Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #9, 12 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

Tutte polynomials in superspace

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Abstract. We associate a quotient of superspace to any hyperplane arrangement by considering the differential closure of a "power ideal" (a particular ideal generated by powers of certain homogeneous linear forms). This quotient is a superspace analogue of the external zonotopal algebra of Holtz and Ron and also contains the central zonotopal algebra. We show that the bigraded Hilbert series of this quotient is equal to an evaluation of the Tutte polynomial. We also construct an explicit basis for the Macaulay inverse. These results generalize previous work of Ardila–Postnikov and Holtz–Ron.

Keywords: Hyperplane arrangement, power ideal, superspace, Tutte polynomial

1 Introduction

Consider commutative generators $\boldsymbol{x}_n = (x_1, \ldots, x_n)$ and *anticommutative* generators $\boldsymbol{\theta}_n = (\theta_1, \ldots, \theta_n)$. We define *rank n superspace* Ω_n to be $\mathbb{C}[\boldsymbol{x}_n] \otimes \wedge\{\boldsymbol{\theta}_n\}$. This ring arises in physics where the two sets of variables represent states of bosons and fermions, respectively. This explains why the variables are often referred to as *bosonic* and *fermionic* variables. Additionally, Ω_n is the Hochschild homology of the polynomial ring $\mathbb{C}[\boldsymbol{x}_n]$ and, as such, may be considered as the ring of polynomial-valued holomorphic differential forms on $\mathbb{C}^n$.

In the last few years, there has been great interest in quotients of superspace. So far, the primary impetus has come from Macdonald polynomials; in particular, a still-open conjecture of Zabrocki [13] suggests a connection between the now-proven "Delta conjecture" of Haglund–Remmel–Wilson [6] and a module that generalizes the diagonal coinvariants by introducing fermionic variables. See [3, 8] for more recent work.

We instead approach superspace from the perspective of hyperplane arrangements (or, equivalently, realizable matroids). Our point of entry is the theory of power ideals,

*Brendon Rhoades was partially supported by NSF Grant DMS-2246846.
†Vasu Tewari was partially supported by NSF Grant DMS-2246961.
‡Andy Wilson was partially supported by AMS–Simons PUI Grant 434651.

as developed by Ardila–Postnikov [1] and Holtz–Ron [7]. As we hope to demonstrate, power ideals have a superspace generalization that merits investigation.

To construct a power ideal, we begin with a *size m multiarrangement* $\mathcal{A} = \{H_1, \ldots, H_m\}$ of hyperplanes in $\mathbb{C}^n$, i.e. a collection of hyperplanes wherein we allow repeats. Associated with $\mathcal{A}$ is a family of homogeneous ideals $J_{\mathcal{A},k} \subseteq \mathbb{C}[\boldsymbol{x}_n]$ known as *power ideals*:

$$J_{\mathcal{A},k} := \left(\lambda_L^{\rho_{\mathcal{A}}(L)+k} \mid L \subseteq \mathbb{C}^n \text{ a line} \right) \tag{1.1}$$

where λ_L is the linear form corresponding to any line L through the origin and

$$\rho_{\mathcal{A}}(L) := \#\{1 \leq i \leq m \mid L \not\subseteq H_i\}. \tag{1.2}$$

The resulting quotient rings have fascinating mathematical properties with deep ties to numerical analysis, algebra, geometry, and combinatorics, particularly when $k = -1$, 0, or 1 [5, 9, 10]. These cases correspond to internal, central, and external zonotopal algebras, respectively [7].

We will move power ideals into superspace by taking their "differential closures." Regarding Ω_n as a ring of differential forms, we have the *Euler operator* (or *total derivative*) $d : \Omega_n \to \Omega_n$ defined by

$$df := \sum_{i=1}^{n} (\partial f / \partial x_i) \cdot \theta_i \tag{1.3}$$

where, in the evaluation of $(\partial f / \partial x_i)$, the $\boldsymbol{\theta}$-variables are treated as constants. Throughout, we treat the $\boldsymbol{x}$-variables and $\boldsymbol{\theta}$-variables as elements of degree $(1,0)$ and $(0,1)$, respectively. Let $I \subseteq \mathbb{C}[\boldsymbol{x}_n]$ be a homogeneous ideal. The *differential closure* $I^\theta \subseteq \Omega_n$ of I is the smallest ideal in Ω_n containing I and closed under the operator d. If $g_1, \ldots, g_r \in \mathbb{C}[\boldsymbol{x}_n]$ are homogeneous generators of I, we have[1]

$$I^\theta = (g_1, \ldots, g_r, dg_1, \ldots, dg_r) \subseteq \Omega_n. \tag{1.4}$$

As the ideal $I^\theta \subseteq \Omega_n$ is bihomogeneous, the quotient space Ω_n / I^θ acquires a bigrading wherein the θ-degree 0 piece is the original graded quotient $\mathbb{C}[\boldsymbol{x}_n]/I$. If the commutative quotient $\mathbb{C}[\boldsymbol{x}_n]/I$ has interesting properties, it is natural to ask whether and how these properties extend to the supercommutative quotient Ω_n / I^θ.

In this article, we focus on the differential closure of $J_{\mathcal{A},1}$ inside Ω_n, which we denote $I_{\mathcal{A}}$, and the resulting quotient space $\mathcal{E}_{\mathcal{A}} = \Omega_n / I_{\mathcal{A}}$. Our first main result relates the bigraded Hilbert series of $\mathcal{E}_{\mathcal{A}}$ to the the Tutte polynomial $T_{\mathcal{A}}(x, y)$.

Theorem 1.1. *For any rank r multiarrangement $\mathcal{A}$ of size m in $\mathbb{C}^n$ we have the bigraded Hilbert series*

$$\operatorname{Hilb}(\mathcal{E}_{\mathcal{A}}; q, t) = (1+t)^r q^{m-r} T_{\mathcal{A}} \left(\frac{1+q+t}{1+t}, \frac{1}{q} \right).$$

[1] This is due to the product rule $d(f \cdot g) = df \cdot g \pm f \cdot dg$ for bihomogeneous $f, g \in \Omega_n$ together with the relation $d \circ d = 0$, which follows from the anticommutativity of the fermionic variables.

where q tracks bosonic degree and t tracks fermionic degree.

Setting $t = 0$ recovers results of Ardila–Postnikov [1] and Holtz–Ron [7]. The proof of Theorem 1.1 involves a short exact sequence (3.7) that witnesses a deletion-restriction type recursion. We discuss corollaries of this theorem involving objects such as zonotopal algebras, face vectors, and bigraphical arrangements in Section 4.

Our second main result provides a basis of the Macaulay inverse space $I_{\mathcal{A}}^{\perp}$, which is isomorphic to $\mathcal{E}_{\mathcal{A}}$ as a bigraded vector space. Let $\mathcal{B}$ denote the set of bases for the matroid $\mathfrak{M}_{\mathcal{A}}$ corresponding to $\mathcal{A}$. Given $B \in \mathcal{B}$, we let $EP_{\mathcal{A}}(B)$, $IP_{\mathcal{A}}(B)$, $IA_{\mathcal{A}}(B)$ denote the sets of externally passive elements, internally passive elements, and internally active elements, respectively. These notions are defined more precisely in Section 3.4.

Theorem 1.2. *For $H \in \mathcal{A}$, let α_H denote the*[2] *homogeneous linear form with zero set H. The following set forms a basis for $I_{\mathcal{A}}^{\perp}$:*

$$M_{\mathcal{A}} := \bigcup_{B \in \mathcal{B}} \left\{ \prod_{e \in E} \alpha_e \times \prod_{i \in I} d\alpha_i \times \prod_{s \in S} \alpha_s \times \prod_{t \in T} d\alpha_t \ : \ \begin{array}{c} E = EP_{\mathcal{A}}(B),\ I \subseteq IP_{\mathcal{A}}(B), \\ S, T \subseteq IA_{\mathcal{A}}(B),\ S \cap T = \varnothing \end{array} \right\}. \tag{1.5}$$

In the rest of this abstract, we provide necessary background in Section 2 before discussing our main results in Section 3 and some of their consequences and related open questions in Section 4.

2 Background

2.1 Superspace

As mentioned in the introduction, superspace Ω_n is the tensor product of the symmetric algebra $\mathbb{C}[\boldsymbol{x}_n]$ in n bosonic variables with the exterior algebra $\wedge\{\boldsymbol{\theta}_n\}$ in n fermionic variables. A *monomial* in Ω_n is defined to be a product of a monomial in the $\boldsymbol{x}$-variables with a nonzero monomial in the $\boldsymbol{\theta}$-variables. Monomials in the $\boldsymbol{\theta}$-variables, i.e. fermionic monomials, are indexed up to sign by subsets $J = \{j_1 < \cdots < j_r\} \subseteq [n]$. Given such a J, we set $\theta_J := \theta_{j_1} \cdots \theta_{j_r}$. If $x_1^{a_1} \cdots x_n^{a_n} \theta_J$ is a monomial in Ω_n, its *bosonic* degree is $a_1 + \cdots + a_n$, its *fermionic* degree is $|J|$, and its bidegree is $(a_1 + \cdots + a_n, |J|)$. The $\mathbb{C}$-algebra Ω_n admits a direct sum decomposition

$$\Omega_n = \bigoplus_{i,j \geq 0} (\Omega_n)_{i,j} \tag{2.1}$$

where $(\Omega_n)_{i,j} = \mathbb{C}[\boldsymbol{x}_n]_i \otimes \wedge^j \{\boldsymbol{\theta}_n\}$ consists of bihomogeneous elements of bidegree (i, j).

[2] This linear form is unique up to a nonzero scalar.

For $1 \leq i \leq n$, the usual partial derivative $\partial_i := \partial/\partial x_i$ acts on the first tensor factor of Ω_n while the *contraction operator* $\partial_i^\theta := \partial/\partial\theta_i$ acts on the second factor by extending its action on *fermionic* monomials: given distinct indices $1 \leq j_1 < \cdots < j_r \leq n$, we let

$$\partial_i^\theta(\theta_{j_1} \cdots \theta_{j_r}) = \begin{cases} (-1)^{s-1}\theta_{j_1} \cdots \widehat{\theta_{j_s}} \cdots \theta_{j_r} & j_s = i \text{ for some } 1 \leq s \leq r \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

where the hat denotes omission. The operators ∂_i and ∂_i^θ satisfy

$$\partial_i \partial_j = \partial_j \partial_i \qquad \partial_i \partial_j^\theta = \partial_j^\theta \partial_i \qquad \partial_i^\theta \partial_j^\theta = -\partial_j^\theta \partial_i^\theta \tag{2.3}$$

for $1 \leq i, j \leq n$. Given $f = f(x_1, \ldots, x_n, \theta_1, \ldots, \theta_n) \in \Omega_n$, we therefore have a well-defined differential operator $\partial f := f(\partial_1, \ldots, \partial_n, \partial_1^\theta, \ldots, \partial_n^\theta)$ of superspace on itself by $f \odot g := \partial f(g)$. Given a bihomogeneous ideal $I \subseteq \Omega_n$, the *Macaulay inverse system* $I^\perp$ is the bigraded subspace of Ω_n given by

$$I^\perp := \{g \in \Omega_n \,:\, f \odot g = 0 \text{ for all } f \in I\ \}. \tag{2.4}$$

Let $V = \bigoplus_{i,j \geq 0} V_{i,j}$ be a bigraded complex vector space with each piece $V_{i,j}$ finite-dimensional. The *bigraded Hilbert series* of V is

$$\mathrm{Hilb}(V; q, t) := \sum_{i,j \geq 0} \dim_{\mathbb{C}}(V_{i,j}) \cdot q^i t^j \tag{2.5}$$

in the variables q, t. For our purposes, V will always be a bihomogeneous subspace or quotient of superspace Ω_n, the variable q will track bosonic degree, and the variable t will track fermionic degree.

If $I \subseteq \Omega_n$ is a bihomogeneous ideal, we have the bigraded direct sum $\Omega_n = I \oplus \overline{I^\perp}$, where the $\bar{\cdot}$ operator reverses the order of the θ-monomials and takes the complex conjugate of the coefficients. The bigraded Hilbert series of the quotient ring Ω_n/I therefore coincides with that of the inverse system $I^\perp$:

$$\mathrm{Hilb}(\Omega_n/I; q, t) = \sum_{i,j \geq 0} \dim_{\mathbb{C}} (\Omega_n/I)_{i,j} \cdot q^i t^j = \sum_{i,j \geq 0} \dim_{\mathbb{C}}(I^\perp) \cdot q^i t^j = \mathrm{Hilb}(I^\perp; q, t). \tag{2.6}$$

2.2 Multiarrangements

A *linear hyperplane* H is a codimension one subspace of $\mathbb{C}^n$. Any linear hyperplane H is the zero set of a homogeneous linear form $\alpha_H = a_1 x_1 + \cdots a_n x_n$ for $(a_1, \ldots, a_n) \in \mathbb{C}^n \setminus \{0\}$. We refer to α_H as the *normal vector* of H; it is unique up to a nonzero scalar.

An *affine hyperplane* H is an affine translate of a linear hyperplane; the normal vector of an affine hyperplane is that of its linear translate. A *multiarrangement* $\mathcal{A}$ of hyperplanes is a collection $\{H_1, \ldots, H_m\}$ of hyperplanes of *size* m where we allow repeats. For a given hyperplane H in $\mathcal{A}$, we refer to the number of times it appears in the collection $\mathcal{A}$ as its *multiplicity*. An arrangement is *simple* if the multiplicity of any hyperplane in it is equal to 1. Henceforth, we employ the term "arrangement" for multiarrangements.

Let $\mathcal{A}$ be an arrangement and let $H \in \mathcal{A}$ be a hyperplane. The *deletion* $\mathcal{A} - H$ is the arrangement obtained from $\mathcal{A}$ by removing one copy of H. The *restriction* $\mathcal{A} \mid H$ is the arrangement

$$\mathcal{A} \mid H := \{H' \cap H \,:\, H' \in \mathcal{A} - \{H\},\ H' \cap H \neq \varnothing\}. \tag{2.7}$$

Given a matroid $\mathfrak{M}$ on a ground set E with rank function $\mathsf{rk} : 2^{\mathsf{E}} \to \mathbb{Z}_{\geq 0}$, its *Tutte polynomial* $T_{\mathfrak{M}}(x,y)$ is

$$T_{\mathfrak{M}}(x,y) = \sum_{A \subseteq \mathsf{E}} (x-1)^{r-\mathsf{rk}(A)}(y-1)^{|A|-\mathsf{rk}(A)}, \tag{2.8}$$

where $r := \mathsf{rk}(\mathfrak{M})$. Tutte [12] famously showed that this implies a combinatorial expansion in terms of *internal* and *external activities* as we range over matroid bases of $\mathfrak{M}$:

$$T_{\mathfrak{M}}(x,y) = \sum_{\text{basis } B} x^{\mathrm{ia}(B)} y^{\mathrm{ea}(B)} \in \mathbb{N}[x,y]. \tag{2.9}$$

We postpone the formal definition of the notion of activity (and passivity) until Section 3.4. If a matroid $\mathfrak{M}$ is the vector matroid of all normal vectors of hyperplanes in $\mathcal{A}$, we may talk about the Tutte polynomial $T_{\mathcal{A}}$ without any ambiguity.

3 Main construction and results

3.1 Superpower ideals

Given $(\ell_1, \ldots, \ell_n) \in \mathbb{C}^n \setminus 0$, consider the line $L = \mathbb{C} \cdot (\ell_1, \ldots, \ell_n)$ in $\mathbb{C}^n$ and let λ_L be the linear form

$$\lambda_L := \ell_1 x_1 + \cdots + \ell_n x_n \in \mathbb{C}[\boldsymbol{x}_n]. \tag{3.1}$$

The linear form λ_L is defined up to a nonzero scalar.

Let $\mathcal{A} = \{H_1, \ldots, H_m\}$ be a arrangement of linear hyperplanes in $\mathbb{C}^n$ and let $k \geq -1$ be an integer. Ardila–Postnikov [1, Section 3.1] defined the ideal $J_{\mathcal{A},k} \subseteq \mathbb{C}[\boldsymbol{x}_n]$ by

$$J_{\mathcal{A},k} := \left(\lambda_L^{\rho_{\mathcal{A}}(L)+k} \mid L \subseteq \mathbb{C}^n \text{ a line}\right) \tag{3.2}$$

where

$$\rho_{\mathcal{A}}(L) := \#\{1 \leq i \leq m \mid L \not\subseteq H_i\}. \tag{3.3}$$

The cases $k \in \{-1, 0, 1\}$ are of particular interest, as the respective quotients $\mathbb{C}[\boldsymbol{x}_n]/J_{\mathcal{A},k}$ are the *internal*, *central* and *external zonotopal algebras* [1, 7]. The (singly-graded) Hilbert series of these algebras are obtained as univariate specializations of the Tutte polynomial of $\mathfrak{M}_{\mathcal{A}}$. More precisely, if r is the rank of $\mathcal{A}$ we have [1, 7]

$$\mathrm{Hilb}(\mathbb{C}[\mathbf{x}_n]/J_{\mathcal{A},0};q) = q^{m-r}T_{\mathcal{A}}(1,q^{-1}), \tag{3.4}$$

$$\mathrm{Hilb}(\mathbb{C}[\mathbf{x}_n]/J_{\mathcal{A},1};q) = q^{m-r}T_{\mathcal{A}}\left(1+q,q^{-1}\right). \tag{3.5}$$

The following superspace ideals are our object of study.

Definition 3.1. Let $\mathcal{A}$ be a arrangement in $\mathbb{C}^n$. We let $I_{\mathcal{A}} \subseteq \Omega_n$ be the differential closure of the ideal $J_{\mathcal{A},1}$. In terms of generators, we have

$$I_{\mathcal{A}} := \left(\lambda_L^{\rho_{\mathcal{A}}(L)+1}, d\lambda_L^{\rho_{\mathcal{A}}(L)+1} \,:\, L \subseteq \mathbb{C}^n \text{ a line}\right) \subseteq \Omega_n. \tag{3.6}$$

The ideal $I_{\mathcal{A}}$ is bihomogeneous. We write $\mathcal{E}_{\mathcal{A}} := \Omega_n/I_{\mathcal{A}}$ for the associated bigraded quotient ring and $I_{\mathcal{A}}^{\perp} \subseteq \Omega_n$ for the inverse system of $I_{\mathcal{A}}$.

We will be interested in the bigraded Hilbert series $\mathrm{Hilb}(\mathcal{E}_{\mathcal{A}};q,t)$. By setting the θ-variables equal to zero, we see from Equation (3.5) that

$$\mathrm{Hilb}(\mathcal{E}_{\mathcal{A}};q,0) = \mathrm{Hilb}(I_{\mathcal{A}}^{\perp};q,0) = \mathrm{Hilb}(\mathbb{C}[\mathbf{x}_n]/J_{\mathcal{A},1};q) = q^{m-r}T_{\mathcal{A}}\left(1+q,q^{-1}\right)$$

Less obviously (see Corollary 4.1), the top t-degree of $\mathrm{Hilb}(\mathcal{E}_{\mathcal{A}};q,t)$ is a polynomial in q coinciding with $\mathrm{Hilb}(\mathbb{C}[\mathbf{x}_n]/J_{\mathcal{A},0};q)$. Various features of $\mathcal{E}_{\mathcal{A}}$ interpolate between the external and central zonotopal algebras.

Example 3.2. Consider the hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^2$ determined by the hyperplane $x_1 - x_2 = 0$. For $L = \mathbb{C}\cdot(a,b)$ with $a \neq b$, we know that $\rho_{\mathcal{A}}(L) = 1$. If $a = b \neq 0$, on the other hand, then $\rho_{\mathcal{A}}(L) = 0$. In turn, this means that the ideal $I_{\mathcal{A}}$ is generated by $(ax_1+bx_2)^2$ where $a \neq b$ are not both 0, x_1+x_2, as well as their differentials, i.e.

$$I_{\mathcal{A}} = (x_1^2, x_2^2, x_1+x_2, \theta_1 x_1, \theta_2 x_2, \theta_1+\theta_2).$$

It can be checked that $\{1, x_1, \theta_1\}$ is a monomial basis for $\mathcal{E}_{\mathcal{A}}$, so $\mathrm{Hilb}(\mathcal{E}_{\mathcal{A}};q,t) = 1+q+t$.

We find the Hilbert series of $\mathcal{E}_{\mathcal{A}}$ by studying $I_{\mathcal{A}}^{\perp}$. First, we give a spanning set for $I_{\mathcal{A}}^{\perp}$.

3.2 A spanning set via an exact sequence

Let us consider the case of an arrangement $\mathcal{A} = \{H_1, H_2, \ldots, H_m\}$ of m linear hyperplanes in $\mathbb{C}^n$. Changing coordinates if necessary, *we may assume that H_1 is given by $x_1 = 0$ and we will work under this assumption for the remainder of this section.*

Let $\mathcal{A} - H_1 = \{H_2, \ldots, H_m\}$ and $\mathcal{A} \mid H_1 = \{H_j \cap H_1 \,:\, j \geq 2\}$ be the deletion and restriction of $\mathcal{A}$ with respect to H_1. We claim that the following is an exact sequence

$$0 \to I_{\mathcal{A}-H_1}^{\perp} \xrightarrow{\varphi} I_{\mathcal{A}}^{\perp} \xrightarrow{\psi} I_{\mathcal{A}|H_1}^{\perp} \oplus I_{\mathcal{A}|H_1}^{\perp} \to 0 \tag{3.7}$$

where

- the map $\varphi : I_{\mathcal{A}-H_1}^{\perp} \to I_{\mathcal{A}}^{\perp}$ is given by $\varphi(f) := x_1 \cdot f$, and
- the map $\psi : I_{\mathcal{A}}^{\perp} \to I_{\mathcal{A}|H_1}^{\perp} \oplus I_{\mathcal{A}|H_1}^{\perp}$ is given by

$$\psi(f) := (f \mid_{x_1=0,\theta_1=0}, \theta_1 \odot f \mid_{x_1=0}). \tag{3.8}$$

In the second coordinate of ψ, the notation $\theta_1 \odot f \mid_{x_1=0}$ is justified because differentiation $\theta_1 \odot (-)$ with respect to θ_1 commutes with evaluation $(-) \mid_{x_1=0}$ at $x_1 = 0$. In the full version of this abstract[11], we prove that the maps φ and ψ in (3.7) map into their claimed targets and that the sequence (3.7) is indeed exact, which implies

$$\mathrm{Hilb}(\mathcal{E}_{\mathcal{A}}; q, t) = q \cdot \mathrm{Hilb}(\mathcal{E}_{\mathcal{A}-H_1}; q, t) + (1+t) \cdot \mathrm{Hilb}(\mathcal{E}_{\mathcal{A}|H_1}; q, t). \tag{3.9}$$

We will use (3.9) to relate the Hilbert series of $\mathcal{E}_{\mathcal{A}}$ to the Tutte polynomial of $\mathcal{A}$.

Example 3.3. Let $\mathcal{A}$ be the arrangement in $\mathbb{C}^2$ determined by two copies of the hyperplane H given by $x_1 = 0$. The reader may check that a basis for $I_{\mathcal{A}}^{\perp}$ is given by

$$\{1, x_1, x_1^2, \theta_1, \theta_1 x_1\}.$$

The arrangement $\mathcal{A} - H$ is determined by a single copy of H and $I_{\mathcal{A}-H}^{\perp}$ has basis $\{1, x_1, \theta_1\}$. Multiplying each element by x_1 produces the subset $\{x_1, x_1^2, x_1\theta_1\}$ of the basis for $I_{\mathcal{A}}^{\perp}$. Restricting $\mathcal{A}$ to H produces a "degenerate" hyperplane which may be deleted. Thus $I_{\mathcal{A}|H}^{\perp} = \mathbb{C}\{1\}$. Finally, observe that the only basis element that survives the specialization $x_1 = 0$ and $\theta_1 = 0$ is 1, and the unique element that that survives the $\odot$ action of θ_1 and then the specialization $x_1 = 0$ is θ_1. Each element of the given basis for $I_{\mathcal{A}}^{\perp}$ has appeared exactly once in this description.

3.3 Hilbert series and Tutte polynomials

We can now describe the consequences of the exact sequence (3.7).

Theorem 3.4. *For any rank r arrangement $\mathcal{A}$ of size m in $\mathbb{C}^n$, we have*

$$\mathrm{Hilb}\left(\mathcal{E}_{\mathcal{A}}; q, t\right) = \mathrm{Hilb}(I_{\mathcal{A}}^{\perp}; q, t) = (1+t)^r q^{m-r} T_{\mathcal{A}}\left(\frac{1+q+t}{1+t}, \frac{1}{q}\right)$$

where q tracks bosonic degree and t tracks fermionic degree.

Proof. It suffices to consider the Hilbert series for $I_{\mathcal{A}}^{\perp}$. We interpret $\mathrm{Hilb}(-; q, t)$ as a function from the class of realizable matroids over $\mathbb{C}$ (which are equivalent to arrangements $\mathcal{A}$ that allow for degenerate hyperplanes) to $\mathbb{N}[q, t]$ and proceed to show that it meets the criteria of a *Tutte–Grothendieck invariant* as outlined in [4].[3]

[3] Note that the class of realizable matroids is a minor-closed family, which means we are justified in applying this strategy.

Begin by noting that, if $\mathcal{A}$ is the empty arrangement, then $\mathrm{Hilb}(I_{\mathcal{A}}^{\perp}; q, t)$ is indeed equal to 1. So we may suppose that $\mathcal{A}$ is nonempty. Note further that if H is a degenerate hyperplane in $\mathcal{A}$ then $\mathrm{Hilb}(I_{\mathcal{A}}^{\perp}; q, t) = \mathrm{Hilb}(I_{\mathcal{A}-H}^{\perp}; q, t)$.

Now suppose that $\mathcal{A}$ contains a hyperplane H that is not degenerate. Then (3.7) gives

$$\mathrm{Hilb}(I_{\mathcal{A}}^{\perp}; q, t) = q \cdot \mathrm{Hilb}(I_{\mathcal{A}-H}^{\perp}; q, t) + (1+t) \cdot \mathrm{Hilb}(I_{\mathcal{A}|H}^{\perp}; q, t). \tag{3.10}$$

If we were to further assume that H is a coloop, i.e. its normal vector belongs to every basis in $\mathfrak{M}_{\mathcal{A}}$, then (3.10) may be rewritten as

$$\mathrm{Hilb}(I_{\mathcal{A}}^{\perp}; q, t) = (1+q+t) \cdot \mathrm{Hilb}(I_{\mathcal{A}|H}^{\perp}; q, t). \tag{3.11}$$

Finally observe that the preceding equalities inductively imply that if arrangements $\mathcal{A}$ and $\mathcal{B}$ determine isomorphic matroids $\mathfrak{M}_{\mathcal{A}}$ and $\mathfrak{M}_{\mathcal{B}}$ then $\mathrm{Hilb}(I_{\mathcal{A}}^{\perp}; q, t) = \mathrm{Hilb}(I_{\mathcal{B}}^{\perp}; q, t)$.

Now our claim follows from the fact that the Tutte polynomial is a *universal Tutte–Grothendieck invariant*, which implies Hilb is a specialization of the Tutte polynomial. The precise specialization can be read off from the preceding equations. □

Example 3.5. Consider the arrangement $\mathcal{A}$ in $\mathbb{R}^2$ determined by the hyperplanes $x_1 = 0$, $x_2 = 0$, and $x_1 + x_2 = 0$, shown on the left in Figure 1. One may check that

$$I_{\mathcal{A}} = (x_1^3, x_2^3, (x_1 - x_2)^3, x_1^2 x_2^2, \theta_1 x_1^2, \theta_2 x_2^2, (\theta_1 - \theta_2)(x_1 - x_2)^2, x_1 x_2(\theta_1 x_2 + \theta_2 x_1)).$$

One can then compute the following monomial basis for $\mathcal{E}_{\mathcal{A}}$:

$$\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1x_2^2, \theta_1, \theta_2, x_1\theta_1, x_1\theta_2, x_2\theta_1,$$
$$x_2\theta_2, x_1x_2\theta_1, x_1x_2\theta_2, x_2^2\theta_1, \theta_1\theta_2, x_1\theta_1\theta_2, x_2\theta_1\theta_2\}.$$

We thus see that

$$\mathrm{Hilb}(\mathcal{E}_{\mathcal{A}}; q, t) = (1 + 2q + 3q^2 + q^3) + t(2 + 4q + 3q^2) + t^2(1 + 2q).$$

The Tutte polynomial of $\mathfrak{M}_{\mathcal{A}}$ is $T_{\mathcal{A}}(x, y) = x^2 + x + y$, and it can be checked that substituting as per Theorem 3.4 produces $\mathrm{Hilb}(\mathcal{E}_{\mathcal{A}}; q, t)$.

3.4 A basis for the Macaulay inverse

Fix a total order on the hyperplanes of $\mathcal{A}$; this induces a lexicographical order on the subsets of $\mathcal{A}$. Given a matroid basis B of $\mathcal{A}$ and a hyperplane $H \in \mathcal{A} - B$, the hyperplane H is *externally active with respect to B* if B is the lexicographically largest basis contained in $B \cup H$. Otherwise, the hyperplane H is *externally passive with respect to B*. Similarly, a hyperplane $H \in B$ is *internally active with respect to B* if B is the lexicographically smallest basis containing $B - H$. Otherwise H is *internally passive with respect to B*. By Theorem 3.4, we have

$$\mathrm{Hilb}(I_{\mathcal{A}}^{\perp}; q, t) = \sum_{B \in \mathcal{B}} (1+q+t)^{\mathrm{ia}(B)} (1+t)^{\mathrm{ip}(B)} q^{\mathrm{ep}(B)}. \tag{3.12}$$

Example 3.6. Consider the hyperplane arrangement from Example 3.5 with the normal vectors to the hyperplanes $H_1 = \{x_1 = 0\}$, $H_2 = \{x_2 = 0\}$, and $H_3 = \{x_1 + x_2 = 0\}$ recorded as columns of a 2×3 matrix below. Assume our total order on hyperplanes in $\mathcal{A}$ is obtained by reading the columns left to right, i.e. $H_1 < H_2 < H_3$.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

For the basis $\{H_1, H_2\}$, the hyperplane H_3 is externally passive whereas H_1 and H_2 are internally active. For the basis $\{H_1, H_3\}$, the hyperplane H_2 is externally passive, H_1 is internally active while H_3 is internally passive. Finally for the basis $\{H_2, H_3\}$, the hyperplane H_1 is externally active whereas both H_2 and H_3 are internally passive. Thus the right-hand side of (3.12) equals

$$(1+q+t)^2 q^1 (1+t)^0 + (1+q+t)^1 q^0 (1+t)^1 + (1+q+t)^0 q^0 (1+t)^2,$$

which the reader may verify is the same as the Hilbert series obtained earlier.

Recall from the introduction that we write $EA_{\mathcal{A}}(B), EP_{\mathcal{A}}(B), IA_{\mathcal{A}}(B)$, and $IP_{\mathcal{A}}(B)$ for the externally active, externally passive, internally active, and internally passive hyperplanes in $\mathcal{A}$ with respect to a given basis B. Note also that, if we have a total order on the hyperplanes in an arrangement $\mathcal{A}$, then the deleted and restricted arrangement $\mathcal{A} - H$ and $\mathcal{A} \mid H$ both inherit total orders. We consider the family $M_{\mathcal{A}}$ of superspace elements

$$M_{\mathcal{A}} := \bigcup_B \left\{ \prod_{e \in E} \alpha_e \times \prod_{i \in I} d\alpha_i \times \prod_{s \in S} \alpha_s \times \prod_{t \in T} d\alpha_t \,:\, \begin{array}{c} E = EP_{\mathcal{A}}(B),\ I \subseteq IP_{\mathcal{A}}(B), \\ S, T \subseteq IA_{\mathcal{A}}(B),\ S \cap T = \varnothing \end{array} \right\} \tag{3.13}$$

where the union is over matroid bases B of $\mathcal{A}$.

Theorem 3.7. *The set $M_{\mathcal{A}}$ forms a monomial basis for the bigraded vector space $I_{\mathcal{A}}^{\perp}$.*

4 Further remarks

4.1 The classical external and central zonotopal algebras

Corollary 4.1. *Let r be the rank of the arrangement $\mathcal{A} \subseteq \mathbb{C}^n$ and let $m = |\mathcal{A}|$.*

(1) $\mathrm{Hilb}(\mathcal{E}_{\mathcal{A}}; q, 0) = \mathrm{Hilb}(\mathbb{C}[\mathbf{x}_n]/J_{\mathcal{A},1}; q)$.

(2) $[t^r]\mathrm{Hilb}(\mathcal{E}_{\mathcal{A}}; q, t) = \mathrm{Hilb}(\mathbb{C}[\mathbf{x}_n]/J_{\mathcal{A},0}; q)$ *where $[t^r](-)$ extracts the coefficient of t^r.*

(3) Let $\mathrm{top}(\mathcal{A})$ *denote the summand of maximal total degree in* $\mathrm{Hilb}(\mathcal{E}_{\mathcal{A}}; q, t) \in \mathbb{N}[q, t]$*. Then*

$$\mathrm{top}(\mathcal{A}) = (-1)^r q^{m-r} t^r \chi_{\mathcal{A}}\left(-\frac{q}{t}\right).$$

where $\chi_{\mathcal{A}}$ is the characteristic polynomial of $\mathcal{A}$.

Going back to Example 3.5, we see that the coefficients of t^0 and t^2 are indeed the Hilbert series for the classical external and central zonotopal algebras. For $\mathrm{top}(\mathcal{A})$, we get $q^3 + 3q^2t + 2qt^2 = q(q+t)(q+2t)$, and this is clearly a homogenized version of the characteristic polynomial of the central arrangement in Figure 1.

Remark 4.2. Corollary 4.1(2) on the bosonic Hilbert series of the top fermionic degree of $\mathcal{E}_{\mathcal{A}}$ may be upgraded to a stronger algebraic fact – namely, that there is a natural isomorphism of graded vector spaces

$$\mathbb{C}[\boldsymbol{x}_n]/J_{\mathcal{A},0} \xrightarrow{\sim} (\mathcal{E}_{\mathcal{A}})_{*,r}$$

identifying the central zonotopal algebra of $\mathcal{A}$ with the top fermionic-degree piece of $\mathcal{E}_{\mathcal{A}}$.

4.2 Real arrangements

Suppose $\mathcal{A}$ is a complexified real arrangement corresponding to a matroid $\mathfrak{M}$ that is realizable over $\mathbb{R}$. Let $\mathcal{A}^{\mathrm{gen}}$ be a *generic* affine arrangement determined by $\mathfrak{M}$. Thus hyperplanes in $\mathcal{A}^{\mathrm{gen}}$ are obtained as affine translates of linear hyperplanes in $\mathcal{A}$ with the constraint that k hyperplanes have a nonempty intersection if and only if the normals to these hyperplanes form a linearly independent set. Let $f_i(\mathcal{A}^{\mathrm{gen}})$ for $0 \leq i \leq n$ denote the number of i-dimensional faces in the polyhedral complex on $\mathbb{R}^n$ induced by $\mathcal{A}^{\mathrm{gen}}$.

Corollary 4.3. *For any complexified real arrangement $\mathcal{A}$,*

$$\mathrm{Hilb}(\mathcal{E}_{\mathcal{A}}; 1, t) = \sum_{i=0}^{n} f_{n-i}(\mathcal{A}^{\mathrm{gen}}) \cdot t^i.$$

Observe that f_n and f_0 give the dimensions of the external and central zonotopal algebras, respectively, which agrees with Corollary 4.1(1)–(2).

Given a matroid $\mathfrak{M}$ and $d \geq 1$, let $d\mathfrak{M}$ denote its *d-fold thickening*, i.e. the matroid obtained obtained by including $(d-1)$ additional parallel elements for each element in $\mathfrak{M}$. Given a central arrangement $\mathcal{A} \subseteq \mathbb{R}^n$, we denote the two-fold thickening of the underlying matroid by $2\mathcal{A}$.

Corollary 4.4. *Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a central arrangement. The following equality holds:*

$$\mathrm{Hilb}(\mathcal{E}_{\mathcal{A}}; q, t)|_{(q,t)=(q^2,q)} = \mathrm{Hilb}(\mathbb{C}[\boldsymbol{x}_n]/J_{2\mathcal{A},1}; q).$$

Thus $\dim(\mathcal{E}_{\mathcal{A}})$ *equals the number of regions in* $2\mathcal{A}^{\mathrm{gen}}$.

As an example, the rightmost arrangement in Figure 1 is the generic hyperplane arrangement in $\mathbb{R}^2$ obtained by "doubling" each hyperplane in the arrangement in Example 3.5. This arrangement has 19 regions, matching the size of the example basis.

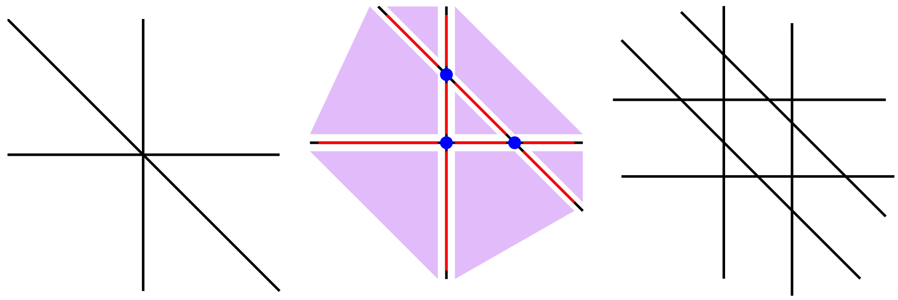

Figure 1: A central arrangement $\mathcal{A}$ (left), the corresponding generic affine arrangment $\mathcal{A}^{\text{gen}}$ (center), and the two-fold thickening $2\mathcal{A}^{\text{gen}}$ (right).

4.3 A conjecture regarding the classical internal zonotopal algebra

The reader may wonder why the internal zonotopal algebra does not make an appearance in this work. We aim to clarify this matter here. Consider the superspace ideal

$$I'_{\mathcal{A}} \coloneqq I_{\mathcal{A},0} \coloneqq \left(\lambda_L^{\rho_{\mathcal{A}}(L)}, d\lambda_L^{\rho_{\mathcal{A}}(L)} \,:\, L \subseteq \mathbb{C}^n \text{ a line}\right) \subseteq \Omega_n. \tag{4.1}$$

This ideal is the differential closure of the ideal $J_{\mathcal{A},0}$ defined in (3.2). In analogy with Corollary 4.1, we expect that the quotient $\Omega_n / I'_{\mathcal{A}}$ contains not just the classical central zonotopal algebra but also the internal zonotopal algebra, as explained below. Even though the Hilbert series of the internal zonotopal algebra is known and, keeping with the theme, equals a specialization of the Tutte polynomial (cf. [1, Proposition 4.15]), there is no known basis for its Macaulay inverse [2].

Conjecture 4.5. *Let $\mathcal{A}$ be a rank r central arrangement in $\mathbb{C}^n$ with $|\mathcal{A}| = m$. The bigraded Hilbert series of $\Omega_n / I'_{\mathcal{A}}$ satisfies the equality*

$$\operatorname{Hilb}\left(\Omega_n / I'_{\mathcal{A}}; q, t\right) = (1+t)^r q^{m-r} T_{\mathcal{A}}\left(\frac{1}{1+t}, \frac{1}{q}\right).$$

At $t = 0$ we recover $\operatorname{Hilb}(\mathbb{C}[\boldsymbol{x}_n]/J_{\mathcal{A},0}; q)$ as is expected. Developing the right-hand side in Conjecture 4.5 following (2.8) gives

$$(1+t)^r q^{m-r} T_{\mathcal{A}}\left(\frac{1}{1+t}, \frac{1}{q}\right) = q^{m-r} \sum_{A \subseteq \mathsf{E}} (-1)^{r-r(A)} t^{r-r(A)} (1+t)^{r(A)} \left(\frac{1}{q} - 1\right)^{|A|-r(A)}, \tag{4.2}$$

and extracting the coefficient of t^r gives $q^{m-r} T_{\mathcal{A}}\left(0, q^{-1}\right)$. This last quantity is the singly graded Hilbert series of the internal zonotopal algebra $\mathbb{C}[\boldsymbol{x}_n]/J_{\mathcal{A},-1}$. In fact, an analogue of Corollary 4.3 holds for real arrangements as well, except that one now records the f-vector of the "bounded" polyhedral complex of $\mathcal{A}^{\text{gen}}$.

Example 4.6. Consider $\mathcal{A} \subseteq \mathbb{R}^2$ determined by the hyperplanes $x_1 = 0$, $x_2 = 0$, and $x_1 + x_2 = 0$. One may check that $I'_{\mathcal{A}} = (x_1^2, x_2^2, x_1x_2, \theta_1x_1, \theta_2x_2, \theta_1x_2 + \theta_2x_1)$ and that $\Omega_n / I'_{\mathcal{A}}$ has monomial basis $\{1, x_1, x_2, \theta_1, \theta_2, \theta_1x_2, \theta_1\theta_2\}$. Hence

$$\mathrm{Hilb}(\Omega_n / I'_{\mathcal{A}}; q, t) = (1 + 2q) + t(2 + q) + t^2.$$

It is easily checked that $(1+t)^2 q T_{\mathcal{A}}(1/(1+t), 1/q)$ for $T_{\mathcal{A}}(x, y) = x^2 + x + y$ is precisely the Hilbert series computed above, agreeing with Conjecture 4.5. Note that setting $q = 1$ gives the polynomial $3 + 3t + t^2$ whose sequence of coefficients agrees with the f-vector corresponding to the unique bounded face of the arrangement in the center of Figure 1.

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