Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #86, 12 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

Plücker inequalities for weakly separated coordinates in TNN Grassmannians

Daniel Soskin*[1] and Prateek Kumar Vishwakarma†[2]

[1]*Institute for Advanced Study, Princeton, NJ, USA*
[2]*Université Laval, Québec, QC, Canada*

Abstract. The fundamental connections of the Grassmannian with both weak separability and Plücker relations are well known. In this work we focus on the totally nonnegative (TNN) part of the Grassmannian, and discover the intrinsic connection between weak separability and Plücker relations. In particular, we show that certain natural sums of terms in a long Plücker relation for pairs of weakly separated Plücker coordinates oscillate around 0 over the TNN Grassmannian. This generalizes the classical oscillating inequalities by Gantmacher–Krein (1941) and recent results on TNN matrix inequalities by Fallat–Vishwakarma (2024). In fact we obtain a characterization of weak separability, by showing that no other pairs of Plücker coordinates satisfy this property. Moreover, our work uncovers the natural connections between weak separability, Plücker relations, as well as Temperley–Lieb immanants, and provides a general and natural class of additive inequalities in Plücker coordinates on the totally nonnegative part of the Grassmannian.

Keywords: Determinantal inequalities, total nonnegativity, Plücker relations, oscillating inequalities, weak separability, totally nonnegative Grassmannian, cluster algebras

1 Introduction and main results

A real matrix is referred to as *totally nonnegative (TNN)* if the determinants of all its square submatrices are nonnegative. Total nonnegativity arose initially in a few different areas: by Gantmacher–Krein [10] in oscillations of vibrating systems, by Fekete–Pólya [8] (following Laguerre) in understanding the variation diminishing property of linear operators, and by Schoenberg and coauthors [1] in applications to the analysis of real roots of polynomials and spline functions. These matrices play significant roles in algebraic

*dsoskin@ias.edu. D.S. was supported by the National Science Foundation under Grant No. DMS-1926686.

†prateek-kumar.vishwakarma.1@ulaval.ca. P.K.V. is supported by the Centre de recherches mathématiques-Laval (CRM-Laval) Postdoctoral Fellowship. He also acknowledges the Pacific Institute for the Mathematical Sciences (PIMS), a SwarnaJayanti Fellowship from DST and SERB, Govt. of India (PI: Apoorva Khare), the SPARC travel support (Scheme for Promotion of Academic and Research Collaboration, MHRD, Govt. of India; PI: Tirthankar Bhattacharyya), and the University of Plymouth (UK).

and enumerative combinatorics, integrable systems, probability, classical mechanics, and many other areas [2, 4, 11, 14]. Lusztig extended the notion of total positivity to reductive Lie groups G [17], where the totally nonnegative part $G^{\geq 0}$ of G is a semialgebraic submonoid of G generated by Chevalley generators. $G^{\geq 0}$ is the subset of G where all elements of the dual canonical basis are nonnegative [18]. This concept could be generalized even further to varieties V. The totally nonnegative subvariety is defined as the subset of V where certain regular functions on V have nonnegative values [3, 9, 12]. Lusztig proved that specializations of elements of the dual canonical basis in the representation theory of quantum groups at $q = 1$ are totally nonnegative polynomials. Thus, it is important to investigate classes of functions on matrices that are nonnegative on totally nonnegative matrices. We will discuss a source of such functions that are closely related to the *(long) Plücker relations*.

This project traces its roots to the classical work by Gantmacher–Krein [10] where they modify the Laplace formula (1772) for the determinant expansion along the first row of the matrices to derive a sequence of inequalities oscillating about 0 and holding for all totally nonnegative matrices. These were expanded into new systems of inequalities by Fallat–Vishwakarma [5], who similarly derived an oscillating set of inequalities from the generalized Laplace identity. To state this result more precisely we introduce some notations: for a matrix A, define $A_{P,Q}$ as the submatrix with rows indexed by P and columns indexed by Q, $[m, n] := \{m, \ldots, n\}$ and $[n] := [1, n]$ for integers $0 \leq m \leq n$.

Theorem 1.1 (Fallat–Vishwakarma [5])**.** *Let $1 \leq d < n$ be integers. Suppose $P_d := [d]$, and $Q_{dk} := [n-d, n] \setminus \{n-d+k\}$, for all $k \in [0, d]$. Then*

$$\sum_{k=0}^{l} (-1)^{l+k} \det A_{P_d, Q_{dk}} \det A_{[n]\setminus P_d, [n]\setminus Q_{dk}} \geq 0, \text{ for all } l \in [0, d],$$

for all $n \times n$ totally nonnegative matrices A.

In this extended abstract we explain why these inequalities refine certain long Plücker relations, via the Temperley–Lieb immanants introduced by Rhoades–Skandera [21]. Another classical identity, which can also be viewed as one of the Plücker relations, is the well-known Karlin's identity [14]. Fallat and Vishwakarma provided a complete refinement of this identity for TNN matrices (see Theorem B in [5]). These two examples suggest investigating the potential existence of a broader set of "Plücker inequalities" over the totally nonnegative Grassmannian. This is exactly the focus of our main theorem; to present it, we establish some classical facts.

Throughout this article we assume $1 \leq m \leq n$ are integers. The Grassmannian $\mathrm{Gr}(m, m+n)$ is the manifold of m-dimensional subspaces of $\mathbb{R}^{m+n}$. Such subspaces can be represented by matrices $X \in \mathbb{R}^{(m+n)\times m}$ of full rank. Let X_I denote the submatrix corresponding to rows indexed by m element subsets $I \subseteq [m+n]$ and columns indexed by $[m]$. Here $\Delta_I(X) := \det X_I$ denotes the corresponding maximal minor of X. Note that right multiplication by an invertible $m \times m$ matrix B rescales each $\Delta_I(X)$ by $\det B$. Thus, the minors $\Delta_I(X)$ together form projective coordinates of $\mathrm{Gr}(m, m+n)$, known as the corresponding *Plücker coordinates.* These are related via *Plücker relations*: defined for m-element ordered subsets $I := (i_1, \dots, i_m), J := (j_1, \dots, j_m)$ of $[m+n]$, and $I_{k,r} := (i_1, \dots, i_{r-1}, j_k, i_{r+1}, \dots, i_m)$ and $J_{k,r} := (j_1, \dots, j_{k-1}, i_r, j_{k+1}, \dots, j_m)$ for $k, r \in [1, m]$ [12, 20]:

$$\Delta_I \Delta_J = \sum_{k \in [m]} \Delta_{I_{k,r}} \Delta_{J_{k,r}} \quad \text{at each point in } \mathrm{Gr}(m, m+n) \quad \text{(and conversely).} \tag{1.1}$$

Here $\Delta_{(i_1,\dots,i_m)}$ is labelled by an ordered sequence, and equals $\Delta_{\{i_1,\dots,i_m\}}$ if $i_1 < \cdots < i_m$. Also, $\Delta_{(i_1,\dots,i_m)} = (-1)^{\mathrm{sgn}(w)} \Delta_{(i_{w(1)},\dots,i_{w(m)})}$ for all permutations $w \in S_m$. The totally nonnegative (TNN) Grassmannian $\mathrm{Gr}^{\geq 0}(m, m+n) \subset \mathrm{Gr}(m, m+n)$ contains those vector subspaces which have a representative matrix $X \in \mathbb{R}^{(m+n) \times m}$ with all $\Delta_I \geq 0$ [20].

The main idea in this paper involves two steps: (1) restricting the Plücker relations to the TNN Grassmannian, and (2) showing that natural families of inequalities can be "extracted" from these relations. That is, we show that for *weakly separated* I, J (Definition 1.2), one can extract inequalities out of (1.1) that look like

$$\Delta_I \Delta_J \leq \sum_{k \in \mathcal{M}} \Delta_{I_{k,r}} \Delta_{J_{k,r}}, \quad \text{and hold for each point in } \mathrm{Gr}^{\geq 0}(m, m+n), \tag{1.2}$$

for "nice" subsets $\mathcal{M} \subseteq [m]$. In fact we describe *all* I, J for which such natural inequalities can be obtained. We now discuss the required notations. For $c_{I,J} \in \mathbb{R}$ and Plücker coordinates Δ_I and Δ_J, we say

$$\sum_{I,J} c_{I,J} \Delta_I \Delta_J \text{ is nonnegative } \big(\geq 0 \big) \quad \text{over} \quad \mathrm{Gr}^{\geq 0}(m, m+n)$$

if for all subspaces $V \in \mathrm{Gr}^{\geq 0}(m, m+n)$ and all representative matrices $X \in \mathbb{R}^{(m+n) \times m}$ of V, $\sum_{I,J} c_{I,J} \Delta_I(X) \Delta_J(X) \geq 0$. The main result in this paper unifies the classical [10] and recent [5] oscillating inequalities mentioned above for TNN matrices, by refining the Plücker relations for $\mathrm{Gr}^{\geq 0}(m, m+n)$. To state it, we require some notations:

Definition 1.2. Fix integers $m, n \geq 1$, and let I and J be ordered m-element subsets of $[m+n]$. Imagine that the elements of I and J are on the circle, among the points $1, 2, \dots, m+n$ which are marked in clockwise order.

1. We call I, J *weakly separated* if the sets $I \setminus J$ and $J \setminus I$ can be separated by a chord in the circle.
2. Suppose $\eta = |I \setminus J| = |J \setminus I|$. Denote $I \setminus J := \{i_1, \dots, i_\eta\}$ and $J \setminus I := \{j_1, \dots, j_\eta\}$, such that $i_1 <_c \cdots <_c i_\eta <_c j_\eta$ and $i_1 <_c j_1 <_c \cdots <_c j_\eta$ in the clockwise order $<_c$, where each order is decided by moving in the clockwise direction on

the circle starting from i_1. (For example: if $m = n = 6$, $I = (1,5,3,4,10,11)$ and $J = (2,6,7,8,9,11)$, then $(i_1,i_2,i_3,i_4,i_5) = (10,1,3,4,5)$ and $(j_1,j_2,j_3,j_4,j_5) = (2,6,7,8,9)$. In particular, we choose i_1 and j_η such that all other i_k, j_k lie in between them while we traverse in the clockwise order starting from i_1.) Now define the following pair of tuples for $k,r \in [1,\eta]$:

$$I_{k,r} := (\dots, j_k, \dots), \quad \text{and} \quad J_{k,r} := (\dots, i_r, \dots), \tag{1.3}$$

where j_k and i_r replace each other in I and J, respectively.

We begin by first describing our main result for weakly separated I, J. Let $l, r \in [\eta]$, then we have a family of inequalities extracted from Plücker relations (1.1):

$$\pm\Big(\sum_{k=1}^{l} \Delta_{I_{k,r}}\Delta_{J_{k,r}} - \Delta_I\Delta_J\Big) \geq 0 \quad \forall\, l \geq \eta - r + 1, \quad \text{over} \quad \mathrm{Gr}^{\geq 0}(m, m+n). \tag{1.4}$$

This provides a novel class of "Plücker-type" determinantal inequalities that hold over the entire TNN Grassmannian for the class of weakly separated I, J. It also suggests two natural follow-up questions: *(1)* Are there analogous Plücker-type inequalities that hold over $\mathrm{Gr}^{\geq 0}(m, m+n)$ for the remaining cases of $l, r \in [1,\eta]$? (Here we continue to work with weakly separated I, J.) *(2)* If yes, then does this also have a converse, in that certain Plücker-type determinantal inequalities do not hold if I, J are not weakly separated? Our main theorem provides an affirmative answer to both of these questions, and in the process also characterizes weak separability. We also provide the precise meaning of "$\pm$" in (1.4), and show that it refers to the *sign* of $\Delta_{I_{l,r}}\Delta_{J_{l,r}}$ in the inequalities, defined via:

$$\mathrm{sgn}(I) = (-1)^{\mathrm{sgn}(w)} \quad \text{where} \quad w \in S_m \quad \text{such that} \quad i_{w(1)} < \cdots < i_{w(m)}$$

for ordered m element subsets $I := (i_1, \dots, i_m)$ of $[m+n]$.

Theorem 1.3 (Main result)**.** *Let I, J be ordered m-element subsets of $[m+n]$; notation as in Definition 1.2. Consider the following system of oscillating inequalities for $l, r \in [1,\eta]$ over $\mathrm{Gr}^{\geq 0}(m, m+n)$:*

$$\begin{aligned} \mathrm{sgn}(I_{l,r})\,\mathrm{sgn}(J_{l,r}) \sum_{k=1}^{l} \Delta_{I_{k,r}}\Delta_{J_{k,r}} \geq 0 \quad \forall\, l < \eta - r + 1, \textit{ and} \\ \mathrm{sgn}(I_{l,r})\,\mathrm{sgn}(J_{l,r})\Big(\sum_{k=1}^{l} \Delta_{I_{k,r}}\Delta_{J_{k,r}} - \Delta_I\Delta_J\Big) \geq 0 \quad \forall\, l \geq \eta - r + 1. \end{aligned} \tag{1.5}$$

This system holds for all $l, r \in [1,\eta]$ if and only if I and J are weakly separated.

One may refer to [23] for several (worked out) novel examples of these oscillating inequalities, including Theorem 1.1 and the refinement of Karlin's identity. Next we recall the tools needed to prove Theorem 1.3, including Temperley–Lieb immanants and Kauffman diagrams. We demonstrate the proof ideas assuming that $m = n = \eta$, and refer the reader to [23] for the the general case.

2 Temperley–Lieb immanants

Some determinantal inequalities can be stated in terms of polynomials $p(\mathbf{x})$ in the matrix entries $\mathbf{x} = (x_{ij})_{i,j=1}^{n}$, for integer $n \geq 1$. Such a polynomial is called *totally nonnegative (TNN)* if it attains nonnegative values on TNN matrices. Next, following Littlewood [16] and Stanley [24], given a function $f : \mathrm{S}_n \to \mathbb{C}$ define the *f-immanant* to be the polynomial

$$\mathrm{Imm}_f(\mathbf{x}) := \sum_{w \in \mathrm{S}_n} f(w) x_{1,w_1} \cdots x_{n,w_n} \in \mathbb{C}[\mathbf{x}]. \tag{2.1}$$

A particular family of immanants are defined for the *Temperley–Lieb algebra*, defined next. Fix $\xi \in \mathbb{C}$; a *Temperley–Lieb algebra $T_n(\xi)$* is generated by $t_1, \ldots, t_{n-1}$ over $\mathbb{C}$ subject to:

$$t_i^2 = \xi t_i \;\text{ for }\; i \in [n-1]; \; t_i t_j t_i = t_i \;\text{ if }\; |i-j| = 1; \text{ and } \; t_i t_j = t_j t_i \;\text{ if }\; |i-j| \geq 2. \tag{2.2}$$

When $\xi = 2$ we have the isomorphism $T_n(2) \cong \mathbb{C}[\mathrm{S}_n]/(1 + s_1 + s_2 + s_1 s_2 + s_2 s_1 + s_1 s_2 s_1)$ [6], [13, Sections 2.1 and 2.11], and [25, Section 7]. Specifically, the isomorphism is via

$$\sigma : \mathbb{C}[\mathrm{S}_n] \to T_n(\xi) \;\text{ with }\; s_i \mapsto t_i - 1. \tag{2.3}$$

Let $\mathcal{B}_n$ be the multiplicative monoid generated by $t_1, \ldots, t_{n-1}$ subject to (2.2), taking $\xi = 1$, i.e. modulo the powers of ξ. This is also known as a standard basis of $T_n(\xi)$. It is known that $|\mathcal{B}_n|$ is the n^{th} Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Diagrams of the basis elements of $T_n(\xi)$, made popular by Kauffman [15, Section 4] are (undirected) graphs with $2n$ vertices and n noncrossing edges, such that each edge lies in the convex hull of these $2n$ vertices (similar to one of the definitions of Catalan numbers involving n noncrossing chords on a circle with $2n$ vertices). Now define the *Temperley–Lieb immanant* $\mathrm{Imm}_\tau(\mathbf{x})$ for each $\tau \in \mathcal{B}_n$ in terms of the function

$$f_\tau : \mathrm{S}_n \to \mathbb{C} \;\text{ with }\; w \mapsto \;\text{coefficient of } \tau \text{ in } \sigma(w), \tag{2.4}$$

and extend it linearly over $\mathbb{C}[\mathrm{S}_n]$. Finally, define the Temperley–Lieb immanants as

$$\mathrm{Imm}_\tau(\mathbf{x}) := \mathrm{Imm}_{f_\tau}(\mathbf{x}) = \sum_{w \in \mathrm{S}_n} f_\tau(w) x_{1,w_1} \cdots x_{n,w_n}.$$

Rhoades–Skandera [21] showed that Temperley–Lieb immanants are a basis of the space

$$\mathrm{span}_{\mathbb{R}}\{\det \mathbf{x}_{P,Q} \det \mathbf{x}_{P^c,Q^c} \mid P, Q \subseteq [n] \text{ with } |P| = |Q|\}, \tag{2.5}$$

and that they are TNN. Infact, these are the extreme rays of the cone of TNN immanants:

Theorem 2.1 ([21]). *Given a function $f : \mathrm{S}_n \to \mathbb{R}$, the immanant*

$$\mathrm{Imm}_f(\mathbf{x}) = \sum_{\substack{P,Q \subseteq [n] \\ |P|=|Q|}} c_{P,Q} \det \mathbf{x}_{P,Q} \det \mathbf{x}_{P^c,Q^c} \tag{2.6}$$

is TNN if and only if it is a nonnegative linear combination of Temperley–Lieb immanants.

In fact, each complementary product of minors is a 0-1 linear combination of Temperley–Lieb immanants [21, Proposition 4.4]. Next we briefly demonstrate how Theorem 2.1 is used in identifying inequalities that involve products of pairs of Plücker coordinates. For this, recall that the $n \times n$ TNN matrices embed inside $\mathrm{Gr}^{\geq 0}(n, 2n)$:

$$\{n \times n \text{ TNN matrices}\} \hookrightarrow \mathrm{Gr}^{\geq 0}(n, 2n) \quad \text{via} \quad \mathbf{x} \mapsto \overline{\mathbf{x}} := \begin{pmatrix} \mathbf{x} \\ W_0 \end{pmatrix}, \tag{2.7}$$

where $W_0 := \big((-1)^{i+1} \cdot \delta_{j,n-i+1}\big)_{i,j=1}^{n}$. This yields a one-to-one correspondence between the minors of $\mathbf{x}$ and the maximal minors of $\overline{\mathbf{x}}$ via $\det \mathbf{x}_{P,Q} = \det \overline{\mathbf{x}}_{I,[m]} = \Delta_I(\overline{\mathbf{x}})$, for all $P \subseteq [n], Q \subseteq [n]$ with $|P| = |Q|$, where $I := P \cup \{2n+1-j \,|\, j \in [n] \setminus Q\}$. This, along with the density of $\mathrm{Gr}^{>}(n, 2n)$ in $\mathrm{Gr}^{\geq 0}(n, 2n)$, and the projective geometry of $\mathrm{Gr}(n, 2n)$, yield a *correspondence* between inequalities in products of pairs of minors of TNN matrices and inequalities in products of pairs of Plücker coordinates over the TNN Grassmannian, which is in fact compatible with the Temperley–Lieb immanant idea:

Theorem 2.2 ([21, 23]). *Suppose I runs over n-element subsets of $[2n]$, and $c_I \in \mathbb{R}$. Then*

$$\sum_I c_I \Delta_I \Delta_{I^c} \geq 0 \quad \textit{over} \quad \mathrm{Gr}^{\geq 0}(n, 2n), \tag{2.8}$$

if and only if $\sum_I c_I \Delta_I(\overline{\mathbf{x}}) \Delta_{I^c}(\overline{\mathbf{x}})$ is a nonnegative linear combination of Temperley–Lieb immanants. Moreover, we have for each I that

$$\Delta_I(\overline{\mathbf{x}}) \Delta_{I^c}(\overline{\mathbf{x}}) = \sum_{\tau \in \mathcal{B}_n} b_\tau \mathrm{Imm}_\tau(\mathbf{x}), \tag{2.9}$$

where $b_\tau = 1$ if each edge in the Kauffman diagram of τ connects elements from I and I^c, and 0 otherwise.

In light of Theorem 2.2, it is necessary and sufficient for the validity of (2.8) that

$$\text{the replacements} \quad \Delta_I \Delta_{I^c} \leftrightarrow \sum_{\tau \in \mathcal{B}_n} b_\tau \mathrm{Imm}_\tau(\mathbf{x}) \quad \text{in} \quad (2.8)$$

yield a nonnegative linear combination of immanants $\mathrm{Imm}_\tau(\mathbf{x})$, where $b_\tau = 1$ if each edge in the Kauffman diagram of τ connects elements from I and I^c, and 0 otherwise. Now, one of the possible methods to keep track of these immanants is to count 2-colorings of the Kauffman diagrams, where each edge connects vertices of opposite colors. We show this via the following example (see [23] for more details).

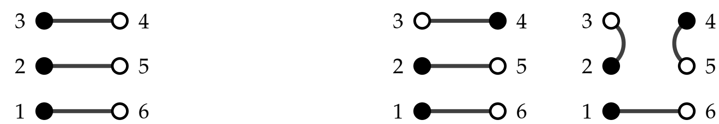

Figure 1: Kauffman diagrams τ_1; τ_1 and τ_2, respectively, as refered to in Example 2.3.

Example 2.3. Suppose we aim to verify if

$$\Delta_{(1,2,4)}\Delta_{(3,5,6)} - \Delta_{(1,2,3)}\Delta_{(4,5,6)} \geq 0 \quad \text{over} \quad \text{Gr}^{\geq 0}(3,6). \tag{2.10}$$

For this, we make the following replacement/identification in (2.10):

$$\Delta_{(1,2,3)}\Delta_{(4,5,6)} \leftrightarrow \text{Imm}_{\tau_1}(\mathbf{x}), \text{ and } \Delta_{(1,2,4)}\Delta_{(3,5,6)} \leftrightarrow \text{Imm}_{\tau_1}(\mathbf{x}) + \text{Imm}_{\tau_2}(\mathbf{x}),$$

where the diagrams τ_1 and τ_2 are shown in Figure 1, and the vertices corresponding to the sets I and I^c are colored in black and white, respectively. The first coloring admits only one bi-colored matching τ_1, while the second coloring admits exactly two bi-colored matchings τ_1 and τ_2. We substitute these replacements in (2.10) to get,

$$\Delta_{(1,2,4)}\Delta_{(3,5,6)} - \Delta_{(1,2,3)}\Delta_{(4,5,6)} \leftrightarrow \big(\text{Imm}_{\tau_1}(\mathbf{x}) + \text{Imm}_{\tau_2}(\mathbf{x})\big) - \text{Imm}_{\tau_1}(\mathbf{x}) = \text{Imm}_{\tau_2}(\mathbf{x}) \geq 0.$$

3 Weak separability and Plücker inequalities

In the theory of cluster algebras, one of the central examples is the cluster algebra of the Grassmannian $\text{Gr}(m, m+n)$ [22]. Clusters of minors are in correspondence with subsets of Plücker coordinates which can be described by set-theoretic properties. To recall, two m-element sets $I, J \subset [m+n]$ are weakly separated if $I \setminus J$ and $J \setminus I$ can be separated by a chord on the circle labeled with $1, 2, \ldots, m+n$ enumerated clockwise. A family of Plücker coordinates is said to be weakly separated if all pairs of Plücker coordinates in it are weakly separated [26]. It is known that maximal (by inclusion) weakly separated sets of Plücker coordinates are in bijection with the clusters consisting of Plücker coordinates in the cluster algebra of $\text{Gr}(m, m+n)$ [7]. This makes weak separability important in the theory of cluster algebras (for instance also see [19]). Now recall our main result: weak separability provides the exact classification of indices I, J for which the Plücker-type inequalities hold. We demonstrate one part of this via the next proof idea.

Proof idea for Theorem 1.3: weak separability $\implies$ the system of inequalities (1.5). First we consider the case when $m = n = \eta = 4$, the weakly separated (ordered) sets are $I = (1,2,3,4)$ and $J = (5,6,7,8)$, and $r = 4$, i.e. $i_r = i_4 = 4$. Thus we have:

$$I_{1,r} = (1,2,3,5),\ J_{1,r} = (4,6,7,8); \qquad I_{2,r} = (1,2,3,6),\ J_{2,r} = (5,4,7,8);$$
$$I_{3,r} = (1,2,3,7),\ J_{3,r} = (5,6,4,8); \qquad I_{4,r} = (1,2,3,8),\ J_{4,r} = (5,6,7,4).$$

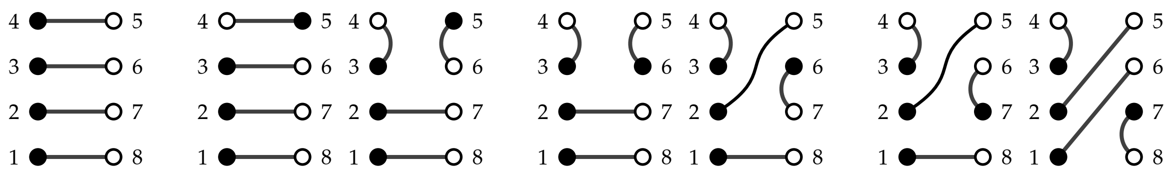

Figure 2: For (3.2): $\Delta_{I^\uparrow}\Delta_{J^\uparrow} \leftrightarrow \mathrm{Imm}_{\tau_0}(\mathbf{x})$, $\Delta_{I^\uparrow_{1,r}}\Delta_{J^\uparrow_{1,r}} \leftrightarrow \mathrm{Imm}_{\tau_0}(\mathbf{x}) + \mathrm{Imm}_{\tau_1}(\mathbf{x})$, $\Delta_{I^\uparrow_{2,r}}\Delta_{J^\uparrow_{2,r}} \leftrightarrow \mathrm{Imm}_{\tau_1}(\mathbf{x}) + \mathrm{Imm}_{\tau_2}(\mathbf{x})$, and $\Delta_{I^\uparrow_{3,r}}\Delta_{J^\uparrow_{3,r}} \leftrightarrow \mathrm{Imm}_{\tau_2}(\mathbf{x}) + \mathrm{Imm}_{\tau_3}(\mathbf{x})$, respectively.

The set of inequalities that we need to verify over $\mathrm{Gr}^{\geq 0}(4,8)$ are

$$\begin{aligned} \Delta_{I^\uparrow_{1,r}}\Delta_{J^\uparrow_{1,r}} - \Delta_{I^\uparrow}\Delta_{J^\uparrow} &\geq 0, \\ -\left(\Delta_{I^\uparrow_{1,r}}\Delta_{J^\uparrow_{1,r}} - \Delta_{I^\uparrow}\Delta_{J^\uparrow}\right) + \Delta_{I^\uparrow_{2,r}}\Delta_{J^\uparrow_{2,r}} &\geq 0, \\ \left(\Delta_{I^\uparrow_{1,r}}\Delta_{J^\uparrow_{1,r}} - \Delta_{I^\uparrow}\Delta_{J^\uparrow}\right) - \Delta_{I^\uparrow_{2,r}}\Delta_{J^\uparrow_{2,r}} + \Delta_{I^\uparrow_{3,r}}\Delta_{J^\uparrow_{3,r}} &\geq 0, \end{aligned} \tag{3.1}$$

and the last inequality is the Plücker relation itself. For this verification, we identify

$$\begin{aligned} \Delta_{I^\uparrow}\Delta_{J^\uparrow} &\leftrightarrow \mathrm{Imm}_{\tau_0}(\mathbf{x}), & \Delta_{I^\uparrow_{1,r}}\Delta_{J^\uparrow_{1,r}} &\leftrightarrow \mathrm{Imm}_{\tau_0}(\mathbf{x}) + \mathrm{Imm}_{\tau_1}(\mathbf{x}), \\ \Delta_{I^\uparrow_{2,r}}\Delta_{J^\uparrow_{2,r}} &\leftrightarrow \mathrm{Imm}_{\tau_1}(\mathbf{x}) + \mathrm{Imm}_{\tau_2}(\mathbf{x}), & \Delta_{I^\uparrow_{3,r}}\Delta_{J^\uparrow_{3,r}} &\leftrightarrow \mathrm{Imm}_{\tau_2}(\mathbf{x}) + \mathrm{Imm}_{\tau_3}(\mathbf{x}), \end{aligned} \tag{3.2}$$

where the corresponding Kauffman diagrams are shown in Figure 2. Indeed these inequalities are true since upon making immanant replacements in (3.1) we get nonnegative linear combinations of Temperley–Lieb immanants.

To show another possibility in the combinatorics of the general proof, we discuss another (slightly more complicated) example. Suppose $m = n = \eta = 5$, the weakly separated sets are $K = (1,2,3,4,10)$ and $L = (5,6,7,8,9)$. Therefore $(k_1,k_2,k_3,k_4,k_5) = (10,1,2,3,4)$ and $(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5) = (5,6,7,8,9)$. Fix $s := r = 3$, i.e. $k_s = 2$. We have:

$$\begin{aligned} &K_{1,s} = (1,5,3,4,10),\ L_{1,s} = (2,6,7,8,9); \quad K_{2,s} = (1,6,3,4,10),\ L_{2,s} = (5,2,7,8,9); \\ &K_{3,s} = (1,7,3,4,10),\ L_{3,s} = (5,6,2,8,9); \quad K_{4,s} = (1,8,3,4,10),\ L_{4,s} = (5,6,7,2,9); \\ &K_{5,s} = (1,9,3,4,10),\ L_{5,s} = (5,6,7,8,2). \end{aligned}$$

The set of inequalities that we need to verify over $\mathrm{Gr}^{\geq 0}(5,10)$ are:

$$\begin{aligned} \Delta_{K^\uparrow_{1,s}}\Delta_{L^\uparrow_{1,s}} &\geq 0, \\ -\Delta_{K^\uparrow_{1,s}}\Delta_{L^\uparrow_{1,s}} + \Delta_{K^\uparrow_{2,s}}\Delta_{L^\uparrow_{2,s}} &\geq 0, \\ \Delta_{K^\uparrow_{1,s}}\Delta_{L^\uparrow_{1,s}} - \Delta_{K^\uparrow_{2,s}}\Delta_{L^\uparrow_{2,s}} + \left(\Delta_{K^\uparrow_{3,s}}\Delta_{L^\uparrow_{3,s}} - \Delta_{K^\uparrow}\Delta_{L^\uparrow}\right) &\geq 0, \\ -\Delta_{K^\uparrow_{1,s}}\Delta_{L^\uparrow_{1,s}} + \Delta_{K^\uparrow_{2,s}}\Delta_{L^\uparrow_{2,s}} - \left(\Delta_{K^\uparrow_{3,s}}\Delta_{L^\uparrow_{3,s}} - \Delta_{K^\uparrow}\Delta_{L^\uparrow}\right) + \Delta_{K^\uparrow_{4,s}}\Delta_{L^\uparrow_{4,s}} &\geq 0, \end{aligned} \tag{3.3}$$

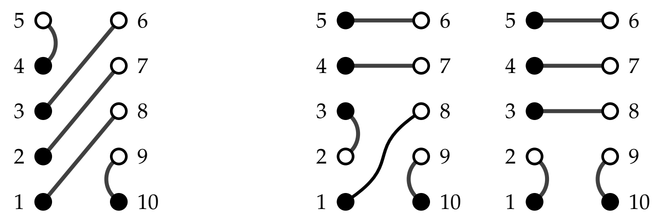

Figure 3: For (3.4): $\Delta_{K^\uparrow}\Delta_{L^\uparrow} \leftrightarrow \mathrm{Imm}_{\tau_0'}(\mathbf{x})$, and $\Delta_{K^\uparrow_{1,s}}\Delta_{L^\uparrow_{1,s}} \leftrightarrow \mathrm{Imm}_{\tau_1'}(\mathbf{x}) + \mathrm{Imm}_{\tau_2'}(\mathbf{x})$.

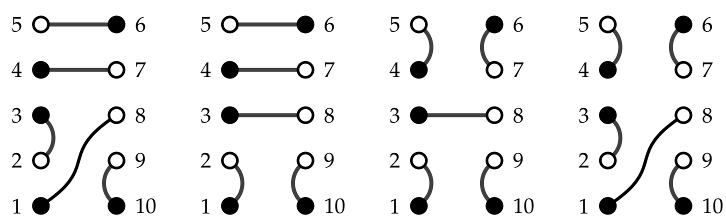

Figure 4: For (3.4): $\Delta_{K^\uparrow_{2,s}}\Delta_{L^\uparrow_{2,s}} \leftrightarrow \mathrm{Imm}_{\tau_1'}(\mathbf{x}) + \mathrm{Imm}_{\tau_2'}(\mathbf{x}) + \mathrm{Imm}_{\tau_3'}(\mathbf{x}) + \mathrm{Imm}_{\tau_4'}(\mathbf{x})$.

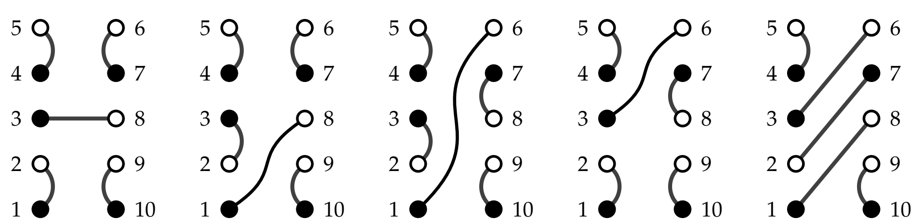

Figure 5: For (3.4): $\Delta_{K^\uparrow_{3,s}}\Delta_{L^\uparrow_{3,s}} \leftrightarrow \mathrm{Imm}_{\tau_3'}(\mathbf{x}) + \mathrm{Imm}_{\tau_4'}(\mathbf{x}) + \mathrm{Imm}_{\tau_5'}(\mathbf{x}) + \mathrm{Imm}_{\tau_6'}(\mathbf{x}) + \mathrm{Imm}_{\tau_0'}(\mathbf{x})$.

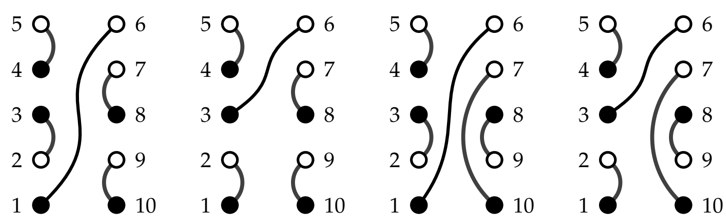

Figure 6: For (3.4): $\Delta_{K^\uparrow_{4,s}}\Delta_{L^\uparrow_{4,s}} \leftrightarrow \mathrm{Imm}_{\tau_5'}(\mathbf{x}) + \mathrm{Imm}_{\tau_6'}(\mathbf{x}) + \mathrm{Imm}_{\tau_7'}(\mathbf{x}) + \mathrm{Imm}_{\tau_8'}(\mathbf{x})$.

and the last inequality is the Plücker relation itself. We perform the replacements:

$$\begin{aligned}
\Delta_{K^\uparrow}\Delta_{L^\uparrow} &\leftrightarrow \mathrm{Imm}_{\tau'_0}(\mathbf{x}),\\
\Delta_{K^\uparrow_{1,s}}\Delta_{L^\uparrow_{1,s}} &\leftrightarrow \mathrm{Imm}_{\tau'_1}(\mathbf{x}) + \mathrm{Imm}_{\tau'_2}(\mathbf{x}),\\
\Delta_{K^\uparrow_{2,s}}\Delta_{L^\uparrow_{2,s}} &\leftrightarrow \mathrm{Imm}_{\tau'_1}(\mathbf{x}) + \mathrm{Imm}_{\tau'_2}(\mathbf{x}) + \mathrm{Imm}_{\tau'_3}(\mathbf{x}) + \mathrm{Imm}_{\tau'_4}(\mathbf{x}),\\
\Delta_{K^\uparrow_{3,s}}\Delta_{L^\uparrow_{3,s}} &\leftrightarrow \mathrm{Imm}_{\tau'_3}(\mathbf{x}) + \mathrm{Imm}_{\tau'_4}(\mathbf{x}) + \mathrm{Imm}_{\tau'_5}(\mathbf{x}) + \mathrm{Imm}_{\tau'_6}(\mathbf{x}) + \mathrm{Imm}_{\tau'_0}(\mathbf{x}),\\
\Delta_{K^\uparrow_{4,s}}\Delta_{L^\uparrow_{4,s}} &\leftrightarrow \mathrm{Imm}_{\tau'_5}(\mathbf{x}) + \mathrm{Imm}_{\tau'_6}(\mathbf{x}) + \mathrm{Imm}_{\tau'_7}(\mathbf{x}) + \mathrm{Imm}_{\tau'_8}(\mathbf{x}),\\
\Delta_{K^\uparrow_{5,s}}\Delta_{L^\uparrow_{5,s}} &\leftrightarrow \mathrm{Imm}_{\tau'_7}(\mathbf{x}) + \mathrm{Imm}_{\tau'_8}(\mathbf{x}),
\end{aligned} \tag{3.4}$$

where the corresponding Kauffman diagrams are shown in Figures 3 to 6. (We skipped the Kauffman diagrams for $\Delta_{K^\uparrow_{5,s}}\Delta_{L^\uparrow_{5,s}}$ as the last inequality is the Plücker relation.) One can see that these replacements yield nonnegative linear combinations of Temperley–Lieb immanants for each inequality in (3.3), as desired. □

Towards the general proof. Suppose we call $i_r = i_4 = 4$ and $k_s = k_3 = 2$ as the *moving indices* in the two examples in the proof idea above. Note that i_r is an end-point of $(i_1, \ldots, i_4)$, while k_s is not so in $(k_1, \ldots, k_5)$. There are a few things that one can observe:

1. The moving indices i_r, k_s provide the "mid-points" (1 for i_r and 3 for k_s) at which the term $\Delta_{I^\uparrow}\Delta_{J^\uparrow}$ and $\Delta_{K^\uparrow}\Delta_{L^\uparrow}$ appear in the systems of inequalities. In particular, the moving index is an end-point if and only if the mid-point is so. And in general it is given by $\eta - r + 1$ for the moving index given by i_r (see the main Theorem 1.3).

2. Suppose we disregard $\mathrm{Imm}_{\tau_0}(\mathbf{x})$ ($\leftrightarrow \Delta_{I^\uparrow}\Delta_{J^\uparrow}$) and $\mathrm{Imm}_{\tau'_0}(\mathbf{x})$ ($\leftrightarrow \Delta_{K^\uparrow}\Delta_{L^\uparrow}$) that refer to "no switch" by the moving indices, in this and (3) below. Then i_r being the end-point and k_s being not so – along with weak separability – causes each $\Delta_{I^\uparrow_{-,r}}\Delta_{J^\uparrow_{-,r}}$ to identify with the sum of either 1 or 2 immanants, and each $\Delta_{K^\uparrow_{-,s}}\Delta_{L^\uparrow_{-,s}}$ with that of 2 or 4. In general, it is $1, 2$ if the moving index is an end-point, and $2, 4$ otherwise.

3. Moreover, one can observe the telescoping phenomena as we successively verified the inequalities in (3.1) and (3.3), respectively via (3.2) and (3.4): the immanant(s) for $\Delta_{*^\uparrow_{1,-}}\Delta_{*^\uparrow_{1,-}}$ constitute a half of $\Delta_{*^\uparrow_{2,-}}\Delta_{*^\uparrow_{2,-}}$; the other half of $\Delta_{*^\uparrow_{2,-}}\Delta_{*^\uparrow_{2,-}}$ match half of those in $\Delta_{*^\uparrow_{3,-}}\Delta_{*^\uparrow_{3,-}}$; the other half of $\Delta_{*^\uparrow_{3,-}}\Delta_{*^\uparrow_{3,-}}$ match half of those in $\Delta_{*^\uparrow_{4,-}}\Delta_{*^\uparrow_{4,-}}$; ...

The proof in the general case demands a bit more work. However, because of the weak separability, the story about the mid-point, the number of immanants (1, 2 or 2, 4), and the telescoping phenomenon – all remain the same. In fact, if one goes back to Gantmacher–Krein [10] and its various refinements in [5], it is this telescoping which is at the center of the theory of oscillating inequalities, and which we completely capture in the main Theorem 1.3 as Plücker inequalities. □

To conclude here, we refer the reader to [23] for the detailed general proof, and the proof of the converse, i.e., *the system of inequalities* (1.5) $\implies$ *weak separability of* I, J.

Acknowledgements

We thank the American Institute of Mathematics (AIM, CalTech Campus) for their stimulating environment during a workshop on Theory and Applications of Total Positivity, held in July 2023. This allowed us to interact fruitfully with several other researchers which led to the beginning of this work. In particular, we thank Projesh Nath Choudhury, Dominique Guillot, Himanshu Gupta, Chi-Kwong Li, and Shahla Nasserasr. We also thank Michael Gekhtman for helpful discussions, and Apoorva Khare for suggesting expository improvements. We sincerely appreciate Steven Karp for writing to us and providing feedback that helped in further improving the work. We also thank Pavlo Pylyavskyy and Shaun Fallat for useful conversations. Finally, we thank the two anonymous referees for their helpful feedback on the paper.

References

[1] M. Aissen, I. J. Schoenberg, and A. M. Whitney. "On the generating functions of totally positive sequences. I". *J. Analyse Math.* **2** (1952), pp. 93–103. DOI.

[2] T. Ando. "Totally positive matrices". *Linear Algebra Appl.* **90** (1987), pp. 165–219. DOI.

[3] A. Berenstein and A. Zelevinsky. "Total positivity in Schubert varieties". *Comment. Math. Helv.* **72**.1 (1997), pp. 128–166. DOI.

[4] S. M. Fallat and C. R. Johnson. *Totally nonnegative matrices*. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2011, xvi+248 pp. DOI.

[5] S. M. Fallat and P. K. Vishwakarma. "Inequalities for totally nonnegative matrices: Gantmacher-Krein, Karlin, and Laplace". *Linear Algebra Appl.* **694** (2024), pp. 360–413. DOI.

[6] C. K. Fan and R. M. Green. "Monomials and Temperley-Lieb algebras". *J. Algebra* **190**.2 (1997), pp. 498–517. DOI.

[7] M. Farber and A. Postnikov. "Arrangements of equal minors in the positive Grassmannian". *Adv. Math.* **300** (2016), pp. 788–834. DOI.

[8] M. Fekete and G. Pólya. "Über ein Problem von Laguerre". *Rend. Circ. Matem. Palermo* **34** (1912), pp. 89–120. DOI.

[9] S. Fomin and A. Zelevinsky. "Double Bruhat cells and total positivity". *J. Amer. Math. Soc.* **12**.2 (1999), pp. 335–380. DOI.

[10] F. P. Gantmacher and M. G. Krein. *Oscillation matrices and kernels and small vibrations of mechanical systems*. Revised edition. AMS Chelsea Publishing, 2002, viii+310 pp. DOI.

[11] "Total positivity and its applications". *Proceedings of the meeting held in Jaca, September 26–30, 1994*. Ed. by M. Gasca and C. A. Micchelli. Vol. 359. Math. Appl. Kluwer Academic Publishers Group, Dordrecht, 1996, x+518 pp. DOI.

[12] M. Gekhtman, M. Shapiro, and A. Vainshtein. *Cluster algebras and Poisson geometry*. Vol. 167. Math. Surveys Monogr. American Mathematical Society, Providence, RI, 2010, xvi+246 pp. DOI.

[13] F. M. Goodman, P. de la Harpe, and V. F. R. Jones. *Coxeter graphs and towers of algebras*. Vol. 14. Mathematical Sciences Research Institute Publications. Springer-Verlag, New York, 1989, x+288 pp. DOI.

[14] S. Karlin. *Total positivity. Vol. I*. Stanford University Press, Stanford, CA, 1968, xii+576 pp. DOI.

[15] L. H. Kauffman. "State models and the Jones polynomial". *Topology* **26**.3 (1987), pp. 395–407. DOI.

[16] D. E. Littlewood. *The theory of group characters and matrix representations of groups*. Reprint of the second (1950) edition. AMS Chelsea Publishing, Providence, RI, 2006, viii+314 pp. DOI.

[17] G. Lusztig. "Total positivity in reductive groups". *Lie theory and geometry*. Vol. 123. Progr. Math. Birkhäuser Boston, Boston, MA, 1994, pp. 531–568. DOI.

[18] G. Lusztig. "Introduction to total positivity". *Positivity in Lie theory: open problems*. Vol. 26. De Gruyter Exp. Math. de Gruyter, Berlin, 1998, pp. 133–145. DOI.

[19] S. Oh, A. Postnikov, and D. E. Speyer. "Weak separation and plabic graphs". *Proc. Lond. Math. Soc. (3)* **110**.3 (2015), pp. 721–754. DOI.

[20] A. Postnikov. "Total positivity, Grassmannians, and networks". 2006. arXiv:math/0609764.

[21] B. Rhoades and M. Skandera. "Temperley-Lieb immanants". *Ann. Comb.* **9**.4 (2005), pp. 451–494. DOI.

[22] J. S. Scott. "Grassmannians and cluster algebras". *Proc. London Math. Soc. (3)* **92**.2 (2006), pp. 345–380. DOI.

[23] D. Soskin and P. K. Vishwakarma. "Plücker inequalities for weakly separated coordinates in totally nonnegative Grassmannians". 2023. arXiv:2310.12916.

[24] R. P. Stanley. "Positivity problems and conjectures in algebraic combinatorics". *Mathematics: frontiers and perspectives*. Amer. Math. Soc., Providence, RI, 2000, pp. 295–319.

[25] B. W. Westbury. "The representation theory of the Temperley-Lieb algebras". *Math. Z.* **219**.4 (1995), pp. 539–565. DOI.

[26] L. K. Williams. "The positive Grassmannian, the amplituhedron, and cluster algebras". *ICM—International Congress of Mathematicians. Vol. 6. Sections 12–14*. EMS Press, Berlin, [2023] ©2023, pp. 4710–4737.