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Quasisymmetric Divided Differences

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Abstract. We develop a quasisymmetric analogue of the combinatorial theory of Schubert polynomials. Indexed binary forests play the role of permutations, and the associated divided difference operators compose according to the "Thompson monoid" governing the Thompson group. Our main application of our theory is to study the ring of quasisymmetric coinvariants and the associated quasisymmetric harmonics spaces. In followup work we describe an algebraic-geometric framework which matches the combinatorial theory.

Résumé. Nous développons un analogue quasisymétrique de la théorie combinatoire des polynômes de Schubert. Les forêts binaires indexés jouent le rôle des permutations, et les opérateurs de différences divisées associés se composent comme le « monoïde de Thompson » correspondant au groupe de Thompson. La principale application de notre théorie est l'étude de l'anneau de covariants quasi-symétriques et des espaces harmoniques quasi-symétriques associés. Dans un travail ultérieur, nous décrirons le pendant en géométrie algébrique de cette théorie combinatoire.

Keywords: Quasisymmetry, Schubert calculus

1 Introduction

The ring of quasisymmetric functions QSym, first introduced in Stanley's thesis [23] and further developed by Gessel [11], is ubiquitous throughout combinatorics; see [1] for a high-level explanation and [12] for thorough exposition. Truncating to finitely many variables $\{x_1, \dots, x_n\}$ gives the ring of *quasisymmetric polynomials* $\mathrm{QSym}_n \subset \mathbb{Z}[x_1, \dots, x_n]$: these are the polynomials such that for any sequence $a_1, \dots, a_k \geq 1$, the coefficients of $x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}$ are equal whenever $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$.

We let $\mathrm{Pol}_n := \mathbb{Z}[x_1, \dots, x_n]$. Letting Sym_n^+ denote the ideal in Pol_n generated by positive degree homogenous symmetric polynomials, the coinvariant algebra $\mathrm{Coinv}_n := \mathrm{Pol}_n \,/\, \mathrm{Sym}_n^+$ has been a central object of study for the past several decades. An important

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reason for this is its distinguished basis of Schubert polynomials [15] and the divided difference operators [6] that interact nicely with this family– see [4, 7, 8, 9, 10, 13, 14, 17] for a sampling of the combinatorics underlying this story. In fact Schubert polynomials lift to a basis of Pol $:= \mathbb{Z}[x_1, x_2, \dots]$. The close relationship between the combinatorics of symmetric and quasisymmetric polynomials leads to the natural question, first posed in [2], of what can be said about the analogous quotient $\mathrm{QSCoinv}_n := \mathrm{Pol}_n / \mathrm{QSym}_n^+$, where QSym_n^+ is the ideal generated by positive degree homogenous quasisymmetric polynomials?

In this paper we develop a quasisymmetric analogue of the **combinatorial theory** of Schubert polynomials $\mathfrak{S}_w$ and the divided differences $\partial_i = \frac{f - s_i \cdot f}{x_i - x_{i+1}}$ which recursively generate them. The reader well-versed with the classical story should refer to Table 2 for a comparison. The role of Schubert polynomials $\mathfrak{S}_w$ is played by the *forest polynomials* $\mathfrak{P}_F$ of [21], and the role of the ∂_i operators are played by certain new *trimming operators* T_i. Just as Schubert polynomials generalize Schur polynomials, the forest polynomials generalize fundamental quasisymmetric polynomials, a distinguished basis of QSym_n. The duality between compositions of trimming operators and forest polynomials allows us to expand any polynomial in the basis of forest polynomials. In fact, a special case of our framework gives a remarkably simple method for directly extracting the coefficients of the expansion of a quasisymmetric polynomial in the basis of fundamental quasisymmetric polynomials.

The interaction between forest polynomials and trimming operators descends nicely to quotients by QSym_n^+, and we thus obtain a basis comprising certain forest polynomials for $\mathrm{QSCoinv}_n$ as well. Our techniques are robust enough to gain a complete understanding even in the case one quotients by homogenous quasisymmetric polynomials of degree at least k for any $k \geq 1$. **In the longer form version of this paper** [18], we use this theory to construct QSym_n^+-harmonics, which turn out to have a basis given by the volume polynomials of certain polytopes, answering a question of Aval–Bergeron–Li [3]. Therein we also generalize the theory to m-ary forests.

Remark 1.1. In Section 8, see Table 1 for a list of small forest polynomials by forest code $c(F)$ and Table 2 for a side by side comparison of our theory to the symmetric story.

In [20] we investigate the underlying **geometric theory**, drawing upon the geometric significance of the ordinary divided difference operator.

2 Quasisymmetric polynomials via R_i and T_i

The following result is at the heart of our understanding of quasisymmetric polynomials. This characterization does not seem to be widely known, although it was implicitly used in the study of the connection between quasisymmetric functions and James spaces by

Pechenik–Satriano [22]. We call this the *Bergeron–Sottile map* as they were the first to introduce this [4] in the context of Schubert calculus (see also [5, 16]).

Definition 2.1. For $f \in \mathrm{Pol}_n$ and $i \in \{1, \dots, n\}$ we define the i'th *Bergeron–Sottile map*

$$\mathsf{R}_i f = f(x_1, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1}). \tag{2.1}$$

In other words, R_i sets $x_i = 0$ and shifts $x_j \mapsto x_{j-1}$ for all $j \geq i+1$. Note that R_i is well-defined on Pol. The following is then an elementary reformulation of quasisymmetry that allows us to work with the condition algebraically.

Theorem 2.2. Let $f \in \mathrm{Pol}_n$. Then $f \in \mathrm{QSym}_n$ if and only if $\mathsf{R}_1 f = \cdots = \mathsf{R}_n f$.

The typical proof that QSym_n is closed under multiplication involves identifying an explicit basis whose multiplication can be explicitly computed. With R_i this is immediate as the equalizer of the algebra maps $\mathsf{R}_i, \mathsf{R}_{i+1}$ is a subring of Pol_n, and the intersection of subrings is a subring.

Corollary 2.3. QSym_n is a subring of Pol_n.

Using these operators we can define our quasisymmetric divided difference operator, which we also dub *trimming* operator for reasons that will become clear shortly.

Definition 2.4. We define the operator $\mathsf{T}_i : \mathrm{Pol} \to \mathrm{Pol}$ by any of the equivalent expressions

$$\mathsf{T}_i f := \mathsf{R}_i \partial_i f = \mathsf{R}_{i+1} \partial_i f = \frac{\mathsf{R}_{i+1} f - \mathsf{R}_i f}{x_i}. \tag{2.2}$$

This is the quasisymmetric divided difference at the core of this work. Because $\mathsf{T}_i(f) = 0$ if and only if $\mathsf{R}_{i+1}(f) = \mathsf{R}_i(f)$, we deduce the following.

Theorem 2.5. $f \in \mathrm{Pol}_n$ is quasisymmetric if and only if $\mathsf{T}_1 f = \cdots = \mathsf{T}_{n-1} f = 0$.

3 Forests and the Thompson monoid

A *binary tree* T has internal nodes $v \in \mathrm{IN}(T)$ that have two children v_L and v_R, and leaf nodes that have no children. We define $|T| = |\mathrm{IN}(T)|$, and let $*$ be the unique tree with $|*| = 0$. An *indexed forest* F is an infinite sequence $T_1, T_2, \dots$ of binary trees where all but finitely many of the trees are $*$. We write For for the set of all indexed forests.

There is a monoid structure on For where we define $F \cdot G \in \mathsf{For}$ by identifying the ith leaf of F with the ith root node of G. The empty forest $\varnothing \in \mathsf{For}$ is the identity element.

Definition 3.1. The Thompson monoid is the monoid

$$\mathsf{ThMon} := \langle 1, 2, \dots \mid i \cdot j = j \cdot (i+1) \text{ for all } i > j \rangle.$$

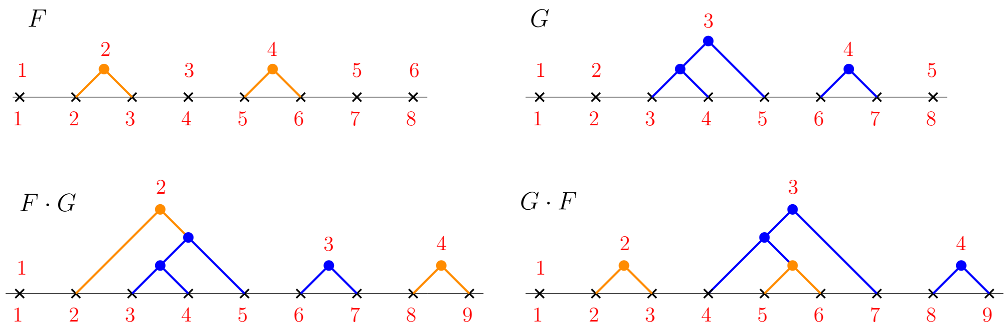

Figure 1: The products $F \cdot G$ and $G \cdot F$ for $F, G \in \mathsf{For}$, with both roots and leaves labeled

This monoid is both left and right cancellative, and its fraction group is naturally identified with Thompson's group F which has the standard presentation

$$\langle x_1, x_2, \ldots \mid x_j^{-1} x_i x_j = x_{i+1} \text{ for } i > j \rangle$$

via the map $i \mapsto x_i^{-1}$. It is a folklore fact in the theory of Thompson groups that $\mathsf{ThMon} \cong \mathsf{For}$ under the map sending

$$i \mapsto \underline{i} := \underbrace{* \cdots *}_{i-1} \cdot \wedge \cdot * \cdot * \cdots$$

where $\wedge$ is the unique binary tree with $|\wedge| = 1$.

Theorem 3.2. The map $i \mapsto \mathsf{T}_i$ is a representation of the Thompson monoid on Pol. Equivalently,

$$\mathsf{T}_i \mathsf{T}_j = \mathsf{T}_j \mathsf{T}_{i+1} \text{ for all } i > j.$$

Because of this, it makes sense to define T_F for $F \in \mathsf{For}$ by declaring

$$\mathsf{T}_F := \mathsf{T}_{i_1} \cdots \mathsf{T}_{i_k} \text{ for any factorization } F = \underline{i_1} \cdots \underline{i_k}.$$

For a permutation w the descent set is the set of all i such that $w(i) > w(i+1)$. The analogue for forests is the *left terminal set* $\mathrm{LTer}(F)$. We call an internal node $v \in \mathrm{IN}(F)$ *terminal* if both of v's children are leaves, and we say that $i \in \mathrm{LTer}(F)$ if there is a terminal node with left child i. This happens if and only if we can write $F = (F/i) \cdot i$ for a (necessarily unique) forest F/i, and so

$$\mathrm{LTer}(F) = \{i \mid F/i \text{ exists}\}.$$

A reduced word for w is a sequence $(i_1, \ldots, i_{\ell(w)})$ such that $w = s_{i_1} \cdots s_{i_{\ell(w)}}$. Analogously, the set of *trimming sequences* for F is defined as

$$\mathrm{Trim}(F) := \{(i_1, \ldots, i_{|F|}) \mid F = \underline{i_1} \cdots \underline{i_{|F|}}\}.$$

Finally, we note that there is a *code map* $c(F) = (a_1, a_2, \ldots)$ where a_i is the number of internal nodes whose leftmost leaf descendent is the i'th leaf. Then $i \in \mathrm{LTer}(F)$ is equivalent to $a_i > 0$ and $a_{i+1} = 0$, and $F \mapsto F/i$ is given by $(a_1, \ldots, a_{i-1} a_i, 0, a_{i+1}, \ldots) \mapsto (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots)$.

We will list small forest polynomials by code in Section 8.

4 Forest polynomials and Trimming operations

Let $S_\infty = \bigcup S_n$, the set of permutations of $\mathbb{N}$ with all but finitely many points fixed. We recall that Schubert polynomials are the unique family of homogenous polynomials indexed by S_∞ such that $\mathfrak{S}_{id} = 1$ and $\partial_i \mathfrak{S}_w = \delta_{i \in \mathrm{Des}(w)} \mathfrak{S}_{ws_i}$. They form a basis of Pol, those with $\mathrm{Des}(w) \subset \{1, \ldots, n\}$ are a $\mathbb{Z}$-basis for Pol_n; those with $\mathrm{Des}(w) \subset \{1, \ldots, n\}$ and $w \notin S_n$ are a $\mathbb{Z}$-basis for Sym_n^+; and the Schubert polynomials with $w \in S_n$ are a $\mathbb{Z}$-basis for $\mathrm{Coinv}_n := \mathrm{Pol}_n \,/\, \mathrm{Sym}_n^+$. Using our theory we arrive at analogous conclusions.

Theorem 4.1. There is a unique family of homogenous polynomials $\left(\mathfrak{P}_F\right)_F$ indexed by $F \in \mathsf{For}$ such that $\mathfrak{P}_\varnothing = 1$ and

$$\mathsf{T}_i \mathfrak{P}_F = \delta_{i \in \mathrm{LTer}(F)} \mathfrak{P}_{F/i}.$$

These are the **forest polynomials** of the first and third authors [21]. Furthermore,

- The forest polynomials are a $\mathbb{Z}$-basis for Pol.
- The forest polynomials with $\mathrm{LTer}(F) \subset \{1, \ldots, n\}$ are a $\mathbb{Z}$-basis for Pol_n.
- The forest polynomials with $\mathrm{LTer}(F) \subset \{1, \ldots, n\}$ and $\operatorname{supp} F \not\subset \{1, \ldots, n\}$ are a $\mathbb{Z}$-basis for QSym_n^+.
- The forest polynomials with $\operatorname{supp} F \subset \{1, \ldots, n\}$ are a $\mathbb{Z}$-basis for $\mathrm{QSCoinv}_n = \mathrm{Pol}_n \,/ \mathrm{QSym}_n^+$.

Corollary 4.2. Every $f \in \mathrm{Pol}$ can be uniquely written as

$$f = \sum_{F \in \mathsf{For}} a_F \mathfrak{P}_F \quad \text{where} \quad a_F = (\mathsf{T}_F f)(0, 0, \ldots).$$

Remark 4.3. The monomial expansion of forest polynomials doesn't play any role in our theory, and no simple seed generates all forest polynomials unlike with Schubert polynomials. For an explicit definition see [21], and [19] for further discussion, and see Section 8 for a list of forest polynomials.

5 Algebraically decomposing QSym_n into fundamentals

Forests F with $\mathrm{LTer}(F) \subset \{n\}$ play an analogous role to the n-Grassmannian permutations (those permutations with $\mathrm{Des}(w) \subset \{n\}$) which index the Schur polynomials s_λ. We call these forests *Zigzag forests*, and denote the set of them ZigZag_n. For an integer sequence $a = (a_1, \dots, a_k)$ with $a_i \geq 1$ we define the set of *compatible sequences*

$$\mathcal{C}(a) := \{(i_1, \dots, i_k) : a_j \geq i_j \geq i_{j+1}, \text{ and if } a_j > a_{j+1} \text{ then } i_j > i_{j+1}\}.$$

Given a sequence $\mathbf{i} = (i_1, \dots, i_k)$ we denote $\mathsf{x}_{\mathbf{i}} := x_{i_1} \cdots x_{i_k}$. Let QSeq_n be the set of sequences $(a_1, \dots, a_k)$ satisfying $n = a_1 \geq \cdots \geq a_k \geq 1$ and $a_i - a_{i+1} \leq 1$ for $1 \leq i \leq k-1$. If $(a_1, \dots, a_k) \in \mathrm{QSeq}_n$ then

$$\mathfrak{F}_a = \sum_{\mathbf{i} \in \mathcal{C}^m(a)} \mathsf{x}_{\mathbf{i}}.$$

is called a *fundamental quasisymmetric polynomial*.

Theorem 5.1. The forest polynomials with $F \in \mathsf{ZigZag}_n$ are a $\mathbb{Z}$-basis for QSym_n, and coincide with the fundamental quasisymmetric polynomials: the mapping $(a_1, \dots, a_k) \mapsto F = a_k \cdots a_1$ is a bijection $\mathrm{QSeq}_n \to \mathsf{ZigZag}_n$ under which we have $\mathfrak{F}_a = \mathfrak{P}_F$.

Example 5.2. Suppose we want to decompose the quasisymmetric polynomial $f(x_1, x_2, x_3) = 2x_1^2x_2 + 2x_1^2x_3 + 2x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 \in \mathrm{QSym}_3$ into fundamental quasisymmetrics. Using Corollary 4.2, we track in Figure 2 the nonzero applications $\mathsf{T}_{i_3}\mathsf{T}_{i_2}\mathsf{T}_{i_1}f$ where $(i_1, i_2, i_3) \in \mathrm{QSeq}_3$, and read off $f = \mathfrak{F}_{332} + 2\mathfrak{F}_{322} - 3\mathfrak{F}_{321}$.

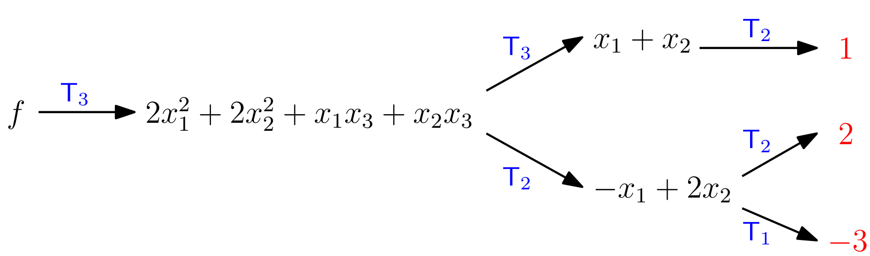

Figure 2: Trimming $f \in \mathrm{QSym}_3$

6 Diagrammatics

We define the *quasisymmetric nil-Hecke algebra* to be the subalgebra of $\mathrm{End}(\mathrm{Pol}_n)$ generated by multiplication by Pol, T_i, and R_i. Many of the relations between these operations are *formal commutation relations*. Write $\mathrm{Pol}_n = V^{\otimes n}$ where $V = \mathrm{Pol}_1$. Then we can write

$$\text{multiplication by } x_i = \mathrm{id}^{\otimes i-1} \otimes x \otimes \mathrm{id}^{\otimes n-i},$$
$$\mathsf{R}_i = \mathrm{id}^{\otimes i-1} \otimes \mathsf{R} \otimes \mathrm{id}^{\otimes n-i}, \text{ and}$$
$$\mathsf{T}_i = \mathrm{id}^{\otimes i-1} \otimes \mathsf{T} \otimes \mathrm{id}^{\otimes n-i-1},$$

where $x : \mathrm{Pol}_1 \to \mathrm{Pol}_1$ is multiplication by x, $\mathsf{R} : \mathrm{Pol}_1 \to \mathrm{Pol}_0 = \mathbb{Z}$ is the map $f \mapsto f(0)$, and $\mathsf{T} : \mathrm{Pol}_2 \to \mathrm{Pol}_1$ is the map $f(x,y) \mapsto \frac{f(z,0)-f(0,z)}{z}$.

We can represent these via diagrams which we should think of taking inputs on the bottom to outputs on the top, and compose two operations $F \circ G$ by stacking the diagram for F on top of the diagram for G. Representing R_i and T_i as in Figure 3, and putting a dot on a strand to represent multiplication by x, isotopy classes of diagrams give the same operations and encode all *formal commutation relations*. These are the Thompson monoid relations $\mathsf{T}_i\mathsf{T}_j = \mathsf{T}_j\mathsf{T}_{i+1}$ for $i > j$ as well as the relations $\mathsf{R}_i\mathsf{R}_j = \mathsf{R}_j\mathsf{R}_{i+1}$ for $i \geq j$, $\mathsf{T}_i\mathsf{R}_j = \mathsf{R}_j\mathsf{T}_{i+1}$ for $i \geq j+2$, $\mathsf{T}_i\mathsf{R}_j = \mathsf{R}_{j-1}\mathsf{T}_i$ for $i \leq j$, $x_ix_j = x_jx_i$, $\mathsf{T}_ix_j = x_j\mathsf{T}_i$ for $i \geq j+1$, $\mathsf{T}_ix_j = x_{j-1}\mathsf{T}_i$ for $i \leq j-2$, $\mathsf{R}_ix_j = x_j\mathsf{R}_i$ for $i \geq j+1$, and $\mathsf{R}_ix_j = x_{j-1}\mathsf{R}_i$ for $i \leq j-1$.

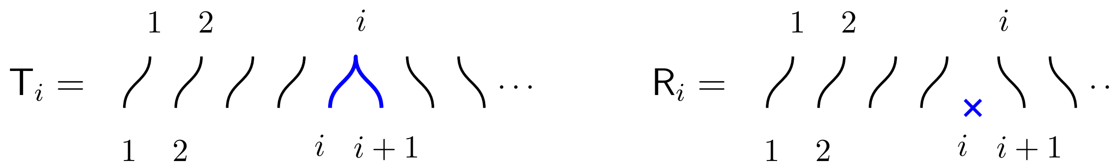

Figure 3: Diagram generators for the quasisymmetric nil-Hecke algebra

Theorem 6.1. The nontrivial relations in the quasisymmetric nil-Hecke algebra are $\mathsf{R}x = 0$, $\mathsf{T}(x \otimes \mathrm{id}) = \mathrm{id} \otimes \mathsf{R}$ and $\mathsf{T}(\mathrm{id} \otimes x) = -\mathsf{R} \otimes \mathrm{id}$, and $x\mathsf{T} = \mathrm{id} \otimes \mathsf{R} - \mathsf{R} \otimes \mathrm{id}$, which have the diagrammatics of Figure 4.

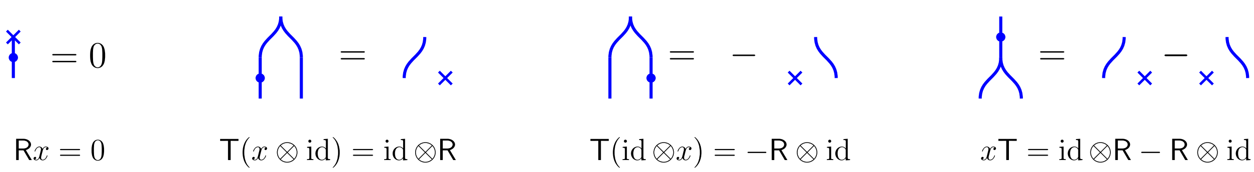

Figure 4: Diagram relations for the quasisymmetric nil-Hecke algebra

7 Positive multiplication and Schubert decompositions

Recall that for Schubert polynomials, it is wide open to establish the nonnegativity of the generalized Littlewood–Richardson coefficients arising in the expansion

$$\mathfrak{S}_u\mathfrak{S}_w = \sum_{v \in S_n} c^v_{u,w}\mathfrak{S}_v$$

in a combinatorial manner. The primary difficulty is that the Leibniz rule

$$\partial_i(fg) = \partial_i(f)g + (s_i \cdot f)\partial_i(g)$$

involves the non-Schubert positive operation $f \mapsto s_i \cdot f$. However, for forest polynomials certain straightening rules for these operations allow us to bypass these problems.

Lemma 7.1 (Twisted Leibniz rule)**.** For $f, g \in \mathrm{Pol}$ we have

$$\mathsf{T}_i(fg) = \mathsf{T}_i(f)\mathsf{R}_{i+1}(g) + \mathsf{R}_i(f)\mathsf{T}_i(g).$$

Theorem 7.2. We have

$$\begin{aligned} \mathsf{R}_i\mathfrak{P}_F &= \sum b^G_{i,F}\mathfrak{P}_G \text{ with } b^G_{i,F} \geq 0; \\ \mathfrak{P}_F\mathfrak{P}_G &= \sum c^H_{F,G}\mathfrak{P}_H \text{ with } c^H_{F,G} \geq 0; \\ \mathfrak{S}_w &= \sum_{F \in \mathsf{For}} a^F_w \mathfrak{P}_F \text{ with } a^F_w \geq 0. \end{aligned}$$

Proof sketch. Denote by $\mathrm{ev}_0 f = f(0, 0, \ldots)$, the constant term map. Suppose Φ is a composite of R_i and T_j operations in some order. The relations

$$\mathsf{T}_i\mathsf{R}_j = \begin{cases} \mathsf{R}_j\mathsf{T}_{i+1} & \text{if } i \geq j+2 \\ \mathsf{R}_{i+1}\mathsf{T}_i + \mathsf{R}_i\mathsf{T}_{i+1} & \text{if } i = j+1 \\ \mathsf{R}_{j-1}\mathsf{T}_i & \text{if } i \leq j \end{cases}$$

allow us to iteratively "move R_j to the left" in any composite it appears in, and then together with the relation $\mathrm{ev}_0 \mathsf{R}_j = \mathrm{ev}_0$ we conclude

$$\mathrm{ev}_0 \Phi = \sum d^\Phi_K \,\mathrm{ev}_0 \mathsf{T}_K \text{ with } d^\Phi_K \geq 0. \tag{7.1}$$

We see that $b^G_{i,F} = \mathrm{ev}_0 \mathsf{T}_G\mathsf{R}_i\mathfrak{P}_F \geq 0$ by applying (7.1) $\mathrm{ev}_0 \mathsf{T}_K\mathfrak{P}_G = \delta_{K,G}$. To see $c^H_{F,G} = \mathrm{ev}_0 \mathsf{T}_H(\mathfrak{P}_F\mathfrak{P}_G) \geq 0$, we apply induction on $|F| + |G|$ together with the twisted Leibniz rule and (7.1) on each of the resulting terms. Finally, by induction on $\ell(w)$ we have for any $i \in \mathrm{LTer}(F)$ that $a^F_w = \mathrm{ev}_0 \mathsf{T}_F\mathfrak{S}_w = \mathrm{ev}_0 \mathsf{T}_{F/i}\mathsf{T}_i\mathfrak{S}_w = \mathrm{ev}_0 \mathsf{T}_{F/i}\mathsf{R}_i\partial_i\mathfrak{S}_w = \delta_{i \in \mathrm{Des}(w)} \mathrm{ev}_0 \mathsf{T}_{F/i}\mathsf{R}_i\mathfrak{S}_{ws_i}$, and we conclude by induction and (7.1). □

Acknowledgements

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8 Tables

$c(F)$	$\mathfrak{P}_F$
$(0,0,0,0,0)$	1
$(1,0,0,0,0)$	x_1
$(0,1,0,0,0)$	x_1+x_2
$(0,0,1,0,0)$	$x_1+x_2+x_3$
$(0,0,0,1,0)$	$x_1+x_2+x_3+x_4$
$(2,0,0,0,0)$	x_1^2
$(1,1,0,0,0)$	x_1x_2
$(1,0,1,0,0)$	$x_1^2+x_1x_2+x_1x_3$
$(1,0,0,1,0)$	$x_1^2+x_1x_2+x_1x_3+x_1x_4$
$(0,2,0,0,0)$	$x_1^2+x_1x_2+x_2^2$
$(0,1,1,0,0)$	$x_1x_2+x_1x_3+x_2x_3$
$(0,1,0,1,0)$	$x_1^2+2x_1x_2+x_2^2+x_1x_3+x_2x_3+x_1x_4+x_2x_4$
$(0,0,2,0,0)$	$x_1^2+x_1x_2+x_2^2+x_1x_3+x_2x_3+x_3^2$
$(0,0,1,1,0)$	$x_1x_2+x_1x_3+x_2x_3+x_1x_4+x_2x_4+x_3x_4$
$(3,0,0,0,0)$	x_1^3
$(2,1,0,0,0)$	$x_1^2x_2$
$(2,0,1,0,0)$	$x_1^2x_2+x_1^2x_3$
$(2,0,0,1,0)$	$x_1^3+x_1^2x_2+x_1^2x_3+x_1^2x_4$
$(1,2,0,0,0)$	$x_1x_2^2$
$(1,1,1,0,0)$	$x_1x_2x_3$
$(1,1,0,1,0)$	$x_1^2x_2+x_1x_2^2+x_1x_2x_3+x_1x_2x_4$
$(1,0,2,0,0)$	$x_1^3+x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_2x_3+x_1x_3^2$
$(1,0,1,1,0)$	$x_1^2x_2+x_1^2x_3+x_1x_2x_3+x_1^2x_4+x_1x_2x_4+x_1x_3x_4$
$(0,3,0,0,0)$	$x_1^3+x_1^2x_2+x_1x_2^2+x_2^3$
$(0,2,1,0,0)$	$x_1^2x_2+x_1^2x_3+x_1x_2x_3+x_2^2x_3$
$(0,2,0,1,0)$	$x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_2x_3+x_2^2x_3+x_1^2x_4+x_1x_2x_4+x_2^2x_4$
$(0,1,2,0,0)$	$x_1x_2^2+x_1x_2x_3+x_1x_3^2+x_2x_3^2$
$(0,1,1,1,0)$	$x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4$
$(4,0,0,0,0)$	x_1^4
$(3,1,0,0,0)$	$x_1^3x_2$
$(3,0,1,0,0)$	$x_1^3x_2+x_1^3x_3$
$(3,0,0,1,0)$	$x_1^3x_2+x_1^3x_3+x_1^3x_4$
$(2,2,0,0,0)$	$x_1^2x_2^2$
$(2,1,1,0,0)$	$x_1^2x_2x_3$
$(2,1,0,1,0)$	$x_1^2x_2^2+x_1^2x_2x_3+x_1^2x_2x_4$
$(2,0,2,0,0)$	$x_1^2x_2^2+x_1^2x_2x_3+x_1^2x_3^2$
$(2,0,1,1,0)$	$x_1^2x_2x_3+x_1^2x_2x_4+x_1^2x_3x_4$
$(1,3,0,0,0)$	$x_1x_2^3$
$(1,2,1,0,0)$	$x_1x_2^2x_3$
$(1,2,0,1,0)$	$x_1x_2^2x_3+x_1x_2^2x_4$
$(1,1,2,0,0)$	$x_1x_2x_3^2$
$(1,1,1,1,0)$	$x_1x_2x_3x_4$

Table 1: Table of forest polynomials $\mathfrak{P}_F$.

	QSym_n	Sym_n
Divided differences	T_i	∂_i
Indexing combinatorics	$F \in \mathsf{For}$ Fully supported forests For_n Forest code $\mathsf{c}(F)$ Left terminal set $\mathrm{LTer}(F)$ F/i for $i \in \mathrm{LTer}(F)$ Trimming sequences $\mathrm{Trim}(F)$ Zigzag forests $Z \in \mathrm{ZigZag}_n$	$w \in S_\infty$ S_n Lehmer code $\mathrm{lcode}(w)$ Descent set $\mathrm{Des}(w)$ ws_i for $i \in \mathrm{Des}(w)$ Reduced words $\mathrm{Red}(w)$ Grassmannian permutations λ
Monoid	Thompson monoid	nilCoxeter monoid
Pol-**basis** **Composites**	Forest polynomials $\mathfrak{P}_F$ $\mathsf{T}_F = \mathsf{T}_{i_1} \cdots \mathsf{T}_{i_k}$ for $\mathbf{i} \in \mathrm{Trim}(F)$	Schuberts $\mathfrak{S}_w$ $\partial_w = \partial_{i_1} \cdots \partial_{i_k}$ for $\mathbf{i} \in \mathrm{Red}(w)$
Pol$_n$-**basis** **Duality**	$\{\mathfrak{P}_F \mid \mathrm{LTer}(F) \subset [n]\}$ $\mathrm{ev}_0\, \mathsf{T}_F \mathfrak{P}_G = \delta_{F,G}$	$\{\mathfrak{S}_w \mid \mathrm{Des}(w) \subset [n]\}$ $\mathrm{ev}_0\, \partial_w \mathfrak{S}_{w'} = \delta_{w,w'}$
Positive expansions	$\mathfrak{P}_F\mathfrak{P}_H = \sum c^G_{F,H}\mathfrak{P}_G,\ c^G_{F,H} \geq 0$	$\mathfrak{S}_u\mathfrak{S}_w = \sum c^v_{u,w}\mathfrak{S}_v,\ c^v_{u,w} \geq 0$
Invariant basis	Fundamental qsyms $\mathfrak{P}_Z$	Schur polynomials s_λ
Coinvariant basis **Coinvariant action**	$\{\mathfrak{P}_F \mid F \in \mathsf{For}_n\}$ $\mathsf{T}_i : \mathrm{QSCoinv}_n \to \mathrm{QSCoinv}_{n-1}$	$\{\mathfrak{S}_w \mid w \in S_n\}$ $\partial_i : \mathrm{Coinv}_n \to \mathrm{Coinv}_n$
Harmonic basis	Forest volume polynomials	Degree polynomials

Table 2: Comparing the symmetric and quasisymmetric stories

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