Yamanouchi *if it is in the shifted plactic class of* $\hat{y}_\nu$ *for some strict partition* ν*.*

Example 5.2. For the strict partition $\nu = (6,4,3)$, we have that its canonical word is

$$\hat{y}_\nu = 3 \quad 2 \quad 1 \quad 4 \quad 3 \quad 2 \quad 1 \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1.$$

We also note that all barely Yamanouchi words are Yamanouchi, but the converse does not hold. Specifically, we show barely Yamanouchi words satisfy the additional condition that, in each final segment, the number of is is at most 1 more than the number of $(i+1)$s; it is in this sense that "barely Yamanouchi" words barely satisfy the Yamanouchi condition. Moreover, we show barely Yamanouchi words satisfy an *interlacing* condition, that between any two instances of i, there is an instance of $i-1$ (unless $i=1$) and an instance of $i+1$. This interlacing property is key to many of our arguments.

Definition 5.3. *Given a strict partition ν of n, we define the* barely Yamanouchi tableau *of shape ν, denoted by $\hat{Y}_\nu$, to be the shifted tableau resulting from the mixed insertion of any barely Yamanouchi word for ν, i.e., $\hat{Y}_\nu = P_{\mathrm{mix}}(\hat{y}_\nu)$.*

We establish a direct tableau-theoretic characterization of barely Yamanouchi tableaux without reference to mixed insertion. For an illustration of the barely Yamanouchi tableau $\hat{Y}_{(7,4,1)}$, see Figure 1. Note in particular that the largest shifted staircase inside ν is filled as in a traditional Yamanouchi tableau (with no primed entries), whereas all other entries are primed and satisfy additional conditions; this is always the case.

We now turn to introducing the *constructed tableaux* that are key to the statement of Theorem 1.1.

Definition 5.4. *A skew shape is a* horizontal strip *if it contains at most one box in each column, and it is a* vertical strip *if it has at most one box in each row. A set of boxes γ is a* generalized rimhook *it can be partitioned into a vertical strip ξ/π and a horizontal strip θ/η with $\xi \subseteq \eta$. We write $\gamma = \xi/\pi \otimes \theta/\eta$. If we can choose the partitions so that $\eta = \xi$, we say γ is a* rimhook.

Let T be a tableau of generalized rimhook shape $\gamma = \xi/\pi \otimes \theta/\eta$. We say that T is a Serrano–Pieri strip *if ξ/π is filled with unprimed letters that increase from top to bottom, θ/η is filled with primed letters that increase from left to right, and each label in ξ/π is less than every label in θ/η. (Note that this is backwards from the ordinary Pieri fillings!)*

Example 5.5. In the tableau

1	$2'$	$3'$	4	5	$7'$	$9'$
	5	$6'$	6	$8'$		
		7				

,

the boxes shaded in blue form a Serrano–Pieri strip. Note that the blue boxes do not form a skew shape, so that it is not a rimhook, but only a generalized rimhook.

Definition 5.6. *Let $\alpha = (\alpha_1, \ldots, \alpha_\ell), \beta = (\beta_1, \ldots, \beta_\ell)$ be sequences of nonnegative integers with $\alpha_i < \beta_i$ for all i. A tableau T of shape λ is* constructible *from $\alpha < \beta$ if*

- *for every j, the letters of $(\alpha_j, \beta_j]$ in T form a Serrano–Pieri strip; and*
- *the unprimed entries of $(\alpha_j, \beta_j]$ occur before the unprimed entries of $(\alpha_i, \beta_i]$ for all $i < j$ when in the same row; and*
- *the primed entries of $(\alpha_j, \beta_j]$ occur before the primed entries of $(\alpha_i, \beta_i]$ for all $i < j$ when in the same column.*

Observe that the horizontal or vertical strip in a Serrano–Pieri strip is allowed to be empty. That is, a horizontal strip is a Serrano–Pieri strip, and likewise a vertical strip.

Definition 5.7. *We say T is* constructed *from $\alpha < \beta$ if it is constructible from $\alpha < \beta$, but none of the Serrano–Pieri strips $\gamma^{(s)} = (\xi^{(s)} / \pi^{(s)}) \otimes (\theta^{(s)} / \eta^{(s)})$ can be extended to a Serrano–Pieri strip $\gamma'^{(s)} = (\xi'^{(s)} / \pi'^{(s)}) \otimes (\theta'^{(s)} / \eta'^{(s)})$ such that $\gamma'^{(s)} \supsetneq \gamma^{(s)}$ and the letters in $\gamma'^{(s)}$, including the primed ones but ignoring their primes, form an interval $(\alpha_j - k, \beta_j + e] \supsetneq (\alpha_j, \beta_j]$ for some $e, k \geq 0$, where $(\alpha_j, \beta_j]$ are the letters (some of them possibly primed) of $\gamma^{(s)}$.*

See Figure 2 for an example of a constructed tableau. Theorem 1.1 then follows from the following result, which is interesting in its own right.

Theorem 5.8. *Let λ, μ, and ν be strict partitions with $|\lambda| + |\mu| = |\nu|$ and $\mu < \nu$. Then, the tableaux T_λ of shape λ such that*

$$T_\lambda \cdot \hat{Y}_\mu = \hat{Y}_\nu$$

(where the $\cdot$ denotes concatenation of shifted plactic words) are exactly the tableaux of shape λ constructed from the partitions $\mu < \nu$.

We now give some examples of the application of the new rule of Theorem 1.1.

Example 5.9. Let $\nu = (11, 9, 5)$, $\mu = (4, 2)$, and $\lambda = (8, 7, 4)$. We need to place the elements of

$$\bigsqcup_i (\mu_i, \nu_i] = (0, 5] \sqcup (2, 9] \sqcup (4, 11]$$

into the shifted shape λ according to our rule.

We start by adding the letters of $(\mu_3, \nu_3] = (0, 5]$. We must insert an initial interval of these letters to form a vertical strip, and then the remaining letters (primed) to form a horizontal strip. There is a unique way of doing so, illustrated below.

1	2′	3′	4′	5′			

Now, we consider the letters of $(\mu_2, \nu_2] = (2, 9]$. This time there are two ways to choose an initial interval and assign its letters to a vertical strip. For the unprimed letters, either we have

$$\begin{ytableau} 1 & 2' & 3' & 3 & 4' & 5' & & \\ \none & 4 & & & & & & \\ \none & \none & & & & \end{ytableau} \quad \text{or} \quad \begin{ytableau} 1 & 2' & 3' & 4' & 5' & & & \\ \none & 3 & & & & & & \\ \none & \none & & & & \end{ytableau}.$$

On the left, since there were primed entries from the first sequence in that row before the placement of 3, it displaces them eastwards; the letters 4 and 5 from the previous sequence are primed, so neither 3 nor 4 from the new sequence can extend the first sequence. Similarly, on the right, 3 cannot extend the previous sequence.

We must add the remaining letters in $(2, 9]$ so that they are primed and constitute a horizontal strip inside D_λ. We must be careful so that each entry i' is inserted weakly west of $(i-1)'$ from the previous sequence. This process yields the tableaux

$$\begin{ytableau} 1 & 2' & 3' & 3 & 4' & 5' & 9' & \\ \none & 4 & 5' & 6' & 7' & 8' & & \\ \none & \none & & & & \end{ytableau}, \begin{ytableau} 1 & 2' & 3' & 3 & 4' & 5' & 8' & 9' \\ \none & 4 & 5' & 6' & 7' & & & \\ \none & \none & & & & \end{ytableau}, \begin{ytableau} 1 & 2' & 3' & 4' & 5' & 7' & 8' & 9' \\ \none & 3 & 4' & 5' & 6' & & & \\ \none & \none & & & & \end{ytableau}.$$

Finally, we place the interval $(\mu_1, \nu_1] = (4, 11]$. For the leftmost tableau above, we get

$$\begin{ytableau} 1 & 2' & 3' & 3 & 4' & 5' & 5 & 9' \\ \none & 4 & 5' & 6' & 7' & 8' & 10' & 11' \\ \none & \none & 6 & 7' & 8' & 9' \end{ytableau} \quad \text{and} \quad \begin{ytableau} 1 & 2' & 3' & 3 & 4' & 5' & 5 & 9' \\ \none & 4 & 5' & 6' & 6 & 7' & 8' & 11' \\ \none & \none & 7 & 8' & 9' & 10' \end{ytableau}.$$

From the middle and rightmost tableaux, respectively, we obtain

$$\begin{ytableau} 1 & 2' & 3' & 3 & 4' & 5' & 8' & 9' \\ \none & 4 & 5' & 5 & 6' & 7' & 10' & 11' \\ \none & \none & 6 & 7' & 8' & 9' \end{ytableau} \quad \text{and} \quad \begin{ytableau} 1 & 2' & 3' & 4' & 5' & 5 & 8' & 9' \\ \none & 3 & 4' & 5' & 6' & 7' & 10' & 11' \\ \none & \none & 6 & 7' & 8' & 9' \end{ytableau}.$$

The reader can check that none of the Serrano–Pieri strips here can be extended, so these are all constructed tableaux.

Since we have computed that there are exactly 4 tableaux of shape $(8, 7, 4)$ constructed from $(4, 2) < (11, 9, 5)$, we learn that $b^{(11,9,5)}_{(8,7,4),(4,2)} = 4$.

Example 5.10. Let $\nu = (7, 3, 2)$, $\mu = (3, 2)$, and $\lambda = (4, 3)$. We consider placing the elements of

$$\bigsqcup_i (\mu_i, \nu_i] = (0, 2] \sqcup (2, 3] \sqcup (3, 7]$$

into the shifted shape λ according to our rule. The unique way to place $(0, 2]$ is

$$\begin{ytableau} 1 & 2' & & \\ \none & & & \end{ytableau}$$

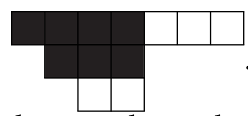
Adding $(2, 3]$ can be done in exactly two ways. Either we obtain

$$\begin{ytableau} 1 & 2' & 3 & \\ \none & & & \end{ytableau} \quad \text{or} \quad \begin{ytableau} 1 & 2' & & \\ \none & 3 & & \end{ytableau}.$$

For the left tableau, the unique way to add $(3, 7]$ gives

$$\begin{ytableau} 1 & 2' & 3 & 4 \\ \none & 5 & 6' & 7' \end{ytableau}.$$

For the right tableau, however, there is no way to validly add $(3, 7]$ as a Serrano–Pieri strip. Thus, we conclude that $b^{(7,3,2)}_{(4,3),(3,2)} = 1$.

The reader familiar with Stembridge's [7] shifted LR rule may note that the calculation of Example 5.10 would have been easier with his rule. In Stembridge's rule, one would have instead considered the shifted skew shape . Stembridge's rule is at its best in settings such as this, where the relevant skew shape consists of a collection of disjoint horizontal segments. In contrast, our rule shines when λ is an uncomplicated shape such as a row or a staircase. In these cases, we find the rule of Theorem 1.1 significantly easier to apply. For example, in the "Pieri case" when λ is a single row, Theorem 1.1 is extremely easy to apply. It is then a straightforward bijection to obtain a new proof of the Hiller–Boe [3] Pieri formula in this case.

We now demonstrate that Theorem 1.1 is more efficient than the earlier Stembridge rule [7] in certain cases. An analysis of a natural algorithmic implementation of Theorem 1.1 shows that the time complexity of Theorem 1.1 is $|\mathrm{SYT}(\lambda)| \cdot \mathcal{O}(|\lambda|^2)$. Now consider the special case where μ and ν are such that $\nu := \mu + \vec{1} = (\mu_1 + 1, \mu_2 + 1, \ldots, \mu_{\ell(\mu)} + 1)$ and λ has staircase shape.

The time complexity of Stembridge's rule is bounded below by $\mathcal{O}(|\mathrm{SYT}(\nu/\mu)|) \cdot |\lambda|$. By the choice of μ and ν, the cardinality of $\mathrm{SYT}(\nu/\mu)$ is found to be

$$\binom{|\lambda|}{\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}} = \frac{|\lambda|!}{\lambda_1! \cdot \cdots \cdot \lambda_{\ell(\lambda)}!}, \text{ modulo priming of the entries.}$$

On the other hand, the complexity of our rule from Theorem 1.1 is a function of $|\mathrm{SYT}(\lambda)|$, which by the shifted hook formula is

$$\frac{|\lambda|!}{\prod_{i \in D_\lambda} h(i)}$$

(again, modulo priming of the entries), where h_i is the classical hook of a cell inside the doubled and unshifted version of the shifted diagram D_λ for λ. One then computes that $|\mathrm{SYT}(\lambda)| \leq \frac{1}{2^{|\lambda|-\ell(\lambda)}}|\mathrm{SYT}(\nu/\mu)|$.

Taking now into account the primed entries, a tableau $T \in \mathrm{SYT}(\lambda)$ may contain primes anywhere off the diagonal. However, a tableau $S \in \mathrm{SYT}(\nu/\mu)$ may have any entries primed except possibly the southmost one. We obtain the sharper estimate that

$$|\mathrm{SYT}(\lambda)| \leq \frac{1}{2^{|\lambda|-\ell(\lambda)}}|\mathrm{SYT}(\nu/\mu)| \cdot \frac{1}{2^{\ell(\lambda)-1}} = \frac{1}{2^{|\lambda|-1}}|\mathrm{SYT}(\nu/\mu)|.$$

Hence, the time complexity of our algorithm is bounded above by

$$|\mathrm{SYT}(\lambda)| \cdot \mathcal{O}(|\lambda|^2) \leq \frac{1}{2^{|\lambda|-1}}|\mathrm{SYT}(\nu/\mu)| \cdot \mathcal{O}(|\lambda|^2),$$

so that for $|\lambda| \gg 0$, the running time of Theorem 1.1 is exponentially faster than the Stembridge formulation in this class of examples.

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