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Type C K-Stanley symmetric functions and Kraśkiewicz–Hecke insertion

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Abstract. We study Type C K-Stanley symmetric functions, which are K-theoretic extensions of the Type C Stanley symmetric functions. Our main contribution is Kraśkiewicz–Hecke insertion (KH), a K-theoretic analogue of Kraśkiewicz insertion. Much like Kraśkiewicz insertion enumerates reduced words for signed permutations, KH enumerates their 0-Hecke expressions. The former enumeration witnesses the Type C Stanley expansion into Schur-Q functions. We conjecture that KH extends this to give the Type C K-Stanley expansion into GQ functions, which are K-theory representatives for the Lagrangian Grassmannian introduced by Ikeda and Naruse. We also show Type C K-Stanleys of top fully commutative signed permutations are skew GQ's. This allows us to prove a conjecture of Lewis and Marberg and to give the first conjectural formulas for the expansion of a skew GQ into GQ's. The latter specializes to a rule for multiplying two GQ functions where one has trapezoid shape. This would extend Buch and Ravikumar's Pieri rule, the only known product rule for GQ's.

Keywords: Lagrangian Grassmannian, set-valued tableaux, insertion algorithms, symmetric functions, Schubert calculus, K-theory

1 Introduction

Although this paper is combinatorial in methods and results, our underlying objective is to understand geometric properties of the Lagrangian Grassmannian. A major line of active research is to understand combinatorially more exotic cohomology theories such as K-theory, which encodes finer data about the boundaries of intersections of Schubert varieties. In the Grassmannian, the K-theory of such intersections was first computed combinatorially by Buch [5], later extended to the orthogonal Grassmannian [9, 7]. The K-theory of the Lagrangian Grassmannian has proved far more difficult to understand,

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with the only progress being Buch and Ravikumar's Pieri rule [6]. We offer a pathway towards understanding such products based on Type C K-Stanley symmetric functions.

The Stanley symmetric functions F_w of the permutation w is a symmetric function that enumerates reduced words for w [17] and expands into the Schur basis with non-negative integral coefficients enumerated by Edelman–Greene insertion tableaux [10]. By choosing the permutation w appropriately, this expansion recovers the Littlewood–Richardson rule for products of Schur functions. Our work continues this line of investigation. We study the *Type C K-Stanley symmetric functions* G^C_w, introduced in [12], which is indexed by a signed permutation w. Our insertion, which we call *Kraśkiewicz–Hecke insertion*, generalizes Kraśkiewicz insertion, the Type C analogue of Edelman–Greene insertion [13]. Our insertion enumerates 0-Hecke words for w and gives a conjectural description of G^C_w into GQ functions, which represent the K-theory of the Lagrangian Grassmannian [11].

Insertion algorithms map a word $(a_1, \ldots, a_p)$ to an insertion tableau P and a recording tableau Q. The recording tableaux for Kraśkiewicz–Hecke insertion are standard shifted set-valued tableaux from [11]. We introduce *strict decomposition tableaux* (see Definition 4.2) to play the role of insertion tableaux. For $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell)$ a strict integer partition, let $\mathrm{ShSet}_n(\lambda)$ be the set of standard shifted set-valued tableaux containing n values of shape λ and $\mathrm{SDT}(\lambda)$ be the set of strict decomposition tableaux of shape λ. For P a strict decomposition tableau, $\rho(P)$ is the usual reading word of P.

Theorem 1.1. *For all $n \in \mathbb{N}$, the map Kraśkiewicz–Hecke insertion KH is a bijection:*

$$KH : \mathbb{N}^n \xrightarrow{\sim} \bigsqcup_{\lambda \vdash m \leq n \text{ strict}} \mathrm{SDT}(\lambda) \times \mathrm{ShSet}_n(\lambda).$$

Moreover, for $KH(\boldsymbol{a}) = (P, Q)$, the words $\boldsymbol{a}$ and $\rho(P)$ are 0-Hecke expressions for the same signed permutation.

The definition of Kraśkiewicz–Hecke insertion is given in Section 4 and is extraordinarily technical. Checking the strict decomposition tableau column condition for two entries requires examining the intermediate segment of the tableau's reading word. As a consequence, the insertion rules depend on entries in two rows.

For w a signed permutation, let $\mathcal{H}_n(w)$ be the set of 0-Hecke expressions for w of length n. Also, let $a^C_w(\lambda)$ be the number of strict decomposition tableaux of shape λ whose reading word is a 0-Hecke expression for w. By Theorem 1.1, we have:

Corollary 1.2. *For w a signed permutation and $n \in \mathbb{N}$, we have*

$$|\mathcal{H}_n(w)| = \sum_{\lambda \vdash m \leq n \text{ strict}} a^C_w(\lambda) \cdot |\mathrm{ShSet}_n(\lambda)|. \tag{1.1}$$

Corollary 1.2 is precisely analogous to the use of Edelman–Greene and Kraśkiewicz insertions for reduced word enumeration and Hecke insertion for 0-Hecke expression enumeration in Type A. All three of these results are enumerative shadows of (K-)Stanley symmetric function expansions. Unfortunately, as discussed in Remark 5.3 an insertion algorithm extending Kraśkiewicz insertion **cannot** be used to directly compute the analogous expansion for G^C_w. That said, we believe KH is correctly identifying GQ-coefficients.

Conjecture 1.3. *For w a signed permutation,*

$$G^C_w = \sum_{\lambda \text{ strict}} (-1)^{|\lambda|-\ell(w)} a^C_w(\lambda) \cdot GQ_\lambda. \tag{1.2}$$

In Section 3, we show $G^C_w \in \mathbb{Z}[GQ_\lambda : \lambda \text{ strict}]$. However, establishing the positivity implied by Conjecture 1.3 is an open problem, even using geometric methods. As evidence, the coefficient of $x_1 \ldots x_n$ on both sides of (1.2) gives Corollary 1.2.

In Section 5, we present several important corollaries of Conjecture 1.3 for GQ expansions. For $a < b$, let $\tau(a,b) = (b+a-1, b+a-3, \ldots, b-a+1)$. Conjecture 1.3 specializes to provide unknown combinatorial GQ-expansions for $GQ_\lambda \cdot GQ_{\tau(a,b)}$. Note $\tau(1,b) = (b)$ is a single row, so this would generalize the Buch–Ravikumar Pieri rule [6]. In forthcoming work, the first author reproves their result using strict decomposition tableaux. To show rules for $GQ_\lambda \cdot GQ_{\tau(a,b)}$ and related results follow from Conjecture 1.3, we show using Stembridge's theory of fully commutative elements [18] that G^C_w can express various GQ-expansions (see Corollary 3.6).

Paper Structure: Section 2 introduces necessary background material on signed permutations, set-valued tableaux and GQ functions. In Section 3, we give a precise definition for Type C K-Stanley symmetric functions and identify the signed permutations whose G^C's are used in our conjectures. We introduce strict decomposition tableaux and define Kraśkiewicz–Hecke insertion in Section 4, giving a sketch of the proof for Theorem 1.1. We discuss Conjecture 1.3 and its applications, most notably Conjecture 5.1, in Section 5.

2 Background

For n a positive integer, let $\overline{n} = -n$, $[n] = \{1, 2, \ldots, n\}$ and $[\overline{n}] = \{\overline{1}, \overline{2}, \ldots, \overline{n}\}$. Define the total order $\prec$ on $\mathbb{Z} - \{0\}$ by $\overline{1} \prec 1 \prec \overline{2} \prec 2 \prec \ldots$.

A *signed permutation* w is a permutation of $[\overline{n}] \cup [n]$ such that $w(i) = -w(\overline{i})$. By antisymmetry, w is determined by $w([n])$. For example $v = \overline{2}31$ is a signed permutation. Signed permutations with composition form W_n, the Coxeter group of Type B/C. The generators of W_n are $s_0, s_1, \ldots s_{n-1}$ where $s_0 = (\overline{1}, 1) = \overline{1}2 \ldots n$ and $s_i = (\overline{i+1}, \overline{i})(i, i+1) = 1 \ldots i{+}1\, i \ldots n$ for $i > 0$. These generators satisfy the relations

(i) $s_i^2 = 1$, (ii) $s_i s_j = s_j s_i$ $|i-j| > 1$, (iii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ $i > 0$, (iv) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$, which are called (*i*) Self-inverse, (*ii*) Commutation, (*iii*) Braid and (*iv*) Long Braid.

The *0-Hecke monoid* $(W_n, \circ)$ is the monoid on signed permutations obtained by replacing the Self-inverse relation with the Idempotent relation $s_i \circ s_i = s_i$. A *0-Hecke expression* for $w \in W_n$ is an expression of the form $w = s_{a_1} \circ s_{a_2} \circ \ldots \circ s_{a_p}$ with associated *Hecke word* $(a_1, a_2, \ldots, a_p)$. Let $\mathcal{H}_p(w)$ be the set of Hecke words for w with p letters and $\mathcal{H}(w) = \cup_{p \geq 0} \mathcal{H}_p(w)$. The *length* $\ell(w)$ of w is minimal so that $\mathcal{H}_{\ell(w)}(w)$ is non-empty, and $\mathcal{H}_{\ell(w)}(w)$ is the set of *reduced words* for w.

We now introduce several important families of signed permutations. A signed permutation w is *Grassmannian* if $w(i) < w(i+1)$ for all $i \in [n-1]$, that is $w_1, \ldots, w_n$ is an increasing sequence. Grassmannian signed permutations are in bijection with strict partitions: Grassmannian w maps to λ with $\lambda_i = \overline{w(i)}$ for each i such that $w(i) < 0$. For example, $w = \overline{41}23$ is a Grassmannian signed permutation associated with $\lambda = (4, 1)$. Note the identity corresponds to the empty partition. A signed permutation w is *vexillary* if it avoids a list of 18 patterns [3, Theorem 7]. Each vexillary permutation has an associated shifted shape $\lambda(w)$; see [3] for an algorithmic construction of $\lambda(w)$.

The *shifted Young Diagram* of strict partition λ is $D_\lambda = \{(i,j) \in \mathbb{Z}^2 : 1 \leq i \leq j \leq \lambda_i + i\}$. A *shifted tableau* is a function T whose domain is D_λ. A *set-valued tableau* T has entries that are finite, non-empty sets of integers. The shifted set-valued tableau T is *standard* if its entries partition $[n]$ and for each entry (assuming such cells exist):

$$\max T_{ij} \leq \min T_{i+1\, j}, \min T_{i\, j+1}.$$

Similarly, T is *semistandard* with entries in $\mathbb{Z} - \{0\}$ if $\max(T_{ij}) \preceq \min(T_{i+1\, j})$ with equality only for negative values and $\max(T_{ij}) \preceq \min(T_{i\, j+1})$ with equality only for positive values. For T a set-valued tableau of shape λ/μ, the *size* of T is $|T| = \sum_{(i,j) \in D_{\lambda/\mu}} |T_{ij}|$ and the *weight* of T is

$$x^T = \prod_{(i,j) \in \lambda(T)} x^{T_{ij}} \quad \text{where} \quad x^S = x_{|s_1|} \ldots x_{|s_k|} \quad \text{for} \quad S = \{s_1, \ldots, s_k\} \subseteq \mathbb{Z}.$$

Example 2.1. For $\lambda = (4, 3, 1)$ we depict, from left to right, D_λ and shifted Young tableaux of shape λ that are standard, set-valued standard and set-valued semistandard:

$$\begin{array}{|c|c|c|c|}\hline & & & \\ \hline \multicolumn{1}{c|}{} & & & \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}\,, \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & 3 & 6 & 8 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 7 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}\,, \quad \begin{array}{|c|c|c|c|}\hline 1 & 2,3 & 4,6 & 8 \\ \hline \multicolumn{1}{c|}{} & 5,7 & 9 & 10 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 11 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}\,, \quad \begin{array}{|c|c|c|c|}\hline 1' & 1,2 & 2,4' & 6' \\ \hline \multicolumn{1}{c|}{} & 3,4' & 6' & 7 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 10 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}\,.$$

The rightmost tableau has size 11 and weight $x_1^2 x_2^2 x_3 x_4^2 x_6^2 x_7 x_{10}$.

Let $\mathrm{ShSet}^*(\lambda/\mu)$ be the set of shifted set valued semistandard Young tableaux of shape λ/μ. Then

$$GQ_{\lambda/\mu} = \sum_{T \in \mathrm{ShSet}^*(\lambda/\mu)} (-1)^{|T| - |\lambda/\mu|} x^T. \tag{2.1}$$

We write $GQ_\lambda := GQ_{\lambda/\varnothing}$. For $\mu = \varnothing$, the GQ's were first introduced in [11] as K-theory representatives for Schubert classes in the Lagrangian Grassmannian. Though it is not obvious, $GQ_{\lambda/\mu}$ is a symmetric in the variables $\mathbf{x}$. For λ a strict partition with $\ell(\lambda) = k$, $\mu = (k-1, \ldots, 1)$ and $\nu = (\lambda_1 - (k-1), \ldots \lambda_k)$, define $GS_\nu = GQ_{\lambda/\mu}$. Note GS functions are generating functions for unshifted marked set-valued tableaux.

3 Type C K-Stanley symmetric functions

In [12], Kirillov and Naruse introduce Type C K-Stanley symmetric functions G_w^C as part of their construction of Type C double Grothendieck polynomials.

Definition 3.1. The *Type C K-Stanley symmetric function* for $w \in W_n$ is

$$G_w^C(\mathbf{x}) = \sum_{p \geq 0} \sum_{\mathbf{a} \in \mathcal{H}_p(w)} \sum_{\mathbf{i} \in \kappa(\mathbf{a})} (-1)^{p-\ell(w)} x_{|i_1|} \ldots x_{|i_p|} \quad \text{where}$$

$$\kappa(\mathbf{a}) = \left\{ (i_1 \preceq \cdots \preceq i_p) \in (\mathbb{Z} - \{0\})^p : i_k = i_{k+1} \text{ implies } \quad \begin{Bmatrix} a_k > a_{k+1} & i_k < 0 \\ a_k < a_{k+1} & i_k > 0 \end{Bmatrix} \right\}.$$

We call the pair $(\mathbf{a}, \mathbf{i})$ a *compatible sequence*.

Proposition 3.2. *Let $w \in W_n$ and $v \in S_{n+m}$ with $v(i) = i$ for $i \in [n]$. Then:*

1. $G_w^C \in \mathbb{Z}[GQ_\lambda : \lambda \text{ strict}]$;
2. *The coefficient of $x_1 \ldots x_p$ in G_w^C is $(-1)^{p-\ell(w)} \cdot 2^n \cdot |\mathcal{H}_p(w)|$;*
3. $G_{wv}^C = G_u^C \cdot G_v^C$;
4. *If w is vexillary, then $G_w^C = GQ_{\lambda(w)}$.*

These properties are not stated in [12]. The first is an easy consequence of their work and [8], while the second and third follow easily from the definition. The fourth follows by specializing a Pfaffian formula of Anderson [1].

Let $w(a, b, n) = 12 \ldots k\, \ell{+}1\, \ell{+}2 \ldots n\, k{+}1\, k{+}2 \ldots \ell$ where $k = n - a - b$ and $\ell = a + k$. Since $w(a, b, n)$ is vexillary of shape $\tau(a, b)$ as asserted in [3], Proposition 3.2 (4) implies:

Corollary 3.3. *For $a < b$ positive integers and $a + b \leq n$,*

$$G_{w(a,b,n)}^C = GQ_{\tau(a,b)}.$$

Combined with Theorem 3.5, this result implies [4, Conjecture 4.23].

Say $w \in W_n$ is *fully commutative* if its reduced words contain no braid relations and *top* if its reduced words also avoid the consecutive subword $(1, 0, 1)$. Top fully commutative elements are in bijection with skew strict shapes, and there is a transparent bijection between shifted standard tableaux of strict shape λ/μ and reduced words for the top element $w_{(\lambda/\mu)}$ due to Stembridge [18]. We extend this bijection to a weight preserving map res from $\mathsf{ShSSYT}_p(\lambda/\mu)$ to compatible sequences for $w(\lambda/\mu)$ of length p.

Example 3.4. To compute res, label each cell (i, j) in $D_{\lambda/\mu}$ with its content $j - i$. Then read the entries of T in increasing order with respect to $\prec$ to form $\mathbf{i}$ and the contents to form $\mathbf{a}$, breaking ties to enforce compatibility. For example, we have

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1_0 & 1_1 & 1_2 & 2'_3 & 3'_4 & 5'_5 \\ \hline \multicolumn{1}{c|}{} & 2_0 & 2_1 & 3'_2 & 34'_3 & \multicolumn{1}{|c}{} \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 4_0 & 4_1 & 4_2 & \multicolumn{1}{|c}{} \\ \cline{3-5} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 5_0 & 5_1 & \multicolumn{1}{|c}{} \\ \cline{4-5} \end{array}, \quad \mathrm{res}(T) = \begin{pmatrix} \mathbf{i} \\ \mathbf{a} \end{pmatrix} = \begin{pmatrix} 1,1,1,\overline{2},2,2,\overline{3},\overline{3},3,\overline{4},4,4,4,\overline{5},5,5 \\ 0,1,2,3,0,1,4,2,3,3,0,1,2,5,0,1 \end{pmatrix}.$$

Here, the contents are depicted using red subscripts inside of T for reference.

As a consequence, we have:

Theorem 3.5. *For $\mu \subseteq \lambda$ strict shapes, $G^C_{w(\lambda/\mu)} = GQ_{\lambda/\mu}$.*

As an important special case, we have:

Corollary 3.6. *Let $\mu \subseteq \lambda$ be strict partitions and ν be a partition (not necessarily strict). Then for appropriately chosen strict partitions ρ, τ the signed permutation $w(\lambda/\mu) \cdot w(\tau/\rho)$ is top and*

$$G^C_{w(\lambda/\mu)\cdot w(\tau/\rho)} = GQ_\lambda \cdot GS_\nu.$$

Remark 3.7. A (possibly equivalent) formula for G^C_w using tableaux with straight shapes appears in [19], though the connection to GQ-functions is not made in that work.

4 Kraśkiewicz–Hecke Insertion

In order to define Kraśkiewicz–Hecke insertion KH, we first define strict decomposition tableaux. These generalize reduced decomposition tableaux from [13, 14] and standard decomposition tableaux from [16][1]. These are the insertion tableaux for KH.

Definition 4.1. Let $R_i = r_i r_{i+1} \ldots r_k$ be the ith row in a shifted tableau T. We say R_i is *(strictly) unimodal* if there exists $i \leq j \leq k$ so that $r_i > r_{i+1} > \cdots > r_j < \cdots < r_k$. Depending on context, we refer to both j and r_j as the *dip* of R_i. The *decreasing* and *increasing parts* of R_i are $R_i^{\downarrow} = r_i \ldots r_{j-1}$ and $R_i^{\uparrow} = r_j \ldots r_k$, respectively. Note the

[1] Confusingly, reduced decomposition tableaux are referred to as ‘standard’ in [14]. We introduce the term ‘reduced’ as the Serrano tableaux more closely resemble the conventional meaning of ‘standard’.

increasing part is always non-empty, which is at odds with the conventions in [14]. For a unimodal row R_i with dip j, it is frequently convenient to view it as

$$\begin{aligned}\mathsf{T}(R_i) &:= -r_i < \cdots < -r_{j-1} < r_j < r_{j+1} < \cdots < r_k, \text{ or}\\ \mathsf{B}(R_i) &:= -r_i < \cdots < -r_{j-1} < -r_j < r_{j+1} \cdots < r_k,\end{aligned}$$

which are increasing sequences of integers.

Definition 4.2. Let λ be a strict partition with $\ell(\lambda) = k$. A *strict decomposition tableau* is a tableau $T : D_\lambda \to \mathbb{N}$ with unimodal rows $R_1 \dots R_k$ such that for each $i \in [k-1]$ every entry of R_{i+1} is less than the first entry of R_i and the following *witness condition* is satisfied. For increasing sequences $\mathsf{T}(R_i) = a_i \dots a_{i+\lambda_i}$, $\mathsf{B}(R_{i+1}) = b_{i+1} \dots b_{i+1+\lambda_{i+1}}$ and for all $i+1 \le j \le i+1+\lambda_{i+1}$,

$$\{\pm a_i, \dots, \pm a_{j-1}, \pm b_{j+1}, \dots, \pm b_{i+1+\lambda_{i+1}}\} \cap (b_j, a_j] = \varnothing. \tag{4.1}$$

Note that a_j appears immediately above b_j in T. If $a_j \le b_j$, then $(b_j, a_j] = \varnothing$, so Equation (4.1) is satisfied vacuously. If x is an element of the LHS of Equation (4.1), we say x *witnesses* $b_j < a_j$, showing T is not a strict decomposition tableau.

The *reading word* of a strict decomposition tableau T is $\rho(T) = R_\ell R_{\ell-1} \dots R_1$.

Lemma 4.3. *A shifted tableau with unimodal rows is a strict decomposition tableau if and only if the tableau avoids the following five configurations:*

$$(i)\ \begin{array}{ccc} a & \cdots & \square \\ & \cdots & b \end{array},\quad (ii)\ \begin{array}{ccccc} \square & \cdots & a & \cdots & \square \\ & \cdots & c & \cdots & b \end{array},\quad (iii)\ \begin{array}{cccccc} \square & \cdots & v & z & \cdots & \square \\ & \cdots & \cdots & x & \cdots & y \end{array},\quad (iv)\ \begin{array}{ccc} y & \cdots & z \\ & \cdots & x \end{array},\quad (v)\ \begin{array}{ccccc} \square & \cdots & y & \cdots & z \\ & \cdots & \square & \cdots & x \end{array}$$

with $a \le b < c$, $x < y \le z$, and $v < z$.

Example 4.4. The leftmost tableau is a strict decomposition tableau, the rest are not:

$$\begin{array}{cccccc} 5&2&1&0&3&4\\ &2&0&1&2&4 \end{array}\quad \begin{array}{ccc} 3&0&2\\ &0&4 \end{array}\quad \begin{array}{cccccc} 5&2&1&0&2&4\\ &3&0&1&2&4 \end{array}\quad \begin{array}{cccccc} 7&2&1&0&5&6\\ &2&0&1&2&5 \end{array}\quad \begin{array}{cccccc} 5&4&1&0&5&6\\ &2&0&1&4&6 \end{array}\quad \begin{array}{cccccc} 5&3&1&0&3&4\\ &2&0&1&2&4 \end{array} \tag{4.2}$$

The entries highlighted in blue are a_j, b_j so that $a_j < b_j$, while their witnesses, if they exist, are in red. Further notice each of the non-examples exhibits the configurations in Lemma 4.3 from left to right.

For λ a strict partition with $\ell(\lambda) = k$, a *reduced decomposition tableau* of shape λ is a tableau $T : D_\lambda \to \mathbb{N}$ with rows $R_1 \dots R_k$ so that R_i is a unimodal subsequence of maximal length in $R_k R_{k-1} \dots R_i$ for all $i \in [k]$ and whose reading word $\rho(T)$ is a reduced word for some signed permutation. For KH to generalize Kraśkiewicz insertion, we require:

Proposition 4.5. *Every reduced decomposition tableau is also a strict decomposition tableau.*

We will now construct Kraśkiewicz–Hecke insertion, which maps words in the alphabet $\mathbb{N} = \{0, 1, 2, \dots\}$ to pairs of tableaux P, Q of the same shape where P is a *strict decomposition tableau* and Q is a *standard set-valued shifted tableau*. We first present the row insertion rule, which is very similar to Kraśkiewicz row insertion but requires a plethora of additional cases to ensure the insertion tableau is a strict decomposition tableau.

Definition 4.6. *(Kraśkiewicz–Hecke) row insertion* is an algorithm with inputs $a \in \mathbb{N}$ and a two-row strict decomposition tableau RS that outputs $a' \in \mathbb{N} \cup \{\infty\}$ and two-row strict decomposition tableau $R'S$. Note S is unchanged and may be empty. Row insertion is a two step procedure, first applying *right insertion* and then *left insertion*.

Let $R = r_1 \dots r_\ell$ with dip r_q and $S = s_2 \dots s_k$ with dip s_p and $k \leq \ell$. Set $r_0, s_1 = -\infty$, $r_{\ell+1} = \infty$ and $s_m = \infty$ for $m > k$. Note r_i appears immediately above s_i in RS.

We first define right insertion, which outputs $a' \in \mathbb{N} \cup \{\infty\}$ and row R'', which is an input for left insertion. If a equals the dip r_q, set $i = q$. Otherwise, let $q < i \leq \ell + 1$ be minimal such that $r_i \geq a$. Set $A_i = \{s_{i+1}, \dots, s_k, r_1, \dots, r_{i-1}\}$.

1. If $a \neq r_i$, then set $a' = r_i$ and create R'' by changing r_i to a.
2. If $a = r_i$ and $r_{i-1} > r_{i+1}$, then set $a' = r_{i+1}$ and create R'' by setting $r_{i+1} = r_i$. Note the resulting tableau will not have unimodal rows, however the left insertion step afterwards will always restore unimodal rows.
3. If $a = r_i$ and $r_{i-1} \leq r_{i+1}$, then let $R'' = R$ and set $a' = \min(\{r_{i+1}\} \cup [(s_i, r_{i+1}) \cap A_i])$.

We now define left insertion. If $a' = \infty$, then set $b = \infty$ and $R' = R''$. Otherwise, let $1 \leq j \leq q$ be minimal such that $r_j \leq a'$ where here r_j indicates the j^{th} entry of R''.

1. If $a' \neq r_j$, set $b = r_j$ and create R' by changing r_j to a'.
2. If $a' = r_j$, then set $R' = R''$.
   (a) If $j < p$, we define
   $$b = \begin{cases} \max(A_j \cap (r_{j+1}, s_j)), & \text{if } j+1 < q \text{ and } A_j \cap (r_{j+1}, s_j) \neq \emptyset \\ \max(A_j \cap (-\infty, s_j)), & \text{if } j+1 \geq q \text{ and } A_j \cap (-\infty, s_j) \neq \emptyset \\ r_{j+1}, & \text{otherwise.} \end{cases}$$
   (b) If $j \geq p$, then define
   $$b = \begin{cases} s_{j+1}, & \text{if } (a)\ j+1 \geq q \text{ and } (b)\ r_{j+1} > s_{j+1} \text{ or } r_{j+2}, s_{j+2} > r_j > s_{j+1} \\ r_{j+1}, & \text{otherwise.} \end{cases}$$

We extend row insertion to an insertion algorithm by repeated application.

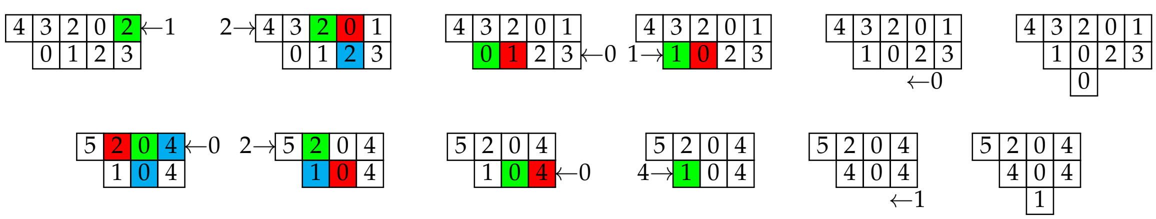
Definition 4.7. For $P = R_1 \dots R_\ell$ a strict decomposition tableau and $a \in \mathbb{N}$, we *Kraśkiewicz–Hecke insert* a into P by row inserting a into $R_1 R_2$, updating R_1 and inserting the output b into $R_2 R_3$ and so on until the output is ∞. The insertion *terminates* in row i where i is the row whose output from row insertion is ∞.

For $\mathbf{a} = (a_1, \dots, a_p) \in \mathbb{N}^p$ and $\mathbf{a}' = (a_1, \dots, a_{p-1})$, we define the *Kraśkiewicz–Hecke insertion tableau* $P_{HK}(\mathbf{a})$ recursively by row inserting a_p into $P_{HK}(\mathbf{a}')$. The *Kraśkiewicz–Hecke recording tableau* $Q = Q_{HK}(\mathbf{a})$ is also constructed recursively from $Q' = Q_{HK}(\mathbf{a}')$. Let λ and λ' be the shapes of $P_{HK}(\mathbf{a})$ and $P_{HK}(\mathbf{a}')$, respectively.

1. If $\lambda \neq \lambda'$, they differ by a single cell (i, j). We obtain Q from Q' by setting $Q_{ij} = \{p\}$.

2. If $\lambda = \lambda'$, then construct Q from Q' by adding p to $Q'_{k\,(\lambda_k + k)}$ where k is the row where insertion of a_p terminated. Note by (R3) that insertion can only terminate in row k if $\lambda_k > \lambda_{k+1} + 1$ or $\lambda_{k+1} = 0$.

Example 4.8. We present two examples of Kraśkiewicz–Hecke insertion:



The arrows on the right side represent right insertion, while the arrows of the left side represent left insertion. Bumping occurs in the green cells, while red cells highlight the next element to be inserted (if distinct from the green cell). Blue cells are relevant for determining which element to insert next.

When inserting $\mathbf{a} = (5, 3, 0, 5, 2, 4, 0, 4, 0)$, we present some insertion tableaux starting with $P((5, 3, 0, 5)) = \boxed{5\,|\,3\,|\,0\,|\,5}$ and ending with $P = P_{KH}(\mathbf{a})$ as well as $Q = Q_{KH}(\mathbf{a})$:

$$\begin{array}{|c|c|c|c|}\hline 5&3&0&2\\\hline \multicolumn{1}{c|}{}&3&\multicolumn{2}{c}{}\\\cline{2-2}\end{array} \quad \begin{array}{|c|c|c|c|c|}\hline 5&3&0&2&4\\\hline \multicolumn{1}{c|}{}&3&\multicolumn{3}{c}{}\\\cline{2-2}\end{array} \quad \begin{array}{|c|c|c|c|c|}\hline 5&3&2&0&4\\\hline \multicolumn{1}{c|}{}&3&0&\multicolumn{2}{c}{}\\\cline{2-3}\end{array} \quad \begin{array}{|c|c|c|c|c|}\hline 5&3&2&0&4\\\hline \multicolumn{1}{c|}{}&3&0&\multicolumn{2}{c}{}\\\cline{2-3}\end{array} \quad P = \begin{array}{|c|c|c|c|c|}\hline 5&4&2&0&4\\\hline \multicolumn{1}{c|}{}&3&0&3&\multicolumn{1}{c}{}\\\cline{2-4}\end{array} \quad Q = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&6,8\\\hline \multicolumn{1}{c|}{}&5&7&9&\multicolumn{1}{c}{}\\\cline{2-4}\end{array}$$

The following proposition implies KH is well-defined.

Proposition 4.9. *Given an $a \in \mathbb{N}$ and T a strict decomposition tableau, the tableau $T' = T \leftarrow a$ is also a strict decomposition tableau.*

Our proof of Proposition 4.9 proceeds by showing if $T' = T \leftarrow a$ contains one of the configurations in Lemma 4.3, then T was not a strict decomposition tableau. The argument is case by case, some of which rely on a lemma showing that row bumping paths of right and left insertion both move weakly to the left. Two words with the same Kraśkiewicz insertion tableaux P are necessarily reduced words for the same signed permutation. Similarly, the Kraśkiewicz–Hecke insertion tableau for a word determines its 0-Hecke product.

Proposition 4.10. *Let T be a SDT and w a signed permutation so that $\rho(T) \in \mathcal{H}(w)$. For $a \in \mathbb{N}$ and $T' = T \leftarrow a$ we have $\rho(T') \in \mathcal{H}(w \circ s_a)$.*

The inverse of Kraśkiewicz–Hecke insertion is defined similarly, first defining inverses to left and right insertion, then using the recording tableau to determine the row and value to begin inverse insertion with. For space considerations, we omit the details.

Since KH is invertible, Proposition 4.9 implies Theorem 1.1. Then Proposition 3.2 (2) and 4.10 imply Corollary 1.2.

5 Conjectures

Our original goal in this project was to compute the GQ_λ-expansion of G_w^C. As we will explain in Remark 5.3, insertion methods exhibit a fundamental inadequacy for the task. Despite this shortcoming, empirically Kraśkiewicz–Hecke insertion computes this expansion as asserted in Conjecture 1.3, which has been tested up to degree 9 for signed permutation in W_4. By taking the $x_1 \dots x_n$ coefficient on each side, we see Conjecture 1.3 and Proposition 3.2 (2) would imply Corollary 1.2:

$$|\mathcal{H}_p(w)| = \sum_{\lambda \text{ strict}} a_w^C(\lambda) \cdot |\mathrm{ShSet}_p(\lambda)|.$$

Note we have removed the factor 2^p from each side, on the right by not allowing barred entries in our tableaux. We view this as very strong evidence for Conjecture 1.3. To see why, assume the conjecture were to fail for $w \in W_n$. By Proposition 3.2, (1) we have

$$G_w^C = \sum_{\lambda \text{ strict}} b_w^C(\lambda) \cdot GQ_\lambda$$

for some coefficients $b_w^C(\lambda) \in \mathbb{Z}$, and conjecturally in $\mathbb{N}$. Then for fixed p, we have

$$\sum_{\lambda \text{ strict}} a_w^C(\lambda) \cdot |\mathrm{ShSet}_p(\lambda)| = \sum_{\lambda \text{ strict}} b_w^C(\lambda) \cdot |\mathrm{ShSet}_p(\lambda)|. \tag{5.1}$$

Additionally, when $|\lambda| = \ell(w)$, we know from [14] that $a_w^C(\lambda) = b_w^C(\lambda)$ for all λ. Therefore, and especially assuming the positivity of $b_w^C(\lambda)$, the failure of Conjecture 1.3 gives a striking relation on the number of shifted set-valued standard tableaux of given sizes.

By combining Conjecture 1.3 with Corollary 3.6, we have:

Conjecture 5.1. *For strict partitions $\mu \subseteq \lambda$, partition ν and appropriate signed permutation w,*

$$GQ_{\lambda/\mu} \cdot GS_\nu = \sum_\rho a_w^C(\rho) \cdot GQ_\rho.$$

As special cases, with $\nu = \varnothing$ we would recover [15, Conjecture 4.36] and with $\mu = \lambda$ we would recover [4, Conjecture 5.14]. Both conjectures are statements of GQ-positivity, and do not combinatorial descriptions of the coefficients. These basic facts about skew GQ's have resisted explanation for several years. As mentioned in the introduction, it is an open problem to find the GQ expansion for products of GQ functions. By Corollary 3.3, Conjecture 5.1 specializes to a rule for computing $GQ_\lambda \cdot GQ_{\tau(a,b)}$.

Example 5.2. For $w = \overline{521}34891067$, we have $a_w^C((10,3,2)) = 2$ with tableaux

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 9&6&4&3&2&1&0&6&7&8\\ \hline \multicolumn{1}{c|}{}&8&1&0&\multicolumn{6}{c}{}\\ \cline{2-4} \multicolumn{2}{c|}{}&0&7&\multicolumn{6}{c}{}\\ \cline{3-4}\end{array}\,, \quad \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 9&6&4&3&2&1&0&6&7&8\\ \hline \multicolumn{1}{c|}{}&8&1&0&\multicolumn{6}{c}{}\\ \cline{2-4} \multicolumn{2}{c|}{}&7&0&\multicolumn{6}{c}{}\\ \cline{3-4}\end{array}\,.$$

This correctly computes the coefficient of GQ_{1032} in $GQ_{521} \cdot GQ_{42}$.

Remark 5.3. For Kraśkiewicz–Hecke insertion to extend to a proof of Conjecture 1.3, we need $\mathbf{a}$ and $Q(\mathbf{a})$ to have the same peak sets. We present an example showing an insertion algorithm generalizing Kraśkiewicz insertion cannot have this property.

For $u = s_1 s_0 s_2$, $v = s_1 s_0 s_2 s_1$, $w = s_1 s_0 s_2 s_1 s_0$,

$$G_u^C = \mathrm{GQ}_{(3)} + \mathrm{GQ}_{(21)} + \beta \mathrm{GQ}_{(31)}\,, \quad G_v^C = \mathrm{GQ}_{(31)}\,, \quad G_w^C = \mathrm{GQ}_{(32)}$$

The strict decomposition tableaux witnessing these expansions are:

$$u:\ U^1 = \begin{array}{|c|c|c|}\hline 1&0&2\\ \hline\end{array}\,, \quad U^2 = \begin{array}{|c|c|}\hline 2&0\\ \hline \multicolumn{1}{c|}{}&1\\ \cline{2-2}\end{array}\,, \quad U^3 = \begin{array}{|c|c|c|}\hline 2&0&2\\ \hline \multicolumn{1}{c|}{}&1&\multicolumn{1}{c}{}\\ \cline{2-2}\end{array} \quad v:\ V = \begin{array}{|c|c|c|}\hline 2&0&1\\ \hline \multicolumn{1}{c|}{}&1&\multicolumn{1}{c}{}\\ \cline{2-2}\end{array} \qquad w:\ W = \begin{array}{|c|c|c|}\hline 2&1&0\\ \hline \multicolumn{1}{c|}{}&1&0\\ \cline{2-3}\end{array}\,.$$

For a word $\mathbf{x} = (x_1, \ldots, x_k)$ and $j \leq k$, let $\mathbf{x}[j] = (x_1, \ldots, x_j)$. Consider the words $\mathbf{a} = (1,0,2,0,1,0), \mathbf{b} = (1,2,0,2,1,0) \in \mathcal{H}(w)$, so $P(\mathbf{a}) = P(\mathbf{b}) = W$ by Proposition 4.10. Similarly, $P(\mathbf{a}[5]) = P(\mathbf{b}[5]) = V$ since v is vexillary. From Kraśkiewicz insertion, we know $P(\mathbf{a}[3]) = U^1$ and $P(\mathbf{b}[3]) = U^2$. Any reasonable generalization of Kraśkiewicz insertion will have $P(\mathbf{a}[4]) = P(\mathbf{b}[4]) = U^3$, with the former since the peak in position 3 must be preserved. However, the associated recording tableaux for these insertions would not preserve peak sets at position 5. We will give a more detailed discussion of this example in the complete version of this paper [2].

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