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The restriction problem and the Frobenius transform

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Abstract. We define an abelian group homomorphism $\mathscr{F}$, which we call the *Frobenius transform*, from the ring of symmetric functions to the ring of the symmetric power series. The matrix entries of $\mathscr{F}$ in the Schur basis are the *restriction coefficients* $r_\lambda^\mu = \dim \operatorname{Hom}_{\mathfrak{S}_n}(V_\mu, \mathbb{S}^\lambda \mathbb{C}^n)$, which are known to be nonnegative integers but have no known combinatorial interpretation.

We compute $\mathscr{F}\{f\}$ when f is an elementary, complete homogeneous, or power sum symmetric function. As a consequence, we prove that $r_\lambda^\mu = 0$ if $|\lambda \cap \hat{\mu}| < 2|\hat{\mu}| - |\lambda|$, where $\hat{\mu}$ is the partition formed by removing the first part of μ. We also prove that $r_\lambda^\mu = 0$ if the Young diagram of μ contains a square of side length greater than $2^{\lambda_1 - 1}$, and this inequality is tight.

Keywords: symmetric functions, representation theory of categories, plethysm, Kronecker product

1 Introduction

Let $n \geq 0$ and let λ be a partition with at most n parts. There is a corresponding irreducible $GL_n(\mathbb{C})$-module: the Schur module $\mathbb{S}^\lambda \mathbb{C}^n$. Because the symmetric group $\mathfrak{S}_n$ embeds in $GL_n(\mathbb{C})$ by permutation matrices, one may ask: how does the restriction of $\mathbb{S}^\lambda \mathbb{C}^n$ to $\mathfrak{S}_n$ decompose into irreducible $\mathfrak{S}_n$-modules?

In other words, let λ and μ be partitions and let $n = |\mu|$. What is the value of the *restriction coefficient*

$$r_\lambda^\mu = \dim \operatorname{Hom}_{\mathfrak{S}_n}(V_\mu, \mathbb{S}^\lambda \mathbb{C}^n),$$

where V_μ is the Specht module corresponding to the partition μ? This problem is known as the *restriction problem*.[1]

While there are many known formulas for the restriction coefficient r_λ^μ, no *combinatorial* formula is known. For example, in 1935, Littlewood proved a plethystic formula for restriction coefficients [8]:

$$r_\lambda^\mu = \langle s_\lambda, s_\mu[1 + h_1 + h_2 + \cdots] \rangle,$$

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[1]The restriction problem is a special case of the more general problem of computing the *inner plethysm coefficients*, which were defined by Littlewood in 1958 [9].

where s_λ denotes the Schur function, h_r denotes the complete homogeneous symmetric function, $\langle \cdot, \cdot \rangle$ denotes the Hall inner product, and the square brackets denote plethysm of symmetric functions. However, this formula does not obviously yield a combinatorial interpretation of r^μ_λ.

The restriction problem has attracted a great deal of interest. Here is a sampling of recent results about the restriction coefficients r^μ_λ. In 2021, Heaton, Sriwongsa, and Willenbring proved the following nonvanishing result for restriction coefficients: for all positive integers $m, n > 1$ and all $\mu \vdash n$, there exists a two-row partition $\lambda = (\lambda_1, \lambda_2) \vdash mn$ such that $\lambda_1 - \lambda_2 \leq m$ and $r^\mu_\lambda > 0$ [5]. In 2021, Orellana and Zabrocki introduced the *irreducible character basis* $\{\tilde{s}_\lambda\}_\lambda$ of the ring of symmetric functions and used it to provide an algorithm for computing r^μ_λ [15]. In 2024, Narayanan, Paul, Prasad, and Srivastava found a combinatorial interpretation for r^μ_λ in the case that μ has one column and λ is either a hook shape or has at most two columns [13].

There are also several recent results about the inverse problem of writing the Specht module V_μ (or, more precisely, the class of V_μ in the representation ring of $\mathfrak{S}_n$) as a $\mathbb{Z}$-linear combination of Schur modules $\mathbb{S}^\lambda \mathbb{C}^n$. In 2019, Assaf and Speyer proved that the coefficients b^μ_λ that arise in this way alternate in sign; that is, $(-1)^{|\mu|-|\lambda|} b^\mu_\lambda \geq 0$ [2]. Ryba categorified this result in 2020 by finding a resolution of V_μ by direct sums of Schur modules [17].

We take the following approach to the restriction problem. Let Λ denote the ring of symmetric functions in the variables $x_1, x_2, x_3, \ldots$ and let $\overline{\Lambda}$ denote the ring of symmetric power series in $x_1, x_2, x_3, \ldots$. In this extended abstract, we will consider the abelian group homomorphism $\mathscr{F} \colon \Lambda \to \overline{\Lambda}$ defined on the basis $\{s_\lambda\}$ of Schur functions by

$$\mathscr{F}\{s_\lambda\} = \sum_\mu r^\mu_\lambda s_\mu.$$

Equivalently, $\mathscr{F}\{s_\lambda\}$ is the result of applying the Frobenius characteristic map to the representation $\bigoplus_n \mathbb{S}^\lambda \mathbb{C}^n$ of $\bigoplus_n \mathbb{C}[\mathfrak{S}_n]$. For this reason, we call $\mathscr{F}$ the *Frobenius transform*. It encodes all information about all the restriction coefficients. We will use this perspective to prove that several of the restriction coefficients r^μ_λ vanish.

The Frobenius transform can alternatively be defined in terms of the irreducible character basis $\{\tilde{s}_\lambda\}_\lambda$ of Orellana and Zabrocki. Namely,

$$\mathscr{F}\{\tilde{s}_\lambda\} = ((1 + h_1 + h_2 + \cdots)^\perp s_\lambda) \cdot (1 + h_1 + h_2 + \cdots)$$

for all partitions λ, where $f^\perp \colon \Lambda \to \Lambda$ denotes the operator adjoint under the Hall inner product to multiplication by f.

Interestingly, it is also possible to think of the Frobenius transform as a decategorification of the *analytic functor* construction defined by Joyal in 1986 [6]. We will not explore this perspective here; see [7, Section 3.1] for more details.

This extended abstract will be organized as follows. Section 3 will cover the basic properties of the Frobenius transform, several of which follow from the work of Orellana and Zabrocki. For example, for any symmetric functions f, g, we have $\mathscr{F}\{fg\} = \mathscr{F}\{f\} * \mathscr{F}\{g\}$, where $*$ is the Kronecker product of symmetric functions.

In Section 4, we will define the *surjective Frobenius transform* $\mathscr{F}_{\text{Sur}} \colon \Lambda \to \Lambda$ by the equation $\mathscr{F}_{\text{Sur}}\{f\} = \mathscr{F}\{f\}/(1 + h_1 + h_2 + \cdots)$. We will discuss the basic properties of $\mathscr{F}_{\text{Sur}}$, such as the fact that $\mathscr{F}_{\text{Sur}}\{f\}$ has the same degree and leading term as f. We will also introduce the inverse function $\mathscr{F}_{\text{Sur}}^{-1}$.

In Section 5, we will state a combinatorial formula for $\mathscr{F}_{\text{Sur}}\{f\}$, where f is any complete homogeneous, elementary, or power sum symmetric function. We will also state a formula for $\mathscr{F}_{\text{Sur}}^{-1}\{f\}$, where f is any elementary symmetric function.

Finally, in Section 6, we will explain how to use the Frobenius transform to prove new and surprising results about the vanishing of the restriction coefficients r_λ^μ. Namely, $r_\lambda^\mu = 0$ if $|\lambda \cap \hat{\mu}| < 2|\hat{\mu}| - |\lambda|$, where $\hat{\mu}$ is the partition formed by removing the first part of μ. Additionally, $r_\lambda^\mu = 0$ if the Young diagram of μ contains a square of side length greater than $2^{\lambda_1 - 1}$, and this inequality is tight.

All of the results in this extended abstract are stated without proof. For proofs of all the main theorems, see [7].

2 Preliminaries

In this extended abstract, we will use the following terminology and notation. Definitions can be found in any standard reference on the theory of symmetric functions [4], [12, Chapter I], [18, Chapter 7], or in the Preliminaries section of the author's published work [7, Section 2].

- The *ring of symmetric functions* Λ and the *ring of symmetric power series* $\overline{\Lambda}$.
- The *monomial*, *elementary*, *homogeneous*, *power sum*, and *Schur* symmetric functions, denoted $m_\lambda, e_\lambda, h_\lambda, p_\lambda, s_\lambda$ respectively.
- The *Hall inner product* $\langle \cdot, \cdot \rangle \colon \Lambda \times \overline{\Lambda} \to \mathbb{Z}$.
- The *plethysm* $f[g]$, where $f, g \in \overline{\Lambda}$.
- The *skewing operator* $f^\perp \colon \Lambda \to \Lambda$, where $f \in \Lambda$.
- The *Littlewood–Richardson coefficient* $c_{\lambda\mu}^\nu$.
- The *Kronecker product* $* \colon \overline{\Lambda} \times \overline{\Lambda} \to \overline{\Lambda}$ and the *Kronecker coefficient* $g_{\lambda\mu\nu}$.
- The *Specht module* V_λ.

- The *Schur functor* $\mathbb{S}^\lambda\colon \mathrm{Vect}_\mathbb{C} \to \mathrm{Vect}_\mathbb{C}$, where $\mathrm{Vect}_\mathbb{C}$ is the category of vector spaces over $\mathbb{C}$.
- The *Frobenius characteristic map* ch_n and the *Frobenius characteristic* of an $\mathfrak{S}_n$-module.

We also use the notation H to refer to the element $1 + h_1 + h_2 + \cdots \in \overline{\Lambda}$.

3 The Frobenius transform: definition and basic properties

Let λ, μ be partitions, and let $n = |\mu|$. Recall from the introduction that the *restriction coefficient*

$$r_\lambda^\mu = \dim \mathrm{Hom}_{\mathfrak{S}_n}(V_\mu, \mathbb{S}^\lambda \mathbb{C}^n)$$

is defined to be the multiplicity of the Specht module V_μ in the Schur module $\mathbb{S}^\lambda\mathbb{C}^n$. It is a long-standing open problem to find a combinatorial interpretation of r_λ^μ. As a potential way to approach this problem, we now define the Frobenius transform, which is the primary object of study in this extended abstract.

Definition 3.1. The *Frobenius transform* is the abelian group homomorphism $\mathscr{F}\colon \Lambda \to \overline{\Lambda}$ defined on the basis $\{s_\lambda\}$ by

$$\mathscr{F}\{s_\lambda\} = \sum_\mu r_\lambda^\mu s_\mu,$$

where the sum is over all partitions μ.

Remark 3.2. By a classical result of Littlewood [8], we have

$$r_\lambda^\mu = \langle s_\lambda, s_\mu[H] \rangle.$$

Hence, $\mathscr{F}$ is adjoint to plethysm by $H = 1 + h_1 + h_2 + \cdots$ under the Hall inner product.

Here is the reason for calling $\mathscr{F}$ the Frobenius transform. Let $n \geq 0$ and let λ be any partition. Then $\mathbb{S}^\lambda\mathbb{C}^n$, considered as an $\mathfrak{S}_n$-module, can be expressed as a direct sum of Specht modules:

$$\mathbb{S}^\lambda \mathbb{C}^n = \bigoplus_{|\mu|=n} r_\lambda^\mu V_\mu.$$

Taking the character of both sides and applying the Frobenius characteristic map, we obtain

$$\mathrm{ch}_n(\chi_{\mathbb{S}^\lambda\mathbb{C}^n}) = \sum_{|\mu|=n} r_\lambda^\mu s_\mu. \tag{3.1}$$

In other words, the degree n part of $\mathscr{F}\{s_\lambda\}$ is equal to the Frobenius characteristic of $\mathbb{S}^\lambda\mathbb{C}^n$.

To summarize, the restriction coefficient r_λ^μ can be written in terms of the Hall inner product in the following three ways:

$$r_\lambda^\mu = \langle \mathscr{F}\{s_\lambda\}, s_\mu\rangle = \langle s_\lambda, s_\mu[H]\rangle = \langle \mathrm{ch}_n(\chi_{\mathbb{S}^\lambda \mathbb{C}^n}), s_\mu\rangle.$$

Example 3.3. Let $r \geq 0$. We will compute $\mathscr{F}\{e_r\}$. First, $e_r = s_\lambda$, where $\lambda = (1^r)$. Hence, for any n, the degree n part of $\mathscr{F}\{e_r\}$ is the Frobenius characteristic of

$$\mathbb{S}^\lambda \mathbb{C}^n = \wedge^r \mathbb{C}^n = \mathrm{Ind}_{\mathfrak{S}_r \times \mathfrak{S}_{n-r}}^{\mathfrak{S}_n}(V_{(1^r)} \otimes V_{(n-r)}),$$

considered as an $\mathfrak{S}_n$-module. Thus, it is equal to $e_r h_{n-r}$ [18, Proposition 7.18.2]. Taking the sum over all n yields $\mathscr{F}\{e_r\} = e_r \cdot H$.

Let us now describe the expansion of $\mathscr{F}\{f\}$ in the power sum basis. Following Orellana and Zabrocki [15], for any partition $\mu = (\mu_1, \ldots, \mu_\ell)$ of n, let us define $\Xi_\mu \in \mathbb{C}^n$ to be the sequence consisting of the roots of unity

$$\begin{gathered} 1, \exp\left(\frac{2\pi i}{\mu_1}\right), \exp\left(\frac{4\pi i}{\mu_1}\right), \ldots, \exp\left(\frac{2(\mu_1 - 1)\pi i}{\mu_1}\right), \\ \vdots \\ 1, \exp\left(\frac{2\pi i}{\mu_\ell}\right), \exp\left(\frac{4\pi i}{\mu_\ell}\right), \ldots, \exp\left(\frac{2(\mu_\ell - 1)\pi i}{\mu_\ell}\right). \end{gathered}$$

Then, we have the following.

Proposition 3.4. *Let $f \in \Lambda$. Then*

$$\mathscr{F}\{f\} = \sum_\mu f(\Xi_\mu) \frac{p_\mu}{z_\mu}.$$

This proposition can be written as

$$\mathscr{F}\{f\} = \phi_0(f) + \phi_1(f) + \phi_2(f) + \cdots,$$

where $\phi_n \colon \Lambda \to \Lambda$ was defined by Orellana and Zabrocki [16, Equation (7)].

Another important property of the Frobenius transform is that it relates the ordinary product of symmetric functions to the Kronecker product $*$.

Proposition 3.5. *Let $f, g \in \Lambda$. Then $\mathscr{F}\{fg\} = \mathscr{F}\{f\} * \mathscr{F}\{g\}$.*

By taking both f and g to be Schur functions, we may think of Proposition 3.5 as a relationship between the Littlewood–Richardson coefficients $c_{\lambda\mu}^\nu$, the Kronecker coefficients $g_{\lambda\mu\nu}$, and the restriction coefficients r_λ^μ, as follows. This is potentially interesting because Littlewood–Richardson coefficients have a known combinatorial interpretation, but Kronecker coefficients and restriction coefficients do not.

Corollary 3.6. *Let λ, μ, ν be partitions. Then*

$$\sum_{\nu'} r_{\nu'}^\nu c_{\lambda\mu}^{\nu'} = \sum_{\lambda', \mu'} r_\lambda^{\lambda'} r_\mu^{\mu'} g_{\lambda'\mu'\nu}.$$

4 The surjective Frobenius transform

Let $f \in \Lambda$. Even though $\mathscr{F}\{f\} \in \overline{\Lambda}$ can have infinitely many nonzero coefficients, it only carries a finite amount of information in the following sense.

Proposition 4.1. *Let $f \in \Lambda$. There exists a symmetric function $\mathscr{F}_{\mathrm{Sur}}\{f\} \in \Lambda$ such that $\mathscr{F}\{f\} = \mathscr{F}_{\mathrm{Sur}}\{f\} \cdot H$. Moreover, $\mathscr{F}_{\mathrm{Sur}}\{f\}$ has the same degree and leading term as f.*

For example, in Example 3.3 we showed that $\mathscr{F}\{e_r\} = e_r \cdot H$. Thus, $\mathscr{F}_{\mathrm{Sur}}\{e_r\} = e_r$. (Note, however, that in general, $\mathscr{F}_{\mathrm{Sur}}$ does not preserve the property of being homogeneous.) We call $\mathscr{F}_{\mathrm{Sur}}$ the *surjective Frobenius transform* because it is related to the category Sur whose objects are finite sets and whose morphisms are surjective functions [7, Proposition 3.15 proof 1].

The function $\mathscr{F}_{\mathrm{Sur}}\colon \Lambda \to \Lambda$ can be described explicitly as the adjoint to plethysm by $H_+ = h_1 + h_2 + h_3 + \cdots$ under the Hall inner product. It is not difficult to show that $\mathscr{F}_{\mathrm{Sur}}$ is invertible:

Corollary 4.2. *There exists a two-sided inverse $\mathscr{F}_{\mathrm{Sur}}^{-1}\colon \Lambda \to \Lambda$ of $\mathscr{F}_{\mathrm{Sur}}$.*

Like the ordinary Frobenius transform, the surjective Frobenius transform can be described in terms of its matrix entries in the Schur basis.

Definition 4.3. Let λ, μ be partitions. Define the *surjective restriction coefficient* t_λ^μ by

$$t_\lambda^\mu = \langle \mathscr{F}_{\mathrm{Sur}}\{s_\lambda\}, s_\mu \rangle$$

and define the *inverse surjective restriction coefficient* u_λ^μ by

$$u_\lambda^\mu = \langle \mathscr{F}_{\mathrm{Sur}}^{-1}\{s_\lambda\}, s_\mu \rangle.$$

By the above, we have $t_\lambda^\mu = u_\lambda^\mu = \delta_{\lambda\mu}$ for $|\mu| \geq |\lambda|$.

5 Computing the Frobenius transform

5.1 Formulas for the surjective Frobenius transform

Recall that one of the overall goals of this project is to find a combinatorial interpretation of the restriction coefficients r_λ^μ. Since the restriction coefficients are the matrix entries of $\mathscr{F}$ in the Schur basis, this is tantamount to finding an explicit combinatorial (i.e. subtraction-free) formula that writes $\mathscr{F}\{s_\lambda\}$ as a linear combination of Schur functions.

Such a formula for $\mathscr{F}\{s_\lambda\}$ is still not known. However, Theorem 5.1 below provides a combinatorial formula for $\mathscr{F}_{\mathrm{Sur}}\{h_\lambda\}$, $\mathscr{F}_{\mathrm{Sur}}\{e_\lambda\}$, and $\mathscr{F}_{\mathrm{Sur}}\{p_\lambda\}$. Since $\mathscr{F}\{f\} = \mathscr{F}_{\mathrm{Sur}}\{f\} \cdot H$ for all $f \in \Lambda$, Theorem 5.1 can also be used to compute $\mathscr{F}\{h_\lambda\}$, $\mathscr{F}\{e_\lambda\}$, and

$\mathscr{F}\{p_\lambda\}$ to any desired degree. A statement equivalent to parts (a) and (b) of Theorem 5.1 has appeared in the work of Orellana and Zabrocki [14, Equation (6)], but part (c) is new.

Theorem 5.1. *Let λ be a partition and let $\ell = \ell(\lambda)$ be its length.*

(a) Then

$$\mathscr{F}_{\mathrm{Sur}}\{h_\lambda\} = \sum_M \prod_{j \in \mathbb{N}^\ell} h_{M(j)},$$

where the sum is over all functions $M\colon \mathbb{N}^\ell \to \mathbb{N}$ such that $M(0,\ldots,0) = 0$ and $\sum_{j\in\mathbb{N}^\ell} j_i M(j) = \lambda_i$ for $i = 1,\ldots,\ell$.

(b) Then

$$\mathscr{F}_{\mathrm{Sur}}\{e_\lambda\} = \sum_M \prod_{j\in\{0,1\}^\ell} \begin{cases} h_{M(j)} & \text{if } j_1+\cdots+j_\ell \text{ is even;} \\ e_{M(j)} & \text{if } j_1+\cdots+j_\ell \text{ is odd,} \end{cases}$$

where the sum is over all functions $M\colon \{0,1\}^\ell \to \mathbb{N}$ such that $M(0,\ldots,0) = 0$ and $\sum_{j\in\{0,1\}^\ell} j_i M(j) = \lambda_i$ for $i = 1,\ldots,\ell$.

(c) Then

$$\mathscr{F}_{\mathrm{Sur}}\{p_\lambda\} = \sum_\pi \prod_{U\in\pi} \left(\sum_{d \mid \gcd\{\lambda_i\colon i\in U\}} d^{|U|-1} p_d \right),$$

where the outer sum is over all partitions π of $\{1,\ldots,\ell\}$ into nonempty sets and the product is over all parts U of π.

Example 5.2. Let us use Theorem 5.1(a) to compute $\mathscr{F}_{\mathrm{Sur}}\{h_{2,2}\}$. First, we list all the functions $M\colon \mathbb{N}^2 \to \mathbb{N}$ such that $M(0,0) = 0$ and $\sum_{j\in\mathbb{N}^2} jM(j) = (2,2)$. There are nine such functions $M_1,\ldots,M_9$. Here are all of their nonzero values.[2]

$$\begin{gathered}
M_1(2,2) = 1 \\
M_2(1,1) = 2 \\
M_3(2,1) = 1 \qquad M_3(0,1) = 1 \\
M_4(1,2) = 1 \qquad M_4(1,0) = 1 \\
M_5(2,0) = 1 \qquad M_5(0,2) = 1 \\
M_6(2,0) = 1 \qquad M_6(0,1) = 2 \\
M_7(0,2) = 1 \qquad M_7(1,0) = 2 \\
M_8(1,1) = 1 \qquad M_8(1,0) = 1 \qquad M_8(0,1) = 1 \\
M_9(1,0) = 2 \qquad M_9(0,1) = 2
\end{gathered}$$

[2] For readers who are familiar with the language of multisets and multiset partitions [15], it can be helpful to remember that such functions M are in bijection with multiset partitions of $\{\{1,1,2,2\}\}$. The multiset partition corresponding to the function M contains $M(j)$ copies of $\{\{1^{j_1}, 2^{j_2}\}\}$ for all $j \in \mathbb{N}^2$. For example, the function M_6 corresponds to the multiset partition $\{\{\{\{1,1\}\},\{\{2\}\},\{\{2\}\}\}\} \Vdash \{\{1,1,2,2\}\}$.

Thus,

$$\begin{aligned}\mathscr{F}_{\mathrm{Sur}}\{h_{2,2}\} &= \underbrace{h_1}_{M_1} + \underbrace{h_2}_{M_2} + \underbrace{h_1^2}_{M_3} + \underbrace{h_1^2}_{M_4} + \underbrace{h_1^2}_{M_5} + \underbrace{h_1h_2}_{M_6} + \underbrace{h_1h_2}_{M_7} + \underbrace{h_1^3}_{M_8} + \underbrace{h_2^2}_{M_9} \\ &= h_1 + h_2 + 3h_{1,1} + 2h_{2,1} + h_{1,1,1} + h_{2,2}.\end{aligned}$$

Example 5.3. Let us use Theorem 5.1(b) to compute $\mathscr{F}_{\mathrm{Sur}}\{e_{5,3}\}$. First, we list all the functions $M\colon \{0,1\}^2 \to \mathbb{N}$ such that $M(0,0) = 0$ and $\sum_{j\in\{0,1\}^2} jM(j) = (5,3)$. There are four such functions M_1, M_2, M_3, M_4. Here are all of their nonzero values.

$$\begin{array}{ccc} & M_1(1,0) = 5 \qquad M_1(0,1) = 3 & \\ M_2(1,1) = 1 & M_2(1,0) = 4 & M_2(0,1) = 2 \\ M_3(1,1) = 2 & M_3(1,0) = 3 & M_3(0,1) = 1 \\ & M_4(1,1) = 3 \qquad M_4(1,0) = 2 & \end{array}$$

Thus,

$$\mathscr{F}_{\mathrm{Sur}}\{e_{5,3}\} = \underbrace{e_5e_3}_{M_1} + \underbrace{h_1e_4e_2}_{M_2} + \underbrace{h_2e_3e_1}_{M_3} + \underbrace{h_3e_2}_{M_4}.$$

Example 5.4. Let us use Theorem 5.1(c) to compute $\mathscr{F}_{\mathrm{Sur}}\{p_{15,10,6}\}$. Take $\lambda = (15,10,6)$ and $\ell = 3$. There are five partitions of $[\ell]$ into nonempty sets: $\{\{1,2,3\}\}$, $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$, $\{\{2,3\},\{1\}\}$, and $\{\{1\},\{2\},\{3\}\}$. Thus,

$$\begin{aligned}\mathscr{F}_{\mathrm{Sur}}\{p_\lambda\} &= \left(\sum_{d\mid\gcd(\lambda_1,\lambda_2,\lambda_3)} d^2p_d\right) + \left(\sum_{d\mid\gcd(\lambda_1,\lambda_2)} dp_d\right)\left(\sum_{d\mid\lambda_3} p_d\right) \\ &\quad + \left(\sum_{d\mid\gcd(\lambda_1,\lambda_3)} dp_d\right)\left(\sum_{d\mid\lambda_2} p_d\right) + \left(\sum_{d\mid\gcd(\lambda_2,\lambda_3)} dp_d\right)\left(\sum_{d\mid\lambda_1} p_d\right) \\ &\quad + \left(\sum_{d\mid\lambda_1} p_d\right)\left(\sum_{d\mid\lambda_2} p_d\right)\left(\sum_{d\mid\lambda_3} p_d\right) \\ &= p_1 + (p_1+5p_5)(p_1+p_2+p_3+p_6) \\ &\quad + (p_1+3p_3)(p_1+p_2+p_5+p_{10}) + (p_1+2p_2)(p_1+p_3+p_5+p_{15}) \\ &\quad + (p_1+p_3+p_5+p_{15})(p_1+p_2+p_5+p_{10})(p_1+p_2+p_3+p_6).\end{aligned}$$

Remark 5.5. One consequence of Theorem 5.1(c) is that the matrix entries of $\mathscr{F}_{\mathrm{Sur}}$ in the power sum basis are all nonnegative integers. The same is not true of $\mathscr{F}$; for example, $\mathscr{F}\{1\} = H = 1 + p_1 + \frac{1}{2}(p_2 + p_1^2) + \cdots$ certainly has some non-integer coefficients.

5.2 Formulas for the inverse surjective Frobenius transform

Now, we turn to the inverse surjective Frobenius transform $\mathscr{F}_{\mathrm{Sur}}^{-1}$. It follows from a 2019 result of Assaf and Speyer [2, Theorem 3] that $\mathscr{F}_{\mathrm{Sur}}^{-1}\{s_\lambda\}$ is Schur-alternating for all partitions λ. That is, we may write

$$\mathscr{F}_{\mathrm{Sur}}^{-1}\{s_\lambda\} = \sum_\mu u_\lambda^\mu s_\mu, \tag{5.1}$$

where the coefficients u_λ^μ (which have appeared in Definition 4.3) satisfy $(-1)^{|\lambda|-|\mu|}u_\lambda^\mu \geq 0$ for all partitions μ [7, Theorem 4.2(c)]. It would be interesting to find a combinatorial interpretation of $(-1)^{|\lambda|-|\mu|}u_\lambda^\mu$, or, equivalently, to find an explicit formula for $\mathscr{F}_{\mathrm{Sur}}^{-1}\{s_\lambda\}$ that is manifestly Schur-alternating.

Such a formula is still not known. However, Theorem 5.11 below provides a formula for $\mathscr{F}_{\mathrm{Sur}}^{-1}\{e_\lambda\}$. The formula implies that $\mathscr{F}_{\mathrm{Sur}}^{-1}\{e_\lambda\}$ is *e-alternating* for all partitions λ. That is, we may write

$$\mathscr{F}_{\mathrm{Sur}}^{-1}\{e_\lambda\} = \sum_\mu d_\lambda^\mu e_\mu, \tag{5.2}$$

where the coefficients d_λ^μ satisfy the property that $(-1)^{|\lambda|-|\mu|}d_\lambda^\mu$ is a nonnegative integer for all μ. Moreover, the formula provides an explicit combinatorial intepretation of the number $(-1)^{|\lambda|-|\mu|}d_\lambda^\mu$.

Before we proceed to the statement of Theorem 5.11, let us recall some definitions from the combinatorics of words. For a more complete introduction, see [10, Chapter 5].

Definition 5.6. Let A be a set. A *word* over the alphabet A is a sequence $w = w_1 \cdots w_n$ with $w_1, \ldots, w_n \in A$. Given a letter $a \in A$, we write $m_a(w)$ to denote the number of times the letter a appears in w.

Definition 5.7 ([11])**.** Let A be a totally ordered set. We say that a nonempty word $w = w_1 \cdots w_n$ over the alphabet A is a *Lyndon word* if it is lexicographically less than its suffix $w_i \cdots w_n$ for $i = 2, \ldots, n$. Let $\mathrm{Lyndon}(A)$ be the set of all Lyndon words over the alphabet A.

Theorem 5.8 (Chen–Fox–Lyndon Theorem [3])**.** *Let A be a totally ordered set. Any word w over the alphabet A has a unique* Lyndon factorization*; that is, an expression as a (lexicographically) non-increasing concatenation of Lyndon words.*

Definition 5.9. Let w be a word over a totally ordered alphabet. Define $\pi(w)$ to be the partition obtained by listing the number of times each Lyndon word appears in the Lyndon factorization of w, and then sorting the resulting positive numbers in decreasing order.

w	Lyndon factorization	$\pi(w)$
11122	(11122)	(1)
11212	(11212)	(1)
11221	(1122)(1)	(1,1)
12112	(12)(112)	(1,1)
12121	(12)(12)(1)	(2,1)
12211	(122)(1)(1)	(2,1)
21112	(2)(1112)	(1,1)
21121	(2)(112)(1)	(1,1,1)
21211	(2)(12)(1)(1)	(2,1,1)
22111	(2)(2)(1)(1)(1)	(3,2)

Figure 1: For each word w over the alphabet [2] that consists of three 1's and two 2's, we compute the Lyndon factorization of w and the partition $\pi(w)$.

Example 5.10. If $A = \{1,2\}$ and $w = 21212121111$, then the Lyndon factorization of w is $w = (2)(12)(12)(12)(1)(1)(1)(1)$. The Lyndon words appearing in this factorization are 2, 12, and 1, which appear once, three times, and four times, respectively, so $\pi(w) = (4,3,1)$.

We are now ready to state our formula for $\mathscr{F}_{\mathrm{Sur}}^{-1}\{e_\lambda\}$.

Theorem 5.11. *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a sequence of nonnegative integers (not necessarily weakly decreasing). Then*

$$\mathscr{F}_{\mathrm{Sur}}^{-1}\{e_\lambda\} = \sum_{w \in W} (-1)^{|\lambda| - |\pi(w)|} e_{\pi(w)},$$

where W is the set of all words w over $[\ell]$ such that $m_i(w) = \lambda_i$ for all $i \in [\ell]$.

In other words, if d_λ^μ is the coefficient of e_μ in $\mathscr{F}_{\mathrm{Sur}}^{-1}\{e_\lambda\}$ as in (5.2), then $(-1)^{|\lambda|-|\mu|} d_\lambda^\mu$ is the number of words w such that $m_i(w) = \lambda_i$ for all $i \in [\ell]$ and $\pi(w) = \mu$.

Example 5.12. Let us use Theorem 5.11 to compute $\mathscr{F}_{\mathrm{Sur}}^{-1}\{e_{3,2}\}$. In Figure 1, we list all the words w over [2] such that $m_i(w) = \lambda_i$ for all $i \in [2]$. These are exactly the words that consist of three 1's and two 2's. For each such word w, we record the Lyndon factorization of w and the partition $\pi(w)$. Reading from the table, we find that

$$\mathscr{F}_{\mathrm{Sur}}^{-1}\{e_{3,2}\} = 2e_1 - 3e_{1,1} + e_{1,1,1} + 2e_{2,1} - e_{2,1,1} + e_{3,2}.$$

We also have a similar alternating formula for $\mathscr{F}_{\mathrm{Sur}}^{-1}\{h_\lambda\}$ [7, Theorem 7.7(b)], but it is more complicated and we will not reproduce it here.

6 Vanishing of restriction coefficients

One important question about r_λ^μ is the following: under what circumstances do we have $r_\lambda^\mu = 0$? Using the Frobenius transform, we can partially answer this question. For any partition μ, let $D(\mu)$ denote the size of the Durfee square of μ. That is, $D(\mu)$ is the largest integer d such that $\mu_d \geq d$ [1, Chapter 8].

Theorem 6.1. *Let μ be a partition and let $k \geq 1$ be an integer. The following are equivalent:*

(A) There exists a partition λ such that $\lambda_1 \leq k$ and $r_\lambda^\mu > 0$.

(B) $D(\mu) \leq 2^{k-1}$.

In particular, if λ and μ are partitions with $D(\mu) > 2^{\lambda_1 - 1}$, then $r_\lambda^\mu = 0$.

We also have the following. For any partitions λ, μ, let $\lambda \cap \mu$ denote the partition whose Young diagram is the intersection of the Young diagrams of μ and λ. Explicitly, $\ell(\lambda \cap \mu) = \min(\ell(\lambda), \ell(\mu))$ and $(\lambda \cap \mu)_i = \min(\lambda_i, \mu_i)$ for all i.

Theorem 6.2. *Let λ, μ be partitions. If the surjective restriction coefficient t_λ^μ does not vanish, then $|\lambda \cap \mu| \geq 2|\mu| - |\lambda|$.*

Theorem 6.3. *Let λ, μ be partitions. If the restriction coefficient r_λ^μ does not vanish, then $|\lambda \cap \hat{\mu}| \geq 2|\hat{\mu}| - |\lambda|$, where $\hat{\mu} = (\mu_2, \ldots, \mu_{\ell(\mu)})$ is the partition formed by removing the first part of μ.*

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