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A degeneration of the brick variety and a mixed subdivision of the associahedron into cubes

Raj Gandhi*[1] and Gabe Udell†[2]

[1]*Department of Mathematics, Cornell University, Ithaca, NY 14853, USA*
[2]*Department of Mathematics, Cornell University, Ithaca, NY 14853, USA*

Abstract. We study a degeneration of Escobar's brick variety in cases where it is toric. We show that the components of the central fibre of the degeneration are reduced toric products of Richardson varieties. We write the moment polytope of the variety, which has the same normal fan as the brick polytope of Pilaud and Santos, as a Minkowski sum of Bruhat interval polytopes and show that the polyhedral subdivision induced by the degeneration is a mixed subdivision. As corollaries, we obtain a characterization of toric Richardsons, and a subdivision of the associahedron into combinatorial cubes.

Keywords: brick polytope, associahedron, polyhedral subdivision, Richardson variety

1 Introduction

Toric geometry provides a bridge between algebraic geometry and discrete geometry. Every projectively embedded toric variety has a moment polytope, and every lattice polytope satisfying certain conditions comes from a toric variety. Characteristics of the moment polytope can be read off from its toric variety (e.g., the dimension of the moment polytope is the dimension of the effective torus action on the variety.)

Our focus is on brick varieties and brick polytopes. Let Q be a word in the simple generators of S_n, and let $w \in S_n$. The *brick polytope* was defined by Pilaud and Santos in [10] as a polytopal realization of certain subword complexes $\Delta(Q, w)$; the *subword complex* was originally defined by Knutson and Miller [6]. The *brick variety* Brick^Q, with respect to Q, was defined by Escobar in [4]. There, Escobar showed that, under certain conditions on Q, the brick variety Brick^Q is a toric variety, and, moreover, that it is the toric variety of the brick polytope. We introduce a slight variation on the brick polytope called the *bulky brick polytope* which is the moment polytope of the brick variety with respect to a different choice of projective embedding from [4]. Our first result is that the bulky brick polytope has a Minkowski sum decomposition into Bruhat interval

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†gru5@cornell.edu. G.U. was supported by the National Science Foundation Graduate Research Fellowship

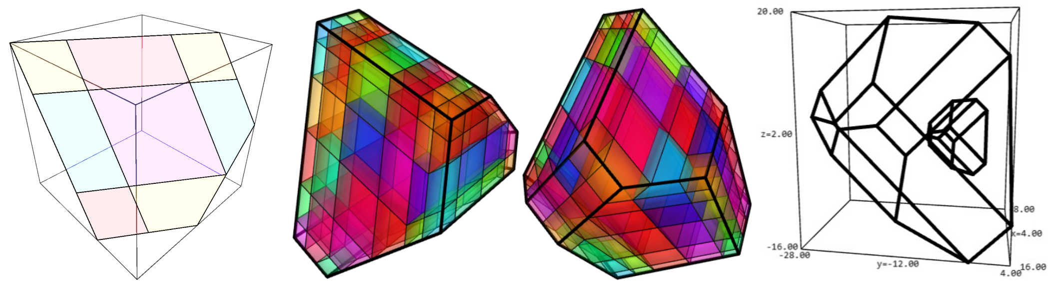


Figure 1: Left: Subdivision of the 2D associahedron into combinatorial cubes. Center: Projection of subdivision of the 3D associahedron into combinatorial cubes. Right: Projections of the skeletons of the 3D associahedron and 3D bulky associahedron.

polytopes (Theorem 4.6). Using this, we prove a new characterization of toric Richardson varieties (Theorem 4.7).

In [13], Sturmfels proved that Gröbner degenerations of toric varieties correspond to regular subdivisions of their moment polytopes. Applying this principle, we describe a degeneration of Brick^Q in Section 5 and study the corresponding subdivision of the brick polytope. We show that the central fibre of this degeneration is a union of reduced products of toric Richardsons, and that the degeneration induces a mixed subdivision of the corresponding brick polytope. In a special case, we apply results of [9] to produce a subdivision of the associahedron into pieces combinatorially equivalent to cubes. See Figure 1 for illustrations of the subdivisions of the 2D and 3D associahedra.

2 The brick polytope

We recall the *brick polytope* of [10] and [11], closely following the exposition in [4], and we define the *bulky brick polytope*, laying the foundations for this paper.

Consider the symmetric group S_n on n letters. The simple transposition $s_i \in S_n$ swaps i and $i+1$ and fixes all other j. Any element $w \in S_n$ can be expressed $w = s_{i_1} \cdots s_{i_k}$, where the s_{i_j} are simple transpositions in S_n. The **length** $\ell(w)$ of an element $w \in S_n$ is the minimal k over all expressions $w = s_{i_1} \cdots s_{i_k}$ of w as a product of simple transpositions s_{i_j} in S_n. Any expression $w = s_{i_1} \cdots s_{i_k}$ with $k = \ell(w)$ is called **reduced**. A **word** $Q = (s_{q_1}, \ldots, s_{q_m})$ in S_n is a sequence of simple transpositions $s_{q_i} \in S_n$, and a **subword** of Q is a subsequence of Q. We can associate to any word $Q = (s_{q_1}, \ldots, s_{q_m})$ a permutation $s_{q_1} \cdots s_{q_m}$. If $v \in S_n$, then a **word for** v is any word $Q = (s_{q_1}, \ldots, s_{q_m})$ in S_n with $v = s_{q_1} \cdots s_{q_m}$. The **Bruhat order** $\leq$ on S_n is the partial order on S_n in which $u \leq v$ if and only if, for any word Q of u, there is a subword of Q that is a word for u. The **longest permutation** w_0 in S_n sends i to $n+1-i$ for all i.

Definition 2.1 ([8, 14]). *The **Bruhat interval polytope** $P_{u,v}$ is the convex hull of the points $(w(1), w(2), \ldots, w(n))$ over all $u \leq w \leq v$.*

Definition 2.2 ([2]). *The **twisted Bruhat interval polytope** $\tilde{P}_{u,v}$ is the convex hull of the points $(n+1-w^{-1}(1), n+1-w^{-1}(2), \ldots, n+1-w^{-1}(n))$ over all $u \leq w \leq v$.*

Remark 2.3. *Observe that $\tilde{P}_{u,v} = P_{w_0 v^{-1}, w_0 u^{-1}}$. A (twisted) Bruhat interval polytope $P_{u,v}$ ($\tilde{P}_{u,v}$) is **toric** if $\dim P_{u,v} = \ell(v) - \ell(u)$ ($\dim \tilde{P}_{u,v} = \ell(v) - \ell(u)$). When $P_{u,v}$ is toric, so are $P_{u^{-1},v^{-1}}$ and $\tilde{P}_{u,v}$, and vice versa [9].*

Fix a word $Q = (s_{q_1}, \ldots, s_{q_m})$ in S_n. We can view a subword J of Q as a sequence $(s_{j_1}, \ldots, s_{j_m})$ obtained from Q by replacing some entries by 1. Denote by $Q \setminus J$ the sequence of length $|Q|$ whose k-th entry equals 1 if $j_k \neq 1$ and equals s_{q_k} otherwise. Denote by $J_{(k)}$ the product of the leftmost k entries in J, and $J_{(0)} = 1$.

Definition 2.4 ([6]). *The **subword complex** $\Delta(Q, w)$, with respect to a word Q and $w \in S_n$, is the simplicial complex on vertex set Q whose facets (faces) are the subwords J of Q such that the product $(Q \setminus J)_{(|Q|)}$ is a reduced expression (contains a reduced expression) for w.*

Definition 2.5 ([6]). *The **Demazure product** $\mathrm{Dem}(Q)$ of a word Q in S_n is defined inductively:*

- $\mathrm{Dem}(\textit{empty word}) = 1$.
- $\mathrm{Dem}((Q, s_i)) = \begin{cases} \mathrm{Dem}(Q) \cdot s_i, & \text{if } \ell(\mathrm{Dem}(Q) \cdot s_i) > \ell(\mathrm{Dem}(Q)), \\ \mathrm{Dem}(Q), & \text{otherwise.} \end{cases}$

Define $[n] := \{1, \ldots, n\}$. Consider the standard basis vectors p_i of $\mathbb{R}^n$, with 1 appearing in entry i and 0 appearing in all other entries. Define the vectors $\omega_i = p_1 + \cdots + p_i$ for $i \in [n]$ and $\alpha_i = p_i - p_{i+1}$ for $i \in [n-1]$. Given a subword complex $\Delta(Q, w_0)$ and a face J of $\Delta(Q, w_0)$, the **weight function** and **bulky weight function**, $w(J, \cdot)\colon \{\text{subwords of } Q\} \to \mathbb{R}^n$ and $\mathbf{w}(J, \cdot)\colon \{\text{subwords of } Q\} \to \mathbb{R}^n$, are defined by

$$w(J,k) := (Q \setminus J)_{(k)}(\omega_{q_k}) \qquad \text{and} \qquad \mathbf{w}(J,k) := (Q \setminus J)_{(k)} \left(\sum_{i=1}^{n} \omega_i \right).$$

The **brick vector** is $B(J) := \sum_{i=1}^{|Q|} w(J,k)$ and the **brick polytope** of Q is

$$\mathcal{B}(Q) := \mathrm{conv}\{B(J) \mid J \in \Delta(Q, w_0) \text{ and } (Q \setminus J)_{(|Q|)} = w_0\}.$$

The **bulky brick vector** is $\mathbf{B}(J) = \sum_{k=0}^{|Q|} \mathbf{w}(J,k)$, and the **bulky brick polytope** of Q is

$$\mathfrak{B}(Q) := \mathrm{conv}\{\mathbf{B}(J) \mid J \in \Delta(Q, w_0) \text{ and } (Q \setminus J)_{(|Q|)} = w_0\}.$$

Remark 2.6. *The brick polytope was originally defined in [10]. The definitions of $w(J,k)$, $B(J)$, and $\mathcal{B}(Q)$ come from [11] with a modification made by [4]. We introduce the bulky versions $\mathbf{w}(J,k)$, $\mathbf{B}(J)$, and $\mathfrak{B}(Q)$ to match our treatment of the brick variety.*

Definition 2.7 ([11]). *Let J be a face of $\Delta(Q, w_0)$. The **root function** is $r(J, \cdot)\colon \{\text{subwords of } Q\} \to \mathbb{R}^n$, with $r(J,k) := (Q \setminus J)_{(k)}(\alpha_{q_k})$. We say Q is **root independent** if for some facet (or all facets) J of $\Delta(Q, w_0)$, the multiset $\{\{r(J,k) : q_k \in J\}\}$ is linearly independent.*

3 A variety of varieties

We recall several geometric objects, including the *brick variety* of [4].

Fix a basis $e = (e_1, \dots, e_n)$ for $\mathbb{C}^n$. Let $G = \mathrm{GL}_n(\mathbb{C})$ be the set of $n \times n$ invertible matrices over $\mathbb{C}$. Consider the set of upper (resp. lower) triangular matrices B_+ (resp. B_-) in G, and the set of diagonal matrices T in G. For $i_1 < \cdots < i_k$, set $F_{\{i_1,\dots,i_k\}} := \mathrm{Span}(e_{i_1}, \dots, e_{i_k})$. Let $\mathrm{Gr}(k, n)$ be the **Grassmannian** of k-planes in $\mathbb{C}^n$, and observe that there is a natural action of G on $\mathrm{Gr}(k, n)$. Coordinate subspaces of the form $F_{\{i_1,\dots,i_k\}}$ are exactly the T-fixed points $\mathrm{Gr}(k, n)^T$ of $\mathrm{Gr}(k, n)$. The **complete flag variety** Fl_n is the variety of complete flags of subspaces in $\mathbb{C}^n$:

$$\mathrm{Fl}_n = \{F_\bullet = (F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{n-1} \subseteq F_n = \mathbb{C}^n) \mid \dim(F_i) = i\}.$$

The standard flag in $\mathbb{C}^n$ is $F_\bullet^{\mathrm{st}} = \left(F_{\{1\}} \subseteq F_{\{1,2\}} \subseteq \cdots \subseteq F_{\{1,\dots,n-1\}} \subseteq \mathbb{C}^n\right)$, and the opposite flag in $\mathbb{C}^n$ is $F_\bullet^{\mathrm{op}} = \left(F_{\{n\}} \subseteq F_{\{n,n-1\}} \subseteq \cdots \subseteq F_{\{n,n-1,n-2,\dots,2\}} \subseteq \mathbb{C}^n\right)$. The group G acts on Fl_n by the action

$$g \cdot F_\bullet := ((g \cdot F_1) \subseteq (g \cdot F_2) \subseteq \cdots \subseteq (g \cdot F_{n-1}) \subseteq \mathbb{C}^n), \quad g \in G,\ F_\bullet \in \mathrm{Fl}_n.$$

For $w \in S_n$ we define the **Schubert variety** $X_w = \overline{B_- w B_+ / B_+}$ and **opposite Schubert variety** $X^w = \overline{B_+ w B_+ / B_+}$ which have the properties that $X_w \subseteq X_v$ iff $w \geq v$, and $X^w \subseteq X^v$ iff $w \leq v$. The **Richardson variety** $X_u^v := X_u \cap X^v \neq \emptyset$ if and only if $u \leq v$. For $\sigma \in S_n$, define $\sigma \cdot F_{\{1,\dots,n\}} := F_{\{\sigma(1),\dots,\sigma(k)\}}$ and $\sigma \cdot F_\bullet^{\mathrm{st}} := ((\sigma \cdot F_{\{1\}}) \subseteq (\sigma \cdot F_{\{1,2\}}) \subseteq \cdots \subseteq (\sigma \cdot F_{\{1,\dots,n-1\}}) \subseteq \mathbb{C}^n)$. The T-fixed points $(\mathrm{Fl}_n)^T$ of Fl_n are $\{\sigma \cdot F_\bullet^{\mathrm{st}} \mid \sigma \in S_n\}$.

The group $G \times G$ acts on $\mathrm{Fl}_n \times \mathrm{Fl}_n$ by $(f, g) \cdot (F_\bullet, G_\bullet) := (f \cdot F_\bullet, g \cdot G_\bullet)$ for all $f, g \in G$, $F_\bullet, G_\bullet \in \mathrm{Fl}_n$. Define $G_\Delta := \{(g, g) \mid g \in G\} \subseteq G \times G$. For $\sigma \in S_n$, the set of pairs of flags in **relative position** σ is the G_Δ-orbit of $(F_\bullet^{\mathrm{st}}, \sigma(F_\bullet^{\mathrm{st}}))$ in $\mathrm{Fl}_n \times \mathrm{Fl}_n$, and we will denote this set by D_σ°. The closure D_σ of D_σ° in $\mathrm{Fl}_n \times \mathrm{Fl}_n$ is $D_\sigma = \sqcup_{\sigma' \leq \sigma} D_{\sigma'}^\circ$ (see, e.g., [3, Lemma 2.1].)

Let $Q = (s_{i_1}, \dots, s_{i_k})$ be a word in S_n. The **brick variety** Brick^Q of [4] is the projective variety consisting of sequences of flags such that the first flag is $F_\bullet^{\mathrm{st}}$ and the last flag is $F_\bullet^{\mathrm{op}}$ and consecutive pairs of flags lie in $D_{s_{i_j}}$:

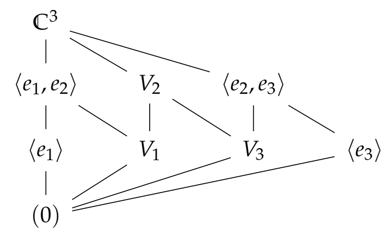
$$\begin{aligned}\mathrm{Brick}^Q &= \{(F_\bullet^0, F_\bullet^1, \dots, F_\bullet^k) \mid F_\bullet^0 = F_\bullet^{\mathrm{st}},\ F_\bullet^k = F_\bullet^{\mathrm{op}}, \text{ and } (F_\bullet^j, F_\bullet^{j+1}) \in D_{s_{i_{j+1}}}, j = 0, \dots, k-1\} \\ &= (D_{s_{i_1}} \times \mathrm{Fl}_n^{k-1}) \cap (\mathrm{Fl}_n \times D_{s_{i_2}} \times \mathrm{Fl}_n^{k-2}) \cap \cdots \cap (\mathrm{Fl}_n^{k-1} \times D_{s_{i_k}}) \cap (F_\bullet^{\mathrm{st}} \times \mathrm{Fl}_n^{k-1} \times F_\bullet^{\mathrm{op}}).\end{aligned}$$

By [4, Theorem 20], the variety Brick^Q is smooth and irreducible, of dimension $|Q| - \ell(w_0)$. Consider the set $\mathcal{J} = \{J \in \Delta(Q, w_0) : (Q \setminus J)_{(|Q|)} = w_0\}$. For $J \in \mathcal{J}$, define the T-fixed point p_J of Brick^Q as

$$p_J := \left(F_\bullet^{\mathrm{st}}, ((Q \setminus J)_{(1)}) \cdot F_\bullet^{\mathrm{st}}, \dots, ((Q \setminus J)_{(m-1)}) \cdot F_\bullet^{\mathrm{st}}, F_\bullet^{\mathrm{op}}\right).$$

By [4, page 7], the map $\mathcal{J} \to (\mathrm{Brick}^Q)^T$ that sends J to p_J is bijective.

Example 3.1 ([4, Example 12])**.** *Consider the word* $Q = (s_1, s_2, s_1, s_2, s_1)$ *in* S_3*. The brick variety* Brick^Q *can be visualized with a* ***Magyar diagram****:*



The points $(F_\bullet^{st}, F_\bullet^1, \ldots, F_\bullet^4, F_\bullet^{op})$ *of* Brick^Q *between* $F_\bullet^{st}$ *and* $F_\bullet^{op}$ *are of the form* $F_\bullet^1 = (V_1 \subseteq \langle e_1, e_2\rangle \subseteq \mathbb{C}^3)$*,* $F_\bullet^2 = (V_1 \subseteq V_2 \subseteq \mathbb{C}^3)$*,* $F_\bullet^3 = (V_3 \subseteq V_2 \subseteq \mathbb{C}^3)$*, and* $F_\bullet^4 = (V_3 \subseteq \langle e_2, e_3\rangle \subseteq \mathbb{C}^3)$*. The point* p_J *corresponding to* $J = (-, -, s_1, s_2, -)$ *has* $V_1 = \langle e_2\rangle$*,* $V_2 = \langle e_2, e_3\rangle$*, and* $V_3 = \langle e_2\rangle$*.*

4 Moment polytopes

4.1 Background

We discuss the moment polytope of the brick variety, following the exposition in [4].

Definition 4.1. *Let* $T' = (\mathbb{C}^\times)^k$*. An algebraic action of* T' *on the projective space* $\mathbb{P}^m$ *is always of the form* $(t_1, \ldots, t_k) \cdot (x_1 : \cdots : x_m) = (x_1 \prod_{i=1}^k t_i^{w_{1,i}} : \cdots : x_m \prod_{i=1}^k t_i^{w_{m,i}})$ *with* $w_{j,i} \in \mathbb{Z}$*. In such a case we say the* T'*-****weight*** *of the fixed point* $(0 : \cdots : 0 : x_j : 0 : \cdots : 0)$ *is the vector* $(w_{j,1}, \ldots, w_{j,k}) \in \mathbb{Z}^k \subseteq \mathbb{R}^k$*. Suppose X is a variety with an algebraic action of* T'*.*

- *If* T' *acts on X with finitely many* T'*-fixed points and there is a* T'*-equivariant embedding* $f\colon X \hookrightarrow \mathbb{P}^m$ *for some m, then the* ***moment polytope*** *of X with respect to f is the convex hull of the* T'*-weights over the points* $f(\gamma)$*, where* γ *is a* T'*-fixed point of X.*
- *The* ***effective torus*** *is* $T'/\mathrm{St}_{T'}(x)$*, where* $\mathrm{St}_{T'}(x)$ *is the* T'*-stabilizer of a general*[1] $x \in X$*.*
- *X is a* ***toric variety*** *with respect to the action of* T' *if X contains a dense* T'*-orbit. Equivalently, by orbit-stabilizer, X is toric if and only if* $\dim(X) = \dim(T'/\mathrm{St}_{T'}(x))$*.*

The T-weight of the image of the fixed point $\sigma \cdot F_{\{1,\ldots,n\}}$ of $\mathrm{Gr}(k, n)$ under the "Plücker embedding" $\psi_k\colon \mathrm{Gr}(k, n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$ is $\sigma(\omega_k)$. The torus T acts diagonally on the variety $\prod_{k=1}^n \mathrm{Gr}(k, n)$, and there is a projective embedding ψ of $\prod_{k=1}^n \mathrm{Gr}(k, n)$ into a projective space, given by the product of Plücker embeddings $\psi_k\colon \mathrm{Gr}(k, n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$, followed by the "Segre embedding". The T-weight of the fixed point $(\sigma_1 \cdot F_{\{1\}}, \ldots, \sigma_n \cdot F_{\{1,\ldots,n\}})$ under ψ, where $\sigma_k \in S_n$ for $k = 1, \ldots, n$, is $\sum_{k=1}^n \sigma_k(\omega_k)$. There is a T-equivariant embedding $\phi_n\colon \mathrm{Fl}_n \hookrightarrow \prod_{k=1}^n \mathrm{Gr}(k, n)$, where $F_\bullet = (F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \mathbb{C}^n)$ is sent to $(F_1, F_2, \ldots, F_n)$. The T-weight of the fixed point $\sigma \cdot F_\bullet^{\mathrm{st}}$ under ϕ_n is $\sigma(\omega_1 + \cdots + \omega_n)$.

[1] *general* in the sense that there is an open subset $U \subseteq X$ such that $\mathrm{St}_{T'}(x)$ is constant for all $x \in U$.

Remark 4.2. *Henceforth, we assume that a T-invariant subvariety of $\mathrm{Gr}(k,n)$ is embedded in $\mathbb{P}^{\binom{n}{k}-1}$ via ψ_k, and a T-invariant subvariety of Fl_n is embedded in $\prod_{k=1}^n \mathrm{Gr}(k,n)$ via ϕ_n. We will use μ_Z to denote the map sending a T-invariant subvariety X (or T-fixed point) of Z to its moment polytope (or T-weight), where Z will always be a single Grassmannian, a product of Grassmannians, a single flag variety, or a product of flag varieties. We always have $\mu_Z(X) = \mathrm{conv}_{x\in X^T}\,\mu_Z(x)$, where X^T denotes the set of T-fixed points of X.*

Remark 4.3. *Observe that $\sigma(\omega_1+\cdots+\omega_n) = \sigma(n, n-1, \ldots, 2, 1) = (n+1-\sigma^{-1}(1), n+1-\sigma^{-1}(2), \ldots, n+1-\sigma^{-1}(n))$. The T-fixed points in the Richardson variety X_u^v are precisely the coordinate flags $w\cdot F_\bullet^{st}$, with $u \le w \le v$. Thus, the moment polytope of X_u^v is the twisted Bruhat interval polytope $\tilde{P}_{u,v}$. The dimension of a T-invariant subvariety matches the dimension of its moment polytope precisely when the variety is toric, so $\tilde{P}_{u,v}$ is toric exactly when X_u^v is.*

Let $Q = (s_{q_1}, \ldots, s_{q_{|Q|}})$ be a word in S_n with $\mathrm{Dem}(Q) = w_0$. Then Brick^Q is a T-invariant subvariety of $\mathrm{Fl}_n^{|Q|+1}$ and thus has a moment polytope $\mu_{\mathrm{Fl}_n^{|Q|+1}}(\mathrm{Brick}^Q)$ given by the convex hull of the resulting T-weights of its T-fixed points.

Alternatively, we may view Brick^Q more directly as a subvariety of a product of Grassmannians. We define $\pi_i : \mathrm{Fl}_n^{|Q|+1} \to \mathrm{Fl}_n$ as the projection onto the i-th Fl_n factor (with indexing of i starting at 0) and $\nu_i : \mathrm{Fl}_n \xhookrightarrow{\phi_n} \prod_{j=1}^n \mathrm{Gr}(j,n) \to \mathrm{Gr}(i,n)$ as the composition of ϕ_n with the projection onto $\mathrm{Gr}(i,n)$. We define $\rho_i : \mathrm{Brick}^Q \to \mathrm{Gr}(q_i, n)$ by $\rho_i = \nu_{q_i}\circ\pi_i$ and obtain the T-equivariant embeddings

$$\rho\colon \mathrm{Brick}^Q \hookrightarrow \prod_{i=1}^{|Q|} \mathrm{Gr}(q_i,n), \qquad (F_\bullet^0, \ldots, F_\bullet^{|Q|}) \mapsto (\rho_1(F_\bullet^1), \ldots, \rho_n(F_\bullet^{|Q|})),$$

$$\pi\colon \mathrm{Brick}^Q \to \prod_{i=0}^{|Q|}\prod_{j=1}^{n} \mathrm{Gr}(j,n), \qquad (F_\bullet^0, \ldots, F_\bullet^{|Q|}) \mapsto (\phi_n(\pi_0(F_\bullet^1)), \ldots, \phi_n(\pi_{|Q|}(F_\bullet^{|Q|}))).$$

Let J be a subword of Q, and recall the fixed point p_J of Brick^Q corresponding to J. The T-weight of p_J under the embedding ρ is the brick vector $B(J) = \sum_{i=1}^{|Q|}(Q\setminus J)_{(i)}(\omega_{q_i})$, and the T-weight of the fixed point p_J under the embedding π is the bulky brick vector $\mathbf{B}(J) = \sum_{i=0}^{|Q|}(Q\setminus J)_{(i)}(\omega_1+\cdots+\omega_n)$. Thus, the moment polytope of Brick^Q under ρ is the brick polytope $\mathcal{B}(Q)$, and the moment polytope of Brick^Q under the embedding obtained by applying ϕ_n to each Fl_n factor is the bulky brick polytope $\mathfrak{B}(Q)$. As the brick polytope and bulky brick polytope are moment polytopes of Brick^Q with respect to two different projective embeddings, it follows from the theory of toric varieties (see, e.g., [5]) that $\mathcal{B}(Q)$ and $\mathfrak{B}(Q)$ have the same normal fan[2].

Theorem 4.4 ([4, Theorem 15])**.** Brick^Q *is a toric variety if and only if Q is root independent and $\ell(w_0) < |Q| \le \ell(w_0) + \dim(T)$. Further,* Brick^Q *is the toric variety of $\mathcal{B}(Q)$ (and of $\mathfrak{B}(Q)$).*

[2]For exposition on "normal fans", see, for example, [11, Section 5.2].

4.2 Minkowski decomposition

The main results of this subsection are Theorem 4.6, which gives a Minkowski decomposition of $\mathfrak{B}(Q)$, and Theorem 4.7, which characterizes toric Richardson varieties[3].

Lemma 4.5. *Let $\pi_i : \mathrm{Fl}_n^k \to \mathrm{Fl}_n$ be the projection of Fl_n^k onto its i-th factor (with indexing of i starting at 0). Let R, S be words in S_n. Set $i := |R|$, $u := w_0\mathrm{Dem}(S)^{-1}$, and $v := \mathrm{Dem}(R)$.*

1. *We have $\pi_i(\mathrm{Brick}^{R+S}) = X_u^v$, where $+$ denotes concatenation of words.*
2. *If Brick^{R+S} is toric, then X_u^v is toric.*

Proof. (1) follows from slight modifications of the proof of [4, Theorem 26]. For (2), assume Brick^{R+S} is toric. Then it contains a dense T-orbit $\mathcal{O}$. As π_i is T-equivariant, surjective onto X_u^v, and continuous, it follows that $\pi_i(\mathcal{O})$ is a dense T-orbit in X_u^v. □

Theorem 4.6. *For $\mathcal{J} = \{J \in \Delta(Q, w_0) : (Q \setminus J)_{(|Q|)} = w_0\}$ and $\lambda_\ell = \min(\{k : k > \ell, q_k = q_\ell\} \cup \{|Q|\}) - \ell$, we have*

$$\mathfrak{B}(Q) = (|Q|+1)\omega_n + \sum_{j=1}^{n-1} \min(\{k : q_k = j\})\omega_j + \sum_{\ell=1}^{|Q|} \lambda_\ell \operatorname*{conv}_{J\in\mathcal{J}} w(J,\ell) \tag{4.1}$$

$$= \tilde{P}_{1,1} + \sum_{k=1}^{|Q|} \tilde{P}_{w_0\mathrm{Dem}(s_{q_{k+1}},\ldots,s_{q_{|Q|}})^{-1},\mathrm{Dem}(s_{q_1},\ldots,s_{q_k})}. \tag{4.2}$$

Proof sketch. Manipulating the formula for $\mathbf{B}(J)$, one finds

$$\mathbf{B}(J) = \sum_{k=1}^{|Q|} \mathbf{w}(J,k) = (|Q|+1)\omega_n + \left(\sum_{j=1}^{n-1} \min(\{k : q_k = j\})\omega_j\right) + \sum_{\ell=1}^{|Q|} \lambda_\ell w(J,\ell).$$

When L is the set of facets of $\Delta(Q, w_0)$, [10] showed that $\mathcal{B}(Q) = \mathrm{conv}_{J\in L} \sum_{k=1}^{|Q|} w(J,k) = \sum_{k=1}^{|Q|} \mathrm{conv}_{J\in L}\, w(J,k)$. Further arguments can show that Equation (4.1) follows from their result once we write $\mathfrak{B}(Q)$ as above. One can show that $\mathrm{conv}_{J\in\mathcal{J}}\, w(J,k)$ is the polytope $\mu_{\mathrm{Gr}(q_k,n)}\rho_k(\mathrm{Brick}^Q)$[4]. Manipulating this expression (and using Lemma 4.5), we eventually arrive at $\mathfrak{B}(Q) = \sum_{k=0}^{|Q|}\sum_{j=1}^{n} \mu_{\mathrm{Gr}(j,n)}(\nu_j(\pi_k(\mathrm{Brick}^Q)))$. Applying [14, Proposition 2.7], we have $\mu_{\mathrm{Fl}_n(\mathbb{C})}(X_u^v) = \sum_{i=1}^{n} \mu_{\mathrm{Gr}(i,n)}(\nu_i(X_u^v))$, and obtain

$$\mathfrak{B}(Q) = \sum_{k=0}^{|Q|} \mu_{\mathrm{Fl}_n}\pi_k(\mathrm{Brick}^Q) = \sum_{k=0}^{|Q|} \tilde{P}_{w_0\mathrm{Dem}(s_{q_{k+1}},\ldots,s_{q_{|Q|}})^{-1},\mathrm{Dem}(s_{q_1},\ldots,s_{q_k})}. \qquad \square$$

[3] In future work, we hope to explore how Theorem 4.7 relates to the root theoretic characterization of toric Richardson varieties given in [1, Theorem 4.7].

[4] The polytope $\mu_{\mathrm{Gr}(q_k,n)}\rho_k(\mathrm{Brick}^Q)$ happens to be a *positroid polytope*, as defined in, e.g., [2].

Theorem 4.7. *Let $u \leq v \in S_n$. Let R be any reduced word for v, let S be any reduced word for $u^{-1}w_0$, and let J be any facet of the subword complex $\Delta(S+R, w_0)$. The following are equivalent:*

1. Brick^{R+S} *is a toric variety.*
2. X_u^v *is a toric variety.*
3. $\ell(v) - \ell(u) \leq \dim(T)$ *and the multiset* $\{\{r(J,k) : q_k \in J\}\}$ *is linearly independent.*

Proof. (1) $\iff$ (3) by Theorem 4.4, and (1) $\implies$ (2) by Lemma 4.5. We will prove (2) $\implies$ (1). The hypotheses on R and S ensure Brick^{R+S} is a resolution of singularities of X_u^v by [4], so $\dim(\mathrm{Brick}^{R+S}) = \dim(X_u^v)$. If X_u^v is a toric variety, then $\dim(\tilde{P}_{u,v}) = \dim(X_u^v)$. Since $\mathfrak{B}(R+S)$ has $\tilde{P}_{u,v}$ as a Minkowski summand (by Theorem 4.6), we have $\dim(\mathfrak{B}(R+S)) \geq \dim(X_u^v) = \dim(\mathrm{Brick}^{R+S}) \geq \dim(\mathfrak{B}(R+S))$. The dimension of the moment polytope of a variety is the same as the dimension of the effective torus. This means the effective torus acting on Brick^{R+S} has the same dimension as Brick^{R+S} and thus Brick^{R+S} is a toric variety whenever X_u^v is. $\square$

5 A degeneration of the brick variety

5.1 The degeneration

We define a degeneration of the brick variety and describe its central fibre (Theorem 5.5).

Say $G \times G$ has coordinates $((a_{i,j})_{i,j=1,\dots,n}, (b_{i,j})_{i,j=1,\dots,n})$. Let $t \in \mathbb{C}$. Let G_t be the subvariety of $G \times G$ cut out by the equations

$$a_{i,j} = t^{j-i} b_{i,j}, \quad i<j; \qquad b_{j,i} = t^{j-i} a_{j,i}, \quad i<j; \qquad a_{i,i} = b_{i,i}, \quad i = 1, \dots, n.$$

Observe that $G_1 = G_\Delta$ and $G_0 = B_- \times_T B_+$ (i.e., G_0 is the set of elements in the product $B_- \times B_+$ whose diagonals are equal). Consider the embedding $\rho\colon \mathbb{C}^\times \to T$, sending t to the diagonal matrix whose diagonal is $(1, t^{-1}, \dots, t^{-n+1})$.

For $t \in \mathbb{C}^\times$, we have $G_t = \{(g, \rho(t)g(\rho(t))^{-1}) \mid g \in G\} \subseteq G \times G$. For $\sigma \in S_n$ and $t \in \mathbb{C}^\times$, the variety $D_{\sigma(t)} := (1, \rho(t)) \cdot D_\sigma$ is invariant under the action of G_t on $\mathrm{Fl}_n \times \mathrm{Fl}_n$:

$$(g, \rho(t)g(\rho(t))^{-1}) \cdot (F_\bullet, \rho(t)(G_\bullet)) = ((g \cdot F_\bullet), \rho(t)(g \cdot G_\bullet)) \in D_{\sigma(t)}, \quad g \in G, F_\bullet, G_\bullet \in \mathrm{Fl}_n.$$

Consider the following two varieties, $\mathcal{G} := \cup_{t\in\mathbb{C}}(G_t \times \{t\}) \subseteq (G^2) \times \mathbb{A}^1$ and $\mathcal{D}_\sigma^\circ := \cup_{t\in\mathbb{C}^\times}(D_{\sigma(t)} \times \{t\}) \subseteq (\mathrm{Fl}_n^2) \times \mathbb{A}^1$. Let $\mathcal{D}_\sigma$ be the closure of $\mathcal{D}_\sigma^\circ$ inside $(\mathrm{Fl}_n^2) \times \mathbb{A}^1$, and let $D_{\sigma(0)}$ be the fibre of $\mathcal{D}_\sigma \to \mathbb{A}^1$ over 0. The action of G_t on $D_{\sigma(t)}$ for each $t \in \mathbb{C}^\times$ induces an action of G_0 on $D_{\sigma(0)}$ (cf. [7, Proposition 4.1]) [5]. In particular, $D_{\sigma(0)}$ is a union of orbits of

[5] The orbit degeneration technique of degenerating a group alongside a variety so that the degeneration of the variety is a union of orbit closures of the degeneration of the group comes from [7], which employed the same family of groups $G_t \subseteq G \times G$.

$G_0 = B_- \times_T B_+$. To uncover which union of orbits, observe that $(D_{\sigma(1)})^T \subseteq (D_{\sigma(t)})^T$ for all $t \in \mathbb{C}$, and consider the $B_- \times_T B_+$ orbits of these T-fixed points. This yields the correct answer, given in Theorem 5.1; however, proving this requires geometric techniques.

Theorem 5.1. *The central fibre $D_{\sigma(0)}$ is reduced and equal to* $\bigcup_{w \in S_n \text{ s.t. } \ell(w\sigma)=\ell(w)+\ell(\sigma)} X_w \times X^{w\sigma}$.

Example 5.2. *In S_3, we have $D_{s_1(0)} = (X_1 \times X^{s_1}) \cup (X_{s_2} \times X^{s_2 s_1}) \cup (X_{s_1 s_2} \times X^{s_1 s_2 s_1})$.*

Let $Q = (s_{q_1}, \ldots, s_{q_k})$ be a word in S_n.

Definition 5.3. *A sequence $(u_0, u_1, \ldots, u_{|Q|})$ in S_n is a **brickly sequence** with respect to Q if $u_0 = 1$, $u_{|Q|} = w_0$, and $u_i < u_i s_{q_{i+1}}$ and $u_{i+1} \leq u_i s_{q_{i+1}}$ for all $i = 0, \ldots, |Q| - 1$. We will denote by $L(Q)$ the set of brickly sequences for Q.*

Example 5.4. *In S_3, set $Q = (s_1, s_2, s_1, s_2, s_1)$. Then $u = (1, 1, s_2, s_1, s_1 s_2, s_1 s_2 s_1)$ is a brickly sequence with respect to Q.*

The degeneration of $D_{s_{q_i}}$ induces a degeneration of Brick^Q into a union of reduced products of Richardson varieties (Theorem 5.5). For $t \in \mathbb{C}$, we define $\mathrm{Brick}^Q(t)$ as

$$(D_{s_{q_1}}(t) \times \mathrm{Fl}_n^{k-1}) \cap (\mathrm{Fl}_n \times D_{s_{q_2}}(t) \times \mathrm{Fl}_n^{k-2}) \cap \cdots \cap (\mathrm{Fl}_n^{k-1} \times D_{s_{i_k}}(t)) \cap (F_\bullet^{\mathrm{st}} \times \mathrm{Fl}_n^{k-1} \times F_\bullet^{\mathrm{op}}).$$

Theorem 5.5. *Suppose* Brick^Q *is a toric variety. For $u = (u_0, u_1, \ldots, u_{|Q|}) \in L(Q)$, set*

$$C_u := X_{u_0}^{u_0} \times X_{u_1}^{u_0 s_{q_1}} \times X_{u_2}^{u_1 s_{q_2}} \times \cdots \times X_{u_{|Q|}}^{u_{|Q|-1} s_{q_{|Q|}}}.$$

Then $\mathrm{Brick}^Q(0) = \bigcup_{u \in L(Q)} C_u$*, and $C_u \cap C_{u'}$ is a common face of C_u and $C_{u'}$ for $u, u' \in L(Q)$.*

The first part of Theorem 5.5 follows from the definition of $\mathrm{Brick}^Q(t)$ and Theorem 5.1, while some of the latter parts require more geometric techniques to prove.

5.2 Bricks tricks for mixed subdivisions

The main result of this subsection is Theorem 5.8, which says that the degeneration of the brick variety of Theorem 5.5 induces a mixed subdivision of the brick polytope.

Lemma 5.6. *Suppose $1 = u_0, u_1, \ldots, u_{|Q|} = w_0 \cdot s_1$ is a brickly sequence in S_n. Then for $x = w_0 \mathrm{Dem}(s_{q_{i+1}}, s_{q_{i+2}}, \ldots, s_{q_{|Q|}})^{-1}$ and $y = \mathrm{Dem}(s_{q_1}, s_{q_2}, \ldots s_{q_i})$, the Bruhat interval $[u_i, u_{i-1} s_{q_i}]$ is contained in $[x, y]$; equivalently, $X_{u_i}^{u_{i-1} s_{q_i}} \subseteq X_x^y$.*

Proof sketch. The proof proceeds by induction using subword combinatorics. □

Definition 5.7. *A **full dimensional Minkowski cell** of a Minkowski sum of polytopes $P = \sum_{i=1}^{n} P_i$ is a polytope $C = \sum_{i=1}^{n} C_i$ such that $\dim(C) = \dim(P)$ and C_i is a face of P_i for all i. A mixed subdivision of P is a set of full dimensional Minkowski cells $\{C_j = \sum_{i=1}^{n} C_{j,i} : j \in J\}$ for some index set J such that $P = \bigcup_{j\in J} C_j$, and $C_j \cap C_k$ is a common face of C_j and C_k for any j, k.*

Theorem 5.8. *If Brick^Q is toric, then, in the notation of Lemma 5.6, the moment polytopes of the components of $\mathrm{Brick}^Q(0) \hookrightarrow \mathrm{Fl}_n(\mathbb{C})^{|Q|}$ give a mixed subdivision of $\mathfrak{B}(Q) = \sum_{i=0}^{|Q|} \tilde{P}_{x,y}$.*

Proof sketch. Let $u_\bullet, v_\bullet \in L(Q)$ be arbitrary brickly sequences. Using the fact that $C_u = X_1^1 \times X_{u_1}^{u_0 s_{q_1}} \times \cdots \times X_{u_{|Q|}}^{u_{|Q|-1} q_{|Q|}}$ is a toric variety, we can show $\dim(\mu_{\mathrm{Fl}_n^{|Q|+1}}(C_u))= \dim \tilde{P}_{1,1} + \sum_{i=1}^{|Q|} \dim \tilde{P}_{u_i, u_{i-1}s_{q_i}}$. The dimension of a Minkowski sum is only equal to the sum of the dimensions of its summands when the summands are affinely independent. Hence, the summands of $\mu_{\mathrm{Fl}_n^{|Q|+1}}(C_u)$ are affinely independent; this implies the linear map from the product $\tilde{P}_{1,1} \times \tilde{P}_{u_1, u_0 s_{q_1}} \times \cdots \times \tilde{P}_{u_{|Q|}, u_{Q-1} s_{|Q|-1}}$ to $\mu_{\mathrm{Fl}_n^{|Q|+1}}(C_u)$ is an isomorphism.

By Lemma 5.6, we have that $[u_i, u_{i-1}s_{q_i}]$ is a subinterval of $[x, y]$. By [9], subintervals of a toric Bruhat interval correspond to faces of the (twisted) Bruhat interval polytope. Hence, each $\tilde{P}_{u_i, u_{i-1}s_{q_i}}$ is a face of $\mu_{\mathrm{Fl}_n^{|Q|+1}}(\pi_i(\mathrm{Brick}^Q))$, and $\mu_{\mathrm{Fl}_n^{|Q|+1}}(C_u)$ is a full-dimensional Minkowski cell. These cells union to give all of $\mathfrak{B}(Q)$ by Theorem 5.5.

It was proved in [1, Theorem 1.1] that toric Bruhat intervals are lattices. The intersection of two intervals contained in a lattice is always an interval (or empty), so $[u_i, u_{i-1}s_{q_i}] \cap [v_i, v_{i-1}s_{q_i}] = [a_i, b_i]$ for some $a_i, b_i \in S_n$. Clearly $X_{a_i}^{b_i} \subseteq X_{u_{i+1}}^{u_i s_{q_i}} \cap X_{v_{i+1}}^{v_i s_{q_i}}$ as it is a sub-Richardson variety. Any Richardson variety contained in $X_{u_{i+1}}^{u_i s_{q_i}} \cap X_{v_{i+1}}^{v_i s_{q_i}}$ will have fixed points indexed by permutations contained in $[a_i, b_i]$ and hence it will be contained in $X_{a_i}^{b_i}$. Since intersections of Richardson varieties are reduced unions of Richardson varieties [12, Theorem 2.74], we conclude that $X_{a_i}^{b_i}$ is the desired intersection. So $\mu_{\mathrm{Fl}_n}(X_{a_i}^{b_i}) = \tilde{P}_{a_i, b_i}$ will then be a common face of $\tilde{P}_{u_i, u_{i-1}s_{q_i}}$ and $\tilde{P}_{v_i, v_{i-1}s_{q_i}}$. A product of faces, each drawn from a different polytope, is a face of the product of those larger polytopes. Ergo, using the fact that the Minkowski sums $\sum \tilde{P}_{u_i, u_{i-1}s_{q_i}}$ and $\sum \tilde{P}_{v_i, v_{i-1}s_{q_i}}$ are both linearly isomorphic to the products of the same polytopes, $\mu_{\mathrm{Fl}_n^{|Q|+1}}(C_u \cap C_v) = \sum \tilde{P}_{a_i, b_i}$ will be a face of both $\mu_{\mathrm{Fl}_n^{|Q|+1}}(C_u)$ and $\mu_{\mathrm{Fl}_n^{|Q|+1}}(C_v)$. □

5.3 Subdividing the associahedron into cubes

The main result of this subsection is Theorem 5.11, which describes a subdivision of the associahedron into pieces that are combinatorially equivalent to cubes.

Theorem 5.9 ([9]). *The twisted Bruhat interval polytope $\tilde{P}_{u,v}$ is combinatorially equivalent to a cube if and only if X_u^v is smooth and toric. If $v = cu$ or $v = uc$ for $c = s_1 s_2 \cdots s_j$ or $c = (s_1 s_2 \cdots s_j)^{-1}$, with $j \leq n-1$, and $\ell(v) - \ell(u) = j$, then X_u^v is smooth and toric.*

Remark 5.10. *By [9], for $X_a^b \subseteq X_c^d$ with X_c^d toric, $\tilde{P}_{a,b}$ will be a face of $\tilde{P}_{c,d}$. So when X_c^d is also smooth, $\tilde{P}_{a,b}$ will be a face of a cube and therefore be a cube. This shows that any sub-Richardson of a smooth toric Richardson is smooth and toric.*

Let $\mathbf{c}$ be a **Coxeter element** in S_n, i.e., a word in S_n containing all simple reflections exactly once. Denote by $\mathbf{c}^\infty$ the concatenation of $\mathbf{c}$ to itself ad infinitum. Define $\mathbf{w_0}(\mathbf{c})$ to be the lex-first subword of $\mathbf{c}^\infty$ that is a reduced word for w_0.

Theorem 5.11. *Fix $\mathbf{c} = (s_1, s_2, \ldots, s_{n-1})$ and $\mathbf{c}' = (s_{n-1}, s_{n-2}, \ldots, s_1)$. Let Q be the concatenation $\mathbf{w_0}(\mathbf{c}) + \mathbf{c}'$. Then Brick^Q is toric and $\mathrm{Brick}^Q \simeq \mathrm{Brick}^{\mathbf{c}+\mathbf{w_0}(\mathbf{c})}$. Moreover, we have*

- *$\pi_i(\mathrm{Brick}^Q)$ is a smooth toric Richardson variety for each i, and each component C_u of $\mathrm{Brick}^Q(0)$ is a product of smooth toric Richardsons.*
- *$\mathcal{B}(Q)$ is a translation of Loday's realization of the associahedron, $\mathfrak{B}(Q)$ has the same normal fan as Loday's realization, the summands of the Minkowski decomposition of $\mathfrak{B}(Q)$ (Equation (4.2)) are combinatorial cubes, and the moment polytopes of the components of $\mathrm{Brick}^Q(0)$ give a mixed subdivision of $\mathfrak{B}(Q)$ into pieces which are combinatorially equivalent to cubes.*

Proof sketch. For $Q = \mathbf{c} + \mathbf{w_0}(\mathbf{c})$, Brick^Q is the toric variety of Loday's realization of the associahedron [4, Corollary 16]. The first step here is to prove that $Q = \mathbf{w_0}(\mathbf{c}) + \mathbf{c}'$ can be obtained from $\mathbf{c} + \mathbf{w_0}(\mathbf{c})$ by commuting moves (e.g. $s_1 s_3 = s_3 s_1$). Commuting moves shuffle the order of the Grassmannian factors of the product $\prod_{i=1}^{|Q|} \mathrm{Gr}_{q_i,n} \ni \rho(\mathrm{Brick}^Q)$ but do not change the incidence structure recorded by the Magyar diagram (unlike arbitrary permutations of the letters of Q) and thus leave $\rho(\mathrm{Brick}^Q)$ unchanged aside from this permutation of its factors. Hence, $\mathrm{Brick}^Q \cong \mathrm{Brick}^{\mathbf{c}+\mathbf{w_0}(\mathbf{c})}$, and $\mathcal{B}(\mathbf{w_0}(\mathbf{c}) + \mathbf{c}') = \mathcal{B}(\mathbf{c} + \mathbf{w_0}(\mathbf{c}))$ is the same as Loday's realization of the associahedron up to translation [4].

To show that $\pi_i(\mathrm{Brick}^Q)$ is smooth and toric, there are three cases. For $i < n$, we can show $\pi_i(\mathrm{Brick}^Q) \subseteq X_1^{s_1 s_2 \ldots s_{n-1}}$. When $i \geq |Q| - n$, we show $\pi_i(\mathrm{Brick}^Q) \subseteq X^{w_0}_{w_0(s_1 s_2 \ldots s_{n-1})}$. Both $X_1^{s_1 s_2 \ldots s_{n-1}}$ and $X^{w_0}_{w_0(s_1 s_2 \ldots s_{n-1})}$ are smooth and toric by Theorem 5.9, so $\pi_i(\mathrm{Brick}^Q)$ is smooth and toric. For $n \leq i \leq |Q| - n - 1$, we can show $\pi_i(\mathrm{Brick}^Q) = X_u^v$ with $u = w_0(s_{q_{i+1}} \ldots s_{q_{|Q|}})^{-1}$ and $v = s_{q_1} s_{q_2} \ldots s_{q_i}$. Thus $vu^{-1} = s_{q_1} s_{q_2} \ldots s_{q_i} s_{q_{i+1}} \ldots s_{q_{|Q|}} w_0$ and from that we can obtain $vu^{-1} = s_1 \ldots s_{n-1}$ and thus $v = s_1 \ldots s_{n-1} u$. Thus applying Theorem 5.9, we conclude that for all i, $\pi_i(\mathrm{Brick}^Q)$ is smooth and toric.

Since $\pi_i(\mathrm{Brick}^Q)$ is smooth and toric, and $\pi_i(C_u)$ is a sub-Richardson of $\pi_i(\mathrm{Brick}^Q)$ for each i, we get that $\pi_i(C_u)$ is a smooth toric Richardson. The remaining combinatorial statements within the theorem can be proven from what we have presented already. ☐

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