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Log-concavity and log-convexity via distributive lattices

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Abstract. We prove a result on order ideals in distributive lattices, called the Order Ideal Lemma. We indicate how the Order Ideal Lemma implies log-concavity and log-convexity of various sequences involving lattice paths, intervals in Young's lattice, order polynomials, specializations of Schur and Schur Q-functions, Lucas sequences, descent and peak polynomials of permutations, pattern avoidance, set partitions, and noncrossing partitions.

Keywords: Distributive lattice, lattice path, log-concavity, log-convexity, order ideal

1 Introduction

Let $(a_n)_{n\geq 0} = a_0, a_1, a_2, \ldots$ be a sequence of real numbers. The sequence is *log-concave* if

$$a_n^2 \geq a_{n-1}a_{n+1} \tag{1.1}$$

for all $n \geq 1$. A *log-convex* sequence is one satisfying

$$a_n^2 \leq a_{n-1}a_{n+1} \tag{1.2}$$

for all $n \geq 1$. Log-concave and log-convex sequences abound in combinatorics, algebra, and geometry. The purpose of the present work is to provide a new combinatorial tool for proving log-concavity and log-convexity using order ideals in distributive lattices.

Let us review some basic concepts from the theory of partially ordered sets (posets). All of our posets will be finite. A *lower order ideal* in a poset $(P, \preceq)$ is $I \subseteq P$ such that if $x \in I$ and $y \preceq x$ then $y \in I$. Similarly, an *upper order ideal* is $J \subseteq P$ satisfying $x \in J$ and $y \succeq x$ implies $y \in J$. We will use "order ideal" to refer to a subset which could be either. Say that poset L is a *lattice* if every pair $x, y \in L$ has a greatest lower bound or *meet*, $x \wedge y$, as well as a least upper bound or *join*, $x \vee y$. The lattice is *distributive* if it satisfies either of the two equivalent distributive laws that, for all $x, y, z \in L$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

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or

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

We can now state our fundamental result which we call the Order Ideal Lemma. It is an easy consequence of the FKG Inequality [4]. But since its proof involves concepts not needed in the rest of this abstract, we omit the demonstration and refer interested readers to a forthcoming paper [5] by the present authors for detailed proofs. For a set S we will use both $|S|$ and $\#S$ for its cardinality.

Lemma 1.1 (The Order Ideal Lemma)**.** *Let L be a distributive lattice and suppose that $I, J \subseteq L$ are ideals.*

(a) If I, J are both lower ideals or both upper ideals then

$$|I| \cdot |J| \leq |I \cap J| \cdot |L|.$$

(b) If one of I, J is a lower order ideal and the other is upper then

$$|I| \cdot |J| \geq |I \cap J| \cdot |L|.$$

Our general strategy for proving log-convexity of a sequence $(a_n)_{n\geq 0}$ will be to construct lattices L_n with $|L_n| = a_n$. If we can find inside L_{n+1} two lower order ideals I, J such that $|I| = |J| = a_n$ and $|I \cap J| = a_{n-1}$ then we will be done by part (a) of the Order Ideal Lemma. Similarly, part (b) can be used to prove log-concavity.

The rest of this abstract is structured as follows. In the next section we will use lattice paths to prove log-convexity of sequences involving the Catalan, Motzkin, and large Schröder numbers. We begin Section 3 by showing that for any poset, the sequence obtained by evaluating its (enriched) order polynomial at nonnegative integers is always log-concave. As a consequence, we obtain log-concavity of sequences of specializations of Schur and Schur Q-functions. Section 4 is dedicated to generalized Lucas sequences i.e. those satisfying $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. We show that any such sequence which has positive initial conditions alternates between satisfying (1.1) and (1.2). In Section 5 we consider sequences of descent (peak) polynomials. The focus of Section 6 is set partitions and we show log-concavity of sequences involving Stirling numbers of the second kind and Narayana numbers. The last section contains directions for future research.

Before continuing, we note that not all of our results are included in this extended abstract due to the page limit. For example, we have also used various intervals in Young's lattice to give log-concavity and log-convexity results. Some of these specialize to show that various sequences of binomial coefficients are log-concave. In addition, we have investigated various sequences related to permutations. In this extended abstract, We will usually only provide the construction of the corresponding distributive lattice for a given sequence, while the detailed proof will be omitted. Full details are available in [5].

2 Lattice paths

In this section we will show how lattice paths together with the Order Ideal Lemma can be used to give unified proofs of the log-convexity of the sequences of Catalan, Motzkin, and large Schröder numbers. We begin with a review of some basic definitions.

A *lattice path* is a sequence $P : p_0, p_1, \ldots, p_n$ of points in the integer lattice so $p_i \in \mathbb{Z}^2$ for all i. A *step* of P is the vector $[x_i, y_i]$ from p_{i-1} to p_i. An *up step* is a step $U = [1,1]$ and a *down step* is $D = [1,-1]$. Note that we use brackets for vectors and parentheses for points. A *Dyck path of semilength n* is a lattice path P satisfying

1. P starts at $p_0 = (0,0)$ and ends at $p_{2n} = (2n, 0)$,
2. P uses steps U and D, and no point on P has negative y-coordinate.

Let $\mathcal{D}_n = \{P \mid P \text{ is a Dyck path of semilength } n\}$. It is well known that the cardinality of $\mathcal{D}_n$ is the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{2.1}$$

We wish to turn $\mathcal{D}_n$ into a distributive lattice. If $P \in \mathcal{D}_n$ then we let $A(P)$ be the *area* of P which is the set of all points of $\mathbb{R}^2$ between P and the x-axis. We now define a partial order on $\mathcal{D}_n$ by

$$P \preceq Q \text{ if and only if } A(P) \subseteq A(Q). \tag{2.2}$$

We note that $\mathcal{D}_n$ is a distributive lattice. This follows from the fact that it is isomorphic to an interval in Young's lattice. It is also a consequence of a more general theorem of Ferrari and Pinzani [3] giving a criterion for a family of lattice paths ordered by (2.2) to be a distributive lattice. The next result follows from easy algebraic manipulations of (2.1), and there are other combinatorial proofs that the Catalan sequence is log-convex such as the one given by Sun and Wang [12]. But with the Order Ideal Lemma, the proof is combinatorial and will generalize to other families of paths for which closed-form formulae are not known.

Theorem 2.1. *The sequence $(C_n)_{n\geq 0}$ of Catalan numbers is log-convex.*

Proof. We begin with the distributive lattice $\mathcal{D}_{n+1}$ and note that $|\mathcal{D}_{n+1}| = C_{n+1}$. Let

$$I = \{P \in \mathcal{D}_{n+1} \mid P = UDP' \text{ for some translated Dyck path } P' \text{ of semilength } n\}.$$

It follows that I is a lower order ideal because if $P \in I$ and $Q \preceq P$ then (2.2) forces $Q = UDQ'$ for some Q'. Furthermore, we have an isomorphism of posets $I \cong \mathcal{D}_n$ given by $P = UDP' \mapsto P'$. Thus $|I| = C_n$.

Now consider

$$J = \{P \in \mathcal{D}_{n+1} \mid P = P'UD \text{ for some Dyck path } P' \text{ of semilength } n\}.$$

Similar considerations to those in the previous paragraph show that $|J| = C_n$. Furthermore

$$I \cap J = \{P \in \mathcal{D}_{n+1} \mid P = UDP'UD \text{ for some Dyck path } P' \text{ of semilength } n-1\}$$

so that $|I \cap J| = C_{n-1}$. Now applying part (a) of the Order Ideal Lemma gives

$$C_n^2 = |I| \cdot |J| \le |I \cap J| \cdot |L| = C_{n-1}C_{n+1}$$

finishing the proof. □

We now consider the Motzkin numbers. A *Motzkin path of length n* is a lattice path P which satisfies

1. P starts at $p_0 = (0,0)$ and ends at $p_n = (n,0)$,
2. P uses steps U, D, and horizontal $H = [1,0]$, and no point on P has negative y-coordinate.

Let $\mathcal{M}_n = \{P \mid P \text{ is a Motzkin path of length } n\}$ so that $|\mathcal{M}_n| = M_n$ is the nth *Motzkin number*. The set $\mathcal{M}_n$ ordered by (2.2) is a distributive lattice as demonstrated in [3]. Showing that the Motzkin sequence is log-convex is much like the proof of the previous theorem.

Theorem 2.2. *The sequence $(M_n)_{n\ge0}$ of Motzkin numbers is log-convex.*

Finally, we investigate the large Schröder numbers. A *Schröder path of semilength n* is a lattice path P satisfying

1. P starts at $p_0 = (0,0)$ and ends at $p_n = (n,0)$,
2. P uses steps U, D, and twice horizontal $T = [2,0]$, and no point on P has negative y-coordinate.

If we let $\mathcal{S}_n = \{P \mid P \text{ is a Schröder path of semilength } n\}$ then $|\mathcal{S}_n| = S_n$ is the nth *large Schröder number*. As usual, we order $\mathcal{S}_n$ using (2.2). However, this poset is not covered by the general theorem of [3], although they remark that it can be shown that the poset is a lattice. It is, in fact, distributive.

Lemma 2.3. *The poset $\mathcal{S}_n$ is a distributive lattice.*

The next result follows in the way to which we have become accustomed.

Theorem 2.4. *The sequence $(S_n)_{n\ge0}$ of Schröder numbers is log-convex.*

3 Order polynomials

The order polynomial of a labeled poset was introduced by Stanley in his thesis [10] and has since been shown to be a fundamental invariant. In this section we outline why the sequence of values of the order polynomial of any labeled poset is log-concave. This permits us to prove log-concavity of sequences formed by specializing the Schur function corresponding to any partition.

Let $(P, \preceq)$ be a poset on $[p]$. The reader should be sure to distinguish the use of $\preceq$ for the partial order on P and $\leq$ for the total order on the integers. A *P-partition with range* $[n]$ is a map $f : P \to [n]$ such that for all $x \prec y$ we have

1. $f(x) \leq f(y)$ (that is, f is order preserving), and
2. if $x > y$ then $f(x) < f(y)$.

Let $\mathcal{O}_P(n)$ denote the set of P-partitions with range n. The *order polynomial* of P is $\Omega_P(n) = \#\mathcal{O}_P(n)$.

Theorem 3.1 ([10])**.** *For any P on $[p]$ we have $\Omega_P(n)$ is a polynomial in n.*

We now turn $\mathcal{O}_p(n)$ into a poset by ordering P-partitions component-wise, that is, $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in P$. Our next result was proved in the special case that P is naturally labeled (that is, $x \prec y$ implies $x < y$) by Chan, Pak and Panova [2].

Theorem 3.2. *For any P on $[p]$, the sequence $(\Omega_P(n))_{n\geq 0}$ is log-concave.*

We now use the well-known connection between order polynomials and Schur functions to derive an interesting special case of the previous theorem. If λ is an integer partition then a *semistandard Young tableau (SSYT) of shape* λ is a filling of the boxes of λ with positive integers such that rows weakly increase left-to-right and columns strictly increase top-to-bottom. The partition $\lambda = (5, 3, 1)$ and a semistandard Young tableau T of that shape are displayed in the first row of Figure 1. We let (i, j) be the cell of λ in row i and column j where rows and columns are indexed as in a matrix. Given an SSYT of shape λ we denote by $T_{i,j}$ the element of T in box (i, j). In the tableau of Figure 1 we have $T_{2,3} = 5$. Let SSYT_λ denote the set of SSYT of shape λ. Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a set of variables indexed by the positive integers. The *Schur function* corresponding to λ is the generating function

$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathrm{SSYT}_\lambda} \prod_{(i,j)\in\lambda} x_{T_{i,j}}.$$

The Schur functions are symmetric and form an important basis for the algebra of symmetric functions. For more information about them, see the texts of Sagan [8] or Stanley [9].

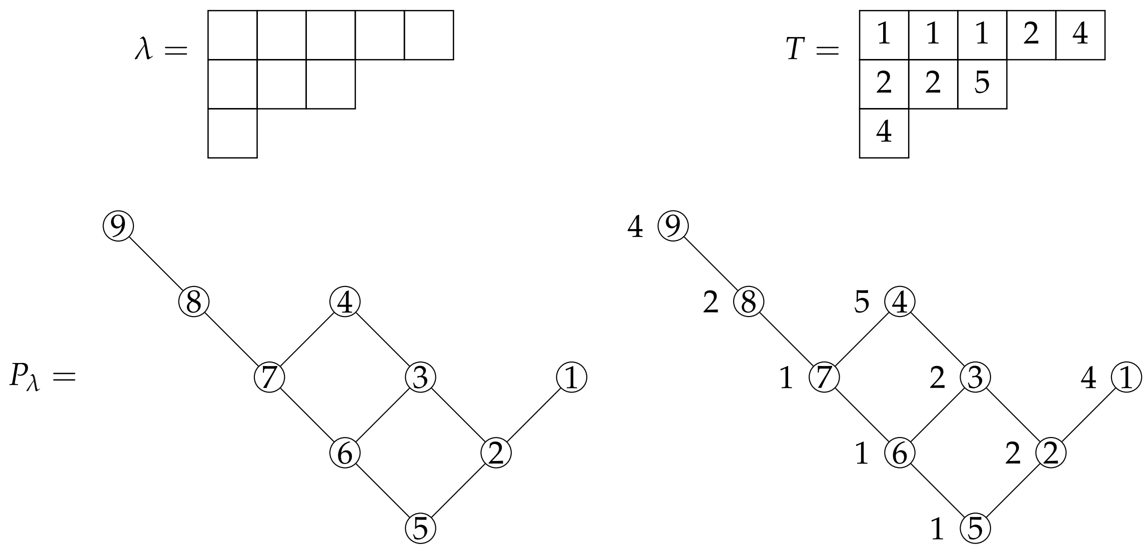

Figure 1: The shape $\lambda = (5,3,1)$, a semistandard Young tableau, T of that shape, as well as the corresponding poset P_λ and P_λ-partition.

To make the connection with P-partitions, we first turn $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ into a poset component-wise, that is, $(i,j) \preceq (i',j')$ if and only if $i \leq i'$ and $j \leq j'$. We now make this a poset P_λ on the interval $[m]$ where $m = \sum_l \lambda_l$ by labeling the last row of λ with $1, 2, \ldots, \lambda_k$ left-to-right (viewing λ as its original Young diagram). Then labeling the penultimate row left-to-right with $\lambda_k + 1, \lambda_k + 2, \ldots, \lambda_k + \lambda_{k-1}$, and so forth. This labeling is displayed in Figure 1 at the bottom left. It is easy to see that a P_λ-partition is the same as an SSYT of shape λ. The partition for the SSYT T in Figure 1 is displayed directly below the tableau. It should now be clear that we have $s_\lambda(1^n) = \Omega_{P_\lambda}(n)$ where 1^n indicates the specialization that $x_i = 1$ for $i \leq n$ and $x_i = 0$ for $i > n$.

As an immediate consequence of Theorem 3.2, we have the following result.

Corollary 3.3. *For any partition λ, the sequence $(s_\lambda(1^n))_{n \geq 0}$ is log-concave.*

We can also apply the Order Ideal Lemma to show that the sequences derived from the enriched order polynomials of Stembridge [11] are log-concave, again independent of the underlying poset. This gives rise to log-concave sequences of specializations of Schur Q-functions.

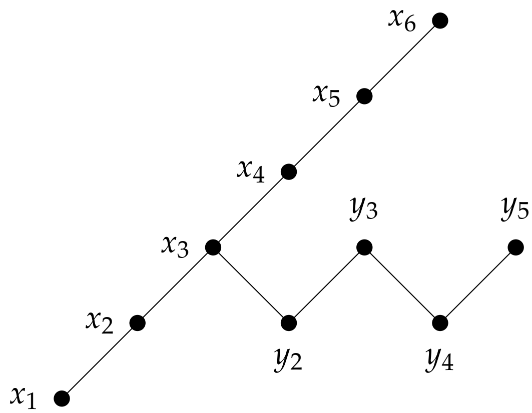

Figure 2: The poset $L_5(3,7)$.

4 Generalized Lucas sequences

A sequence $(l_n)_{n\geq 0}$ of real numbers is a *generalized Lucas sequence* if it satisfies the recursion

$$l_n = l_{n-1} + l_{n-2} \tag{4.1}$$

for $n \geq 2$. These sequences were originally studied by Lucas [6]. Of course, the two most famous examples of such sequences are the *Fibonacci numbers*, $(F_n)_{n\geq 0}$, and *(ordinary) Lucas numbers*, $(L_n)_{n\geq 0}$, with initial conditions $F_0 = F_1 = 1$ and $L_0 = 2$, $L_1 = 1$, respectively.

In this section we will study *positive Lucas sequences* which are generalized Lucas sequences with $l_0, l_1 > 0$. In order to state our result precisely, call a sequence $(a_n)_{n\geq 0}$ *log-concave at index n* if $a_n^2 \geq a_{n-1}a_{n+1}$. Note that this definition says nothing about indices other than n. Similarly define *log-convexity at index n*. We will show that any positive Lucas sequence, suitably reindexed, alternates between being log-concave at odd indices and log-convex at even ones. This generalizes a well-known result about Fibonacci numbers.

It will be convenient in our approach to restrict the initial values even further. To do this, we extend a generalized Lucas sequence to negative indices by insisting that the recurrence relation (4.1) continue to hold for $n < 0$ to give an *extended Lucas sequence* $(l_n)_{n\in\mathbb{Z}}$. Call two extended Lucas sequences $(l_n)_{n\in\mathbb{Z}}$ and $(l'_n)_{n\in\mathbb{Z}}$ *shift equivalent* if there is $k \in \mathbb{Z}$ such that $l_n = l'_{n+k}$ for all $n \in \mathbb{Z}$.

Proposition 4.1. *Suppose that $(l_n)_{n\geq 0}$ is a positive Lucas sequence. Then its extension is shift equivalent to an extended Lucas sequence $(l'_n)_{n\in\mathbb{Z}}$ such that* $0 < 2l'_0 \leq l'_1$.

We will now introduce the posets whose lattices of order ideals will permit us to study the behaviour of positive Lucas sequences $(l_n)_{n\geq 0}$. Say that such a sequence is *well-indexed* if $0 < 2l_0 \leq l_1$. Note that, by the previous proposition, every positive Lucas sequence is shift equivalent to a well-indexed one. To simplify notation, we will relabel $r := l_0$ and $s := l_1$. Define a poset $L_n(r,s)$ to have elements $x_1, \ldots, x_{s-1}$ and $y_2, \ldots, y_n$ and order relation $\preceq$ subject to the covers

1. $x_1 \prec x_2 \prec \ldots \prec x_{s-1}$,
2. $y_2 \prec y_3 \succ y_4 \prec y_5 \succ \ldots$, and
3. $y_2 \prec x_r$.

So the x_i form a chain C_{s-1} and the y_j form what we will call an *alternating poset* A_{n-1}. For example, Figure 2 shows the poset $L_5(3,7)$.

Let $J(P)$ denote the set of lower order ideals of a finite poset P. It is a fundamental result that $J(P)$ is a distributive lattice for any P. In particular, we can use the distributive lattice $J(L_n(r,s))$ to prove the following theorem.

Theorem 4.2. *A well-indexed Lucas sequence $(l_n)_{n\geq 0}$ is log-concave at odd indices and log-convex at even ones.*

5 Descent polynomials

We now prove that sequences of evaluations of descent polynomials are log-concave. The *descent set* of $\pi \in \mathfrak{S}_n$ is $\operatorname{Des}\pi = \{i \mid \pi_i > \pi_{i+1}\}$. Let S be any finite set of positive integers and consider $D_n(S) = \{\pi \in \mathfrak{S}_n \mid \operatorname{Des}\pi = S\}$ as well as $d_n(S) = \#D_n(S)$, where the latter is called the *descent polynomial* corresponding to S. The following is a classic result of MacMahon [7].

Theorem 5.1 ([7]). *For any set S and all $n > \max S$ we have that $d_n(S)$ is a polynomial in n.*

There are two standard partial orders on $\mathfrak{S}_n$: the weak and strong Bruhat orders. Recently, Bouvel, Ferrari and Tenner [1] defined a partial order which they call the middle order because it refines the weak order and is refined by the strong. This order has the advantage of being a distributive lattice and is built using inversions. In order to prove that $(d_n(S))_{n\geq 0}$ is always log-concave we will need a variant of the middle order which considers positions. The *positional inversion table* of $\pi \in \mathfrak{S}_n$ is $\kappa(\pi) = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ where $\kappa_i = \#\{j > i \mid \pi_j < \pi_i\}$. Clearly $0 \leq \kappa_i \leq n - i$ for all $i \in [n]$. Consider

$$\mathcal{K}_n = \{\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n) \mid 0 \leq \kappa_i \leq n - i \text{ for all } i \in [n]\}.$$

The map $\mathcal{K}_n \to \mathfrak{S}_n$ given by $\kappa \mapsto \pi$ where $\kappa(\pi) = \kappa$ is a bijection.

We now define the *κ-middle order* $(\mathfrak{S}_n, \trianglelefteq)$ by $\pi \trianglelefteq \sigma$ if and only if $\kappa(\pi) \leq \kappa(\sigma)$ component-wise. In this partial order we have

$$\mathfrak{S}_n \cong [0, n-1] \times [0, n-2] \times \cdots \times [0, 0],$$

where $[0, i] = \{0, 1, \ldots, i\}$ with the usual total order on the integers. It follows that this order is a distributive lattice. We define a partial order $(D_n(S), \trianglelefteq)$ by restricting the κ-middle order on $\mathfrak{S}_n$ to $D_n(S)$. We need to show that we still have a distributive lattice. In fact, we can show that $D_n(S)$ is a sublattice of $\mathfrak{S}_n$ under $\trianglelefteq$.

Lemma 5.2. *For any S the κ-middle order on $D_n(S)$ forms a distributive lattice.*

Theorem 5.3. *For any set S, the sequence $(d_n(S))_{n \geq 0}$ is log-concave.*

With a proof that closely parallels the one for the descent polynomial, one can show that sequences of evaluations of peak polynomials are log-concave.

6 Set partitions

For our last applications of the Order Ideal Lemma, we will indicate how to use set partitions and noncrossing set partitions to prove log-concavity results about Stirling numbers of the second kind and Narayana numbers. In both cases, it will be convenient to express the distributive lattices in terms of restricted growth functions.

6.1 Stirling numbers of the second kind

A *set partition of* $[n]$, $\beta = B_1/B_2/\ldots/B_k \vdash [n]$, is a family of disjoint subsets B_i called *blocks* whose disjoint union is $\uplus_i B_i = [n]$. In examples, we will eliminate the set braces and commas from the B_i. We will also always write our partitions in *standard form* which means that $1 = \min B_1 < \min B_2 < \ldots < \min B_k$. We let $S([n], k)$ be the set of partitions of $[n]$ with k blocks. The *Stirling numbers of the second kind* are $S(n, k) = \#S([n], k)$.

Set partitions are in bijection with certain sequences called restricted growth functions. A *restricted growth function* (RGF) is a sequence $\rho = \rho_1\rho_2\ldots\rho_n$ of positive integers satisfying

1. $\rho_1 = 1$, and
2. for $i \geq 2$ we have $\rho_i \leq 1 + \max(\rho_1\rho_2\ldots\rho_{i-1})$.

We call n the *length of* ρ and write $|\rho| = n$. We will use the notation $\mathrm{RGF}(n, k)$ for the set of RGFs ρ where $|\rho| = n$ and $\max \rho = k$.

There is a well-known bijection $S([n], k) \to \mathrm{RGF}(n, k)$ defined by sending a set partition $\beta = B_1/B_2/\ldots/B_k$ to $\rho = \rho_1\rho_2\ldots\rho_n$ where $\rho_i = j$ if and only if $i \in B_j$.

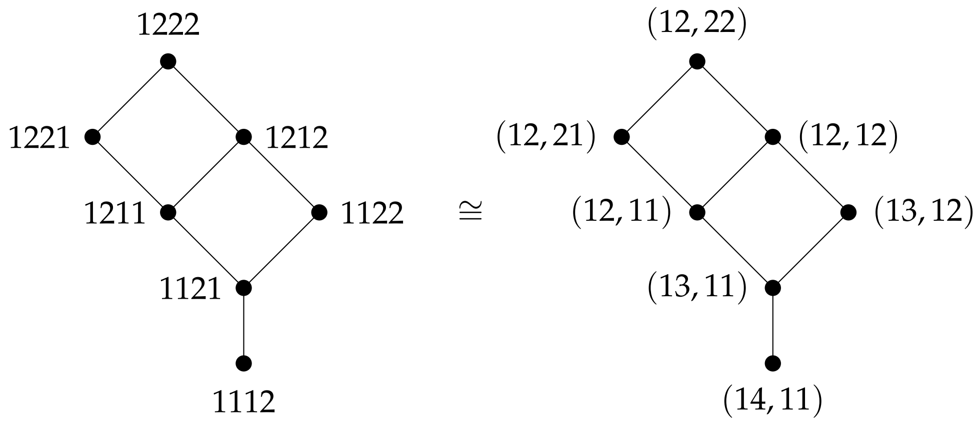

Figure 3: The partial order on RGF(4, 2) both in terms of RGFs and (F, R) pairs.

To describe the partial order on $\mathrm{RGF}(n,k)$ we will need two sequences. If $\rho \in \mathrm{RGF}(n,k)$ then its *sequence of first occurrences* (firsts) is $F(\rho) = f_1 f_2 \ldots f_k$ defined by $f_i = j$ where ρ_j is the first i in ρ. Note that since ρ is an RGF we always have $1 = f_1 < f_2 < \ldots < f_k$. We will also use the *rest of* ρ, denoted by $R(\rho)$, which is ρ with its first occurrences removed. Note that $|R(\rho)| = n - k$. Finally, we define a partial order $(\mathrm{RGF}(n,k), \preceq)$ by $\rho \preceq \tau$ if and only if $F(\rho) \geq F(\tau)$ and $R(\rho) \leq R(\tau)$, where the orders on F and R are component-wise. Figure 3 illustrates this order both on the restricted growth functions $\rho \in \mathrm{RGF}(4,2)$ on the left and on the pairs $(F(\rho), R(\rho))$ on the right.

Lemma 6.1. *The partial order* $(\mathrm{RGF}(n,k), \preceq)$ *is a distributive lattice.*

With this distributive lattice, we can show that a sequence of Stirling numbers of the second kind is log-concave.

Theorem 6.2. *For any* $k \geq 0$*, the sequence* $(S(n,k))_{n \geq 0}$ *is log-concave.*

6.1.1 Narayana numbers

The Narayana numbers can be defined, for $1 \leq k \leq n$, as

$$N(n,k) = \frac{1}{n}\binom{n}{k-1}\binom{n}{k}.$$

They refine the Catalan numbers in that $C_n = \sum_{k=1}^{n} N(n,k)$ and count various refinements of the combinatoiral objects enumerated by C_n. We will give the log-concavity of sequences of Narayana numbers using their interpretation in terms of non-crossing partitions.

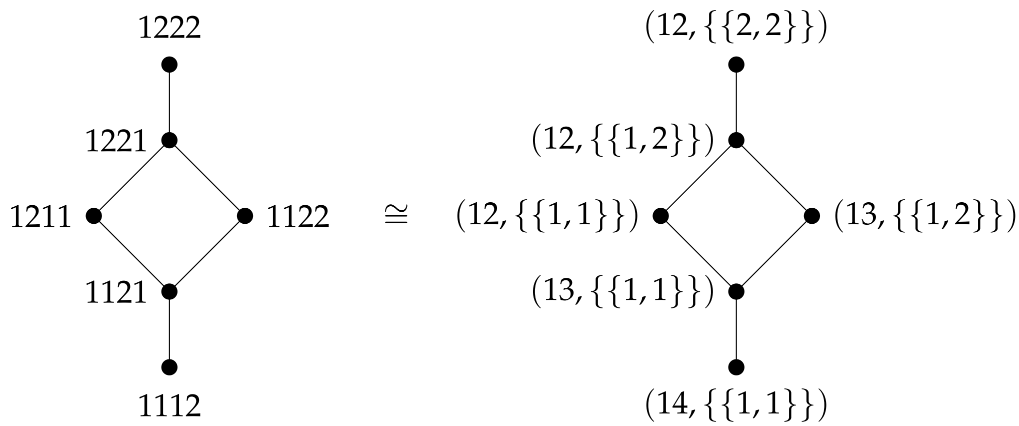

Figure 4: The partial order on $\mathrm{NC}(4, 2)$ both in terms of RGFs and (F, M) pairs.

Call a set partition $\beta = B_1/B_2/\ldots/B_k$ *crossing* if there exist positive integers $a < b < c < d$ with $a, c \in B_i$ and $b, d \in B_j$ for some $i \neq j$, and *non-crossing* otherwise. Clearly a partition is non-crossing if and only if the associate restricted growth function $\rho = r_1 \ldots r_n$ has no subsequence of the form $ijij$. We call such RGFs *non-crossing* as well. For example, in Figure 3 on the left, all the partitions are non-crossing except 1212. We let $\mathrm{NC}(n, k)$ be the set of non-crossing RGFs in $\mathrm{RGF}(n, k)$. It is well known that $N(n, k) = \#\mathrm{NC}(n, k)$.

Define $M(\rho)$ to be the multiset underlying $R(\rho)$. We now partially order $\mathrm{NC}(n, k)$ by letting $\rho \trianglelefteq \sigma$ if and only if $F(\rho) \geq F(\sigma)$ and $M(\rho) \leq M(\sigma)$ where we compare two multisets component-wise after writing them out in weakly increasing order. In Figure 4 we have written out the order on $\mathrm{NC}(4, 2)$ in terms of RGFs (left) and (F, M) pairs (right).

Lemma 6.3. *The poset $(\mathrm{NC}(n, k), \trianglelefteq)$ is a distributive lattice.*

This leads to the main result of this subsection.

Theorem 6.4. *For an $k \geq 0$, the sequence of Narayana numbers $(N(n, k))_{n \geq 0}$ is log-concave.*

7 Future directions

There are various sequences to which it might be possible to apply our method but which have so far resisted proof. For instance, given the nice behaviour of the Stirling numbers of the second kind, one could ask what happens with those of the first. Recall that the *signless Stirling numbers of the first kind* are

$$c(n, k) = \#\{\pi \in \mathfrak{S}_n \mid \pi \text{ has } k \text{ cycles in its disjoint cycle decomposition}\}.$$

We have checked the following conjecture for $1 \leq k \leq n \leq 100$.

Conjecture 7.1. *Given k, there is an integer N_k such that $(c(n,k))_{n\geq 0}$ is log-concave for $n < N_k$ and log-convex for $n \geq N_k$.*

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