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A Bijection Between Marked Bumpless Pipedreams and Compatible Pairs

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Abstract. We construct a bijection between marked bumpless pipedreams with compatible pairs, which are in bijection with not-necessarily-reduced pipedreams. This directly unifies various formulas for Grothendieck polynomials in the literature. Our bijection is a generalization of a variant of the bijection of Gao and Huang in the unmarked, reduced case.

Keywords: Grothendieck polynomials, pipedreams, bumpless pipedreams

1 Introduction

This paper establishes an explicit bijection between compatible pairs and marked bumpless pipedreams, which are two kinds of combinatorial objects that yield combinatorial formulas for the *β-Grothendieck polynomials* $\{\mathfrak{G}_w^{(\beta)} \mid w \in S_\infty := \bigcup_{i=1}^{\infty} S_n\} \subset \mathbb{Z}[x_1, x_2, \dots]$. These polynomials are defined recursively by divided differences in [6, Section 4].

The β-Grothendieck polynomials indexed by $w \in S_n$ form a Schubert basis for the connective K-theory of the flag variety [9]. They have two important specializations: **Grothendieck polynomials** $\mathfrak{G}_w = \mathfrak{G}_w^{(\beta)}|_{\beta=-1}$, which represent structure sheaves of Schubert varieties in the flag variety, and **Schubert polynomials** $\mathfrak{S}_w = \mathfrak{G}_w^{(\beta)}|_{\beta=0}$, which represent the cohomology classes of Schubert varieties [2, 4, 14]. Notably, $\mathfrak{S}_w$ is the coefficient of β^0 in $\mathfrak{G}_w^{(\beta)}$, or equivalently the lowest x-degree component of $\mathfrak{G}_w^{(\beta)}$.

The polynomials $\mathfrak{G}_w^{(\beta)}$ can be expressed as a sum over certain combinatorial objects, where each object contributes one monomial. In such formulas, an object is termed "reduced" if it contributes to the lowest x-degree component of $\mathfrak{G}_w^{(\beta)}$. Restricting to reduced objects yields a combinatorial formula for Schubert polynomials. This paper involves three such combinatorial formulas for $\mathfrak{G}_w^{(\beta)}$:

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- Fomin and Kirillov [6, Proposition 3.3] provided a **compatible pair** formula for $\mathfrak{G}_w^{(\beta)}$, extending the reduced compatible pair formula of Billey, Jockusch, and Stanley [3] for Schubert polynomials.
- Knutson and Miller [10] interpreted $\mathfrak{G}_w^{(\beta)}$ as K-polynomials of matrix Schubert varieties (up to change of variables) and showed via Gröbner degeneration that they can be computed using **pipedreams**, generalizing the reduced pipedream formula of Billey and Bergeron [1] for Schubert polynomials.
- Weigandt [16] reinterpreted Lascoux's formulas [13] for Grothendieck polynomials in terms of alternating sign matrices (ASMs) and obtained the **marked bumpless pipedream (MBPD)** formula for $\mathfrak{G}_w^{(\beta)}$. The reduced version for Schubert polynomials was introduced by Lam, Lee, and Shimozono [12].

It is natural to ask for explicit bijections between these three sets of combinatorial objects. The connection between compatible pairs and pipedreams is classical: Billey and Bergeron [1] provided a bijection between reduced compatible pairs and reduced pipedreams, and their bijection generalizes directly to the non-reduced case. On the other hand, Gao and Huang [7] constructed a bijection between reduced pipedreams and reduced bumpless pipedreams. The goal of this paper is to extend the bijection of Gao and Huang, furnishing a bijection between pipedreams and marked bumpless pipedreams. This completes the connections between the three objects mentioned above. In a subsequent paper, we will show that our bijection restricts to that of [7] in the reduced case.

In fact, long before the discovery of bumpless pipedreams [12], it was known that the 2-enumeration of alternating sign matrices of size n is $2^{n(n-1)/2}$ [15, 5, 11], which coincides with the number of non-reduced pipedreams of size n. Thus, based on Weigandt's observation connecting alternating sign matrices and bumpless pipedreams [16], our bijection can be viewed as a bijection between 2-enumerated ASMs and non-reduced pipedreams. This connection was recently explored by Striker and Huang [8], who established a partial bijection between totally symmetric self-complementary plane partitions and ASMs using the Gao–Huang bijection.

The rest of the paper is structured as follows. In Section 2, we define the three combinatorial objects. In Section 3, we describe our bijection using two pairs of mutually inverse operators f_i^*, e_i^* and f_i, e_i on MBPDs. In Section 4 we define these operators.

2 Combinatorial objects

Fix $n \in \mathbb{Z}_{\geq 0}$. In this section, we define compatible pairs $\mathrm{CP}(n)$, pipedreams $\mathrm{PD}(n)$, and marked bumpless pipedreams $\mathrm{MBPD}(n)$. Let Γ be one of these three sets. Each object

$\gamma \in \Gamma$ has an associated permutation $w(\gamma)$ and composition[1] $\mathrm{wt}(\gamma)$, called its weight. Let $\Gamma(w)$ be the set of $\gamma \in \Gamma$ such that $w(\gamma) = w$. Let $\ell(w)$ be the length of w. Then one may write

$$\mathfrak{G}_w = \sum_{\gamma \in \Gamma(w)} \beta^{|\mathrm{wt}(\gamma)| - \ell(w)} x^{\mathrm{wt}(\gamma)}.$$

Here, $|\mathrm{wt}(\gamma)|$ is the sum of entries in $\mathrm{wt}(\gamma)$, and $x^{\mathrm{wt}(\gamma)}$ is the monomial where the power of x_i is the i^{th} entry in $\mathrm{wt}(\gamma)$. In addition, each such $\Gamma(w)$ has a subset $\Gamma^{\mathrm{red}}(w)$ consisting of reduced γ, meaning that $|\mathrm{wt}(\gamma)| = \ell(w)$. Thus,

$$\mathfrak{S}_w = \sum_{\gamma \in \Gamma^{\mathrm{red}}(w)} x^{\mathrm{wt}(\gamma)}.$$

2.1 Compatible pairs

A *biletter* is an ordered pair of integers (i, a) with $1 \leq i \leq a < n$. A *compatible pair*[2] is a sequence of biletters

$$B = ((i_1, a_1), (i_2, a_2), \ldots, (i_\ell, a_\ell)) \tag{2.1}$$

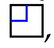
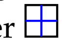
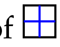
which are strictly decreasing in the order given by $(i_1, a_1) > (i_2, a_2)$ if $i_1 > i_2$ or if $i_1 = i_2$ and $a_1 < a_2$. Let $\mathrm{CP}(n)$ denote the set of compatible pairs. The weight of B is defined by $\mathrm{wt}(B) = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$, where m_i is the number of biletters in B of the form (i, a) for some a. Let $|B|$ be the length of the sequence B, or equivalently $|\mathrm{wt}(B)|$.

The Demazure or 0-Hecke product $*$ on permutations is the unique monoid structure such that for any simple reflection s_i and permutation w,

$$s_i * w = \begin{cases} s_i w & \text{if } s_i w > w, \\ w & \text{otherwise.} \end{cases}$$

Every $B \in \mathrm{CP}(n)$ has an associated permutation $w(B) = s_{a_\ell} * s_{a_{\ell-1}} * \cdots * s_{a_1}$ with notation as in (2.1). Note that the subscripts of a are *decreasing*.

2.2 Pipedreams

Let $[n] = \{1, 2, \ldots, n\}$. A pipedream is a tiling of $\{(i, j) \in [n] \times [n] : i + j \leq n + 1\}$, such that all entries (i, j) with $i + j = n + 1$ have tiles , and all other entries are either  or . We use the matrix-style notation $D_{i,j}$ for the tile in the i-th row and j-th column. The weight $\mathrm{wt}(D)$ is the weak composition whose i^{th} entry is the number of  in row i.

[1] Our compositions are weak, that is, sequences of nonnegative integers that are almost all zero.

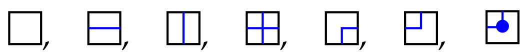
[2] Our definition differs from the definition in the literature by reversing the order of these biletters.

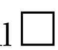
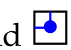
Every such diagram D can be viewed as n pipes entering from the top and exiting to the left. We label the pipe entering from column c as pipe c. We trace the pipes from top to left with the proviso: if two pipes have already crossed, then their subsequent crossings are ignored. Then we read the labels on the left edge of D from top to bottom, obtaining $w(D) \in S_n$.

Remark 2.1. There is a weight-preserving bijection $\mathrm{CP}(n) \to \mathrm{PD}(n)$ that sends the compatible pair $B = ((i_1, a_1), \dots, (i_\ell, a_\ell))$ to the pipedream with crossings at positions $(i_j, a_j - i_j + 1)$ for $1 \le j \le \ell$. This was proved in [1] for the reduced case, but the proof works in the non-reduced setting.

2.3 Marked Bumpless Pipedreams

We work with a set of tiles



named blank, horizontal, vertical, plus, R, J, and marked J. We call  and  *heavy*, and the other tiles *light*.

A *marked bumpless pipedream* is an $[n] \times [n]$ matrix with entries in the above set of tiles such that the pipes are "connected". Moreover, every column has a pipe entering from the bottom but not top, and every row has a pipe exiting from the right but not left. Let $\mathrm{MBPD}(n)$ be the set of $[n] \times [n]$ marked bumpless pipedreams. The weight $\mathrm{wt}(D)$ is the weak composition whose i^{th} entry is the number of heavy tiles in row i.

We label the pipe entering from column i as pipe i. We trace the pipes from bottom to right with the proviso: if two pipes have already crossed, their subsequent crossings are ignored. Then we read the labels on the right edge of D from top to bottom, obtaining $w(D) \in S_n$. For every $w \in S_n$, the *Rothe bumpless pipedream* D_w is the unique element of MBPD_w whose i-th pipe turns only at position $(i, w(i))$ for all i. Clearly, D_{id} is the only element of $\mathrm{MBPD}(n)$ with no heavy tiles. See Section 4.5 for an example of a MBPD.

3 Describing the bijection

In this section we present our main result: a bijection between $\mathrm{CP}(n)$ and $\mathrm{MBPD}(n)$ that preserves $w(\cdot)$ and $\mathrm{wt}(\cdot)$. Our bijection is recursive, requiring several operators on $\mathrm{CP}(n)$ and $\mathrm{MBPD}(n)$. We also describe the effect of the operators on $w(\cdot)$ and $\mathrm{wt}(\cdot)$. Let $\mathbf{e}_i$ be the composition whose i^{th} entry is 1 and other entries are 0.

Definition 3.1. Let $B_{\mathrm{id}} \in \mathrm{CP}(n)$ be the empty biword. For any other $B \in \mathrm{CP}(n)$, the following two steps may be iterated to reduce B to B_{id}. Let (i, a) the first biletter in B.

(CP1) If $i = a$ remove the first biletter (i, i) from B. Write $X_i(B)$ for the resulting biword. Then $w(B) = w(X_i(B)) * s_i$ and $\mathrm{wt}(B) = \mathrm{wt}(X_i(B)) + \mathbf{e}_i$.

(CP2) If $i < a$, replace the first biletter of B by $(i+1, a)$. Call $\uparrow B$ the result of this operation. We have $w(B) = w(\uparrow B))$ and $\mathrm{wt}(B) = \mathrm{wt}(\uparrow B) + \mathbf{e}_i - \mathbf{e}_{i+1}$.

The operators X_i and $\uparrow$ on $\mathrm{CP}(n)$ have corresponding left inverses I_i and $\downarrow$.

(RCP1) Let $I_i(B)$ be the result of prepending (i,i) to B.

(RCP2) Let $\downarrow B$ be the result of changing the first biletter of B from (i,a) to $(i-1,a)$.

Apparently, these two operators are not defined on arbitrary elements of $\mathrm{CP}(n)$.

The bijection $\Phi : \mathrm{MBPD}(n) \to \mathrm{CP}(n)$ is defined recursively by specifying the operations on $\mathrm{MBPD}(n)$ which correspond to X_i and $\uparrow$ on $\mathrm{CP}(n)$. They are the operators f_i^* and f_i whose definitions are postponed to the next section. For the base case, let $\Phi(D_{\mathrm{id}}) = B_{\mathrm{id}}$. For $D \in \mathrm{MBPD}(n)$ with at least one heavy tile, let (i,j) be such that $D_{i,j}$ is the heavy tile with i minimum and then j maximum.

(Case 1) There is no  or  in row $i+1$ to the right of column j. In this case the operator f_i^* is defined on D. We define

$$\Phi(D) := I_i \circ \Phi \circ f_i^*(D).$$

(Case 2) Otherwise the operator f_i is defined on D. We define

$$\Phi(D) :=\downarrow \circ \Phi \circ f_i(D).$$

Diagrammatically we may describe Φ as

$$\begin{array}{ccc} D & \xrightarrow{\Phi} & \Phi(D) \\ {\scriptstyle f_i^*}\big\downarrow & & \big\uparrow{\scriptstyle I_i} \\ D' & \xrightarrow[\Phi]{} & \Phi(D') \end{array} \qquad \begin{array}{ccc} D & \xrightarrow{\Phi} & \Phi(D) \\ {\scriptstyle f_i}\big\downarrow & & \big\uparrow{\scriptstyle \downarrow} \\ D' & \xrightarrow[\Phi]{} & \Phi(D') \end{array}$$

where the diagram on the left (resp. right) represents case 1 (resp. 2). We present an example of $\Phi(\cdot)$ in Section 4.5. Since I_i and $\downarrow$ are not defined on all elements of $\mathrm{CP}(n)$, the well-definedness of Φ is highly nontrivial, involving a delicate analysis of the operators f_i^* and f_i.

Proposition 3.2. *The map* $\Phi : \mathrm{MBPD}(n) \to \mathrm{CP}(n)$ *is well-defined.*

To see Φ preserves $w(\cdot)$ and $\mathrm{wt}(\cdot)$, we just need to describe how f_i^* and f_i change $w(\cdot)$ and $\mathrm{wt}(\cdot)$.

Proposition 3.3. *If* $f_i^*(D)$ *is defined then* $w(D) = w(f_i^*(D)) * s_i$ *and* $\mathrm{wt}(D) = \mathrm{wt}(f^*(D)) + \boldsymbol{e}_i$. *If* $f_i(D)$ *is defined then* $w(D) = w(f_i(D))$ *and* $\mathrm{wt}(D) = \mathrm{wt}(f_i(D)) + \boldsymbol{e}_i - \boldsymbol{e}_{i+1}$.

Compared with the effect of X_i and $\downarrow$, Proposition 3.3 shows that Φ preserves $w(\cdot)$ and $\mathrm{wt}(\cdot)$.

The operators f_i^* and f_i have left inverses e_i^* and e_i respectively, which are also defined in the next section. The bijection $\Psi : \mathrm{MBPD}(n) \to \mathrm{CP}(n)$ is defined analogously. As the base case, let $\Psi(B_{\mathrm{id}}) = D_{\mathrm{id}}$. For non-empty $B \in \mathrm{MBPD}(n)$, we let (i, a) be its first biletter. We consider the two cases as in Definition 3.1:

(Case 1) We define $\Psi(B) := e_i^* \circ \Psi \circ X_i(B)$.

(Case 2) We define $\Psi(B) := e_i \circ \Psi \circ \uparrow (B)$.

Similarly, the well-definedness of Ψ is non-trivial. Our main result is:

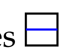
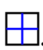
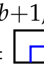
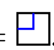
Theorem 3.4. *The maps $\Phi : \mathrm{CP}(n) \to \mathrm{MBPD}(n)$ and $\Psi : \mathrm{MBPD}(n) \to \mathrm{CP}(n)$ are well-defined and mutually inverse maps that preserve both $w(\cdot)$ and $\mathrm{wt}(\cdot)$.*

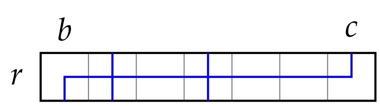
4 Describing the operators on $\mathrm{MBPD}(n)$

In this section, we describe the operators f_r^*, f_r, e_r^* and e_r on $\mathrm{MBPD}(n)$. There will be nine cases for f_r^*, f_r and nine cases for e_r^*, e_r. Readers might refer to Figure 1 for a visual illustration of the cases.

4.1 Some tile notation

Given $D \in \mathrm{MBPD}(n)$ and intervals $I, J \subset [n]$ we denote by $D_{I,J}$ the submatrix of entries $D_{i,j}$ for $i \in I$ and $j \in J$. We write $D_{r,J}$ when $I = \{r\}$ is a singleton row index.

We say $D_{r,[b,c]}$ *is a pipe segment* if the interior part $D_{r,[b+1,c-1]}$ consists solely of tiles  and . A *kink* is a pipe segment $D_{r,[b,c]}$ such that $D_{r,b} =$  and $D_{r,c} =$ .



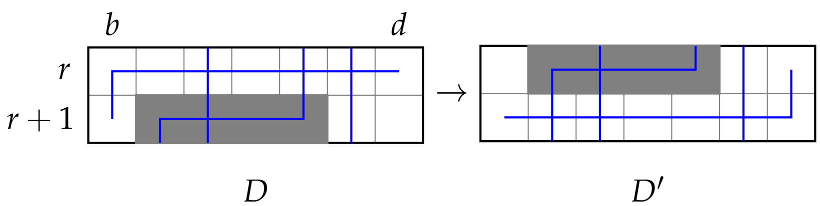
By definition, a non-blank tile $D_{r,[b,b]}$ is always a pipe segment. A *light sequence* in $D \in \mathrm{MBPD}(n)$ is a set of tiles of the form $D_{r,[b,c]}$ which consists solely of light tiles. A light sequence $D_{r,[b,c]}$ is *paired* if $D_{r,b}$ is not connected to the left and $D_{r,c}$ is not connected to the right.

For a row index r and column indices $1 \leq b < d \leq n$, the $(r, [b, d])$-droop and $(r, [b, d])$-undroop are operations that change an MBPD into another, only changing tiles in the "(un)droop rectangle" $[r, r+1] \times [b, d]$. We say that $D \in \mathrm{MBPD}(n)$ admits the $(r, [b, d])$*-droop* if

- The droop rectangle contains only light tiles except possibly a  at $(r+1, d)$.
- $D_{r,[b,d]}$ is a pipe segment.
- $D_{r+1,[b+1,d-1]}$ is a paired light sequence.
- $D_{r,b} \neq$  and $D_{r,d} \neq$ .

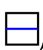
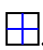
If D admits the $(r, [b, d])$-droop then we may produce $D' \in \mathrm{MBPD}(n)$ as follows; we say that $D \to D'$ is the $(r, [b, d])$-droop.

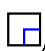
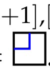
The pipe segment $D_{r,[b,d]}$, which connects down to $D_{r+1,b}$, "droops" to a pipe segment $D'_{r+1,[b,d]}$ which is connected upwards to $D'_{r,d}$. In columns x of D for $b < x < d$, the vertical pipes (x such that $b < x < d$ and $D_{r,x} = D_{r+1,x} =$ ) are unchanged. Each kink in row $r+1$ in D between columns b and d, is shifted up into row r in D'. A shifted kink is shaded gray in the following picture of a droop.

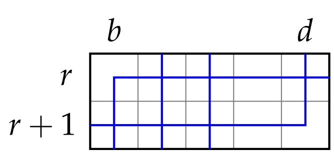


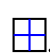
Notice that the tile $D_{r,b}$ might change from  to  or  to . The tile $D_{r+1,d}$ might change from  to  or  to .

By definition, $D \to D'$ is a $(r, [b, d])$-*undroop* if and only if $D' \to D$ is a $(r, [b, d])$-droop. Explicitly, a $(r, [b, d])$-*undroop* $D \to D'$ is defined when

- The undroop rectangle contains only light tiles except possibly a  at (r, b).
- $D_{r+1,[b,d]}$ is a pipe segment.
- $D_{r,[b+1,d-1]}$ is a paired light sequence.
- $D_{r+1,d} \neq$ , $D_{r+1,b} \neq$ .

Finally, for $D \in \mathrm{MBPD}(n)$ the subdiagram $D_{[r,r+1],[b,d]}$ is a *doublecross* if $D_{r,[b,d]}$ and $D_{r+1,[b,d]}$ are pipe segments, $D_{r,b} =$ , and $D_{r+1,d} =$ .



It is not hard to deduce that if $D_{[r,r+1],[b,d]}$ is a double cross, then $D_{r+1,b} = D_{r,d} =$ . This is so named because the pipe through (r, b) and the pipe through $(r+1, d)$ cross twice at (r, d) and $(r+1, b)$.

Remark 4.1. There is at most one doublecross $D_{[r,r+1],[b,d]}$ for each fixed pair (r, b) and also at most one such doublecross for a fixed pair (r, d).

4.2 Describing f_r^* and f_r

Let $D \in \mathrm{MBPD}(n)$. Say that (r, c) is an *f-target* of D if the conditions (f1), (f2), and (f3) hold.

(f1) $D_{r,c}$ is the rightmost heavy tile in row r.

(f2) There is an index $c' > c$ such that $D_{r+1,c'} =$. Let c' be minimum with this property.

(f3) All tiles $D_{r+1,j}$ are light for $j < c'$.

We say that (r, c) is an *f^*-target* of D if the conditions (f1), (f*2), and (f3) hold, where

(f*2) There is no nor in row $r + 1$ to the right of column c. In this case let c' be the maximum index such that $D(r, c') =$.

By abuse of language we will also say that $(D,(r, c))$ is an f-target (resp. f^*-target) to mean that $D \in \mathrm{MBPD}(n)$ and (r, c) is an f-target (resp. f^*-target) of D. We will write F to mean either f or f^*. So $(D,(r, c))$ is an F-target means it is either an f-target or f^*-target.

We define the *window* of an F-target $(D,(r, c))$ to be the two-row rectangle $[r, r + 1] \times [b, c']$ where

$$b = \begin{cases} c & \text{if } D_{r,c} = \square \\ \text{maximum } b < c \text{ with } D_{r,b} = & \text{if } D_{r,c} = \end{cases} \tag{4.1}$$

and c' is defined by (f2) in the case of an f-target and (f*2) in the case of an f^*-target.

There are two trichotomies for an F-target $(D,(r, c))$, giving nine cases in all. They are called left and right because they describe the left and right sides of the window respectively.

The left trichotomy for the F-target $(D,(r, c))$ asserts that exactly one of B, C, or $\not\mathsf{C}$ holds.

(B) (Blank): This holds if $D_{r,c} = \square$.

Otherwise, we know $D_{r,c} =$. There are two more cases:

(C) (Crossing): $D_{r+1,[b,c]}$ is a pipe segment. This is so named because in this case the pipe through (r, c) goes to the left and then crosses vertically with the above pipe segment at $D_{r+1,b} =$.

(C) (Noncrossing): $D_{r+1,[b,c]}$ is not a pipe segment.

The right trichotomy says that for an F-target $(D,(r,c))$, exactly one T, D, or O holds.

(T) (Terminal) The tile (r,c) is an f^*-target of D.

(D) (Doublecross): The tile (r,c) is an f-target of D. Moreover, there is a doublecross $D_{[r,r+1],[d,c']}$ for some d with $c < d < c'$.

(O) (Ordinary): Cases T and D do not hold.

Given an F move, define ρ as d in Case D or c' otherwise. We define $F_r(D) \in \mathrm{MBPD}(n)$ to be the result of the following steps:

- If $D_{r,c} =$  remove the marking.
- In Cases B and C perform the $(r,[c,\rho])$-undroop.
- If the tile at $(r+1,c')$ is , mark it.

4.3 Describing e_r^* and e_r

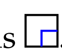
We say the tile $(r+1,c)$ is an *e-target* of $D \in \mathrm{MBPD}(n)$ if the following are satisfied.

(e1) It is the leftmost heavy tile on row $r+1$.

(e2) There is an index $c' < c$ such that $D_{r,c'} =$  and $D_{r+1,[c',c]}$ is not a pipe segment. Let c' be maximum with this property.

(e3) On the right of (r,c') in row r there are no heavy tiles.

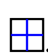
We say the tile $(r+1,c)$ is an e^**-target* of $D \in \mathrm{MBPD}(n)$ if the conditions (e*1), (e2), and (e3) are satisfied, where the condition (e1) for an e-target has been replaced by the condition:

(e*1) There are no heavy tiles in row $r+1$, and c is the largest so that $D_{r,c}$ or $D_{r+1,c}$ is .

Again we abuse language by saying that an *e-target* (resp. e^*-target) is a pair $(D,(r+1,c))$ where $D \in \mathrm{MBPD}(n)$ and $(r+1,c)$ is an e-target (resp. e^*-target) of D. Similar to the F-case, an E-target is either an e-target or e^*-target. We define the *window* of an E-target $(D,(r+1,c))$ to be the two-row rectangle $[r,r+1] \times [c',c]$ with c' as in (e2).

Let $(D,(r+1,c))$ be an E-target. We have 9 cases with three choices for the first symbol and three for the second symbol.

Here is the right trichotomy for E-moves. Exactly one of them holds for an E-target.

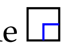
- I (Initial): $(r+1,c)$ is an e^*-target.
- P (Plus): $(r+1,c)$ is an e-target and $D_{r,c} =$ .

- $\not\mathsf{P}$ (No plus): $(r+1,c)$ is an e-target and $D_{r,c} \neq$ .

Define the *right droop column* ρ of the E-target $(D,(r+1,c))$ by

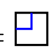
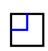
$$\rho = \begin{cases} c & \text{in Cases I and } \not\mathsf{P} \\ \max\left\{d' \mid d' < c \text{ and } D_{r+1,d'} = \square\right\} & \text{in Case P.} \end{cases} \tag{4.2}$$

The left trichotomy for E-moves are the following three cases, exactly one of which holds for any E-target $(D,(r+1,c))$ with window $[r,r+1] \times [c',c]$.

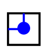
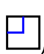
- L (Left turn): $D_{r,[c',c]}$ is not a pipe segment. That is, the pipe leaving the  at (r,c') to the right, must turn upwards (make a left turn) before arriving at (r,c).
- D (Doublecross): $D_{r,[c',c]}$ is a pipe segment and the tile (r,c') is the top-left corner of a doublecross.
- S (Straight): $D_{r,[c',c]}$ is a pipe segment and the tile (r,c') is not the top-left corner of a doublecross.

Define the *left droop column* λ of the E-target $(D,(r+1,c))$ by

$$\lambda = \begin{cases} c' & \text{in Case S} \\ \min\left\{d \mid d > c' \text{ such that } D_{r+1,d} = \square\right\} & \text{in Case D} \\ \min\left\{d \mid d > c' \text{ such that } D_{r,d} = \square\right\} & \text{in Case L.} \end{cases} \tag{4.3}$$

We shall now define E-moves, which are the inverses of F-moves. Given an E-target $(D,(r+1,c))$, define $E_r(D) \in \mathrm{MBPD}(n)$ to be the result of the following steps:

- If $D_{r+1,c} =$ , remove the marking.
- In Cases S and D, perform the $(r,[\lambda,\rho])$-droop.
- If (r,λ) is , mark it.

4.4 F-moves and E-moves are mutually inverses

We give a diagrammatic summary of the F- and E-moves. Small examples of windows of the nine cases for f and f^* are pictured in Figure 1. The green square has coordinate (r,λ) and the red square has coordinate $(r+1,\rho)$. In F-move cases other than $\not\mathsf{C}$ and E-move cases other than L, the green and red squares are the corners of the (un)droop window. The parts of tiles not pictured remain the same. Small examples of the nine cases for e and e^* are pictured in Figure 1.

From the figures, we deduce the following results, which are a refined way of saying that F-moves and E-moves are inverses.

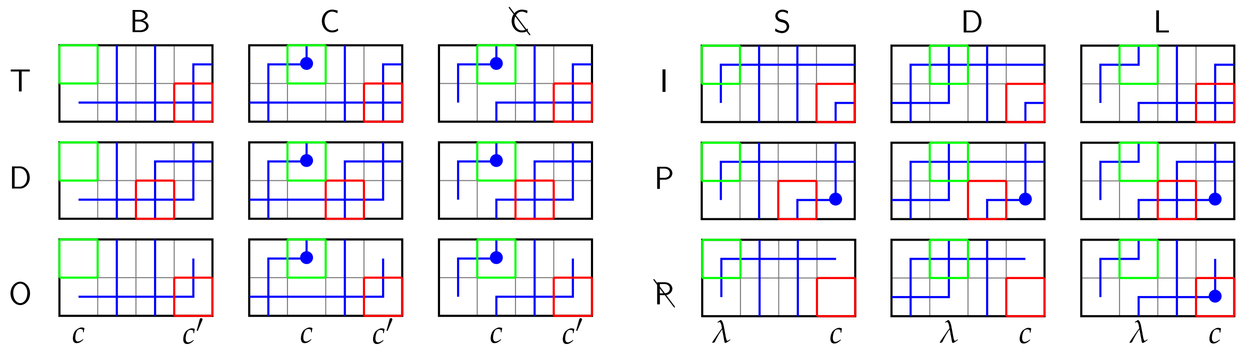


Figure 1: 9 Cases for F-moves and E-moves

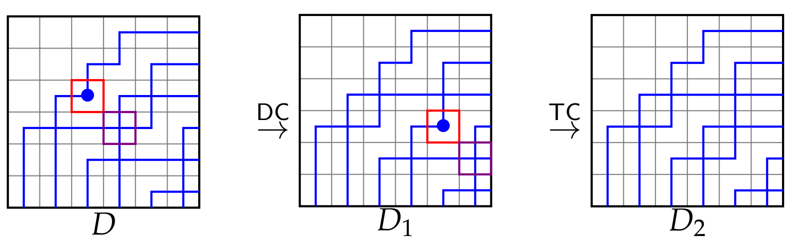
Proposition 4.2. *Let D have F-target (r, c) with window $[r, r+1] \times [b, c']$.*

- *Then $F_r(D)$ has E-target $(r+1, c')$ with the same window.*
- *The case of $(D, (r, c))$ corresponds to the case of $(F_r(D), (r+1, c'))$ according to Figure 1. For instance, if $(D, (r, c))$ is in case* BT, *then $(F_r(D), (r+1, c'))$ is in case* SI.
- *We have $E_r(F_r(D)) = D$.*

Proposition 4.3. *Proposition 4.2 holds after switching the roles of F and E.*

4.5 Example of F-moves and the bijection Φ

Suppose we want to apply Φ on D, which is the leftmost MBPD below. We apply f_3 followed by f_4^*, obtaining D_1 and D_2. In the following diagrams the red square is the target and the violet square is the opposite corner of the undroop.



Suppose we know $\Phi(D_2)$ is the following compatible pair:

$$((3,5),(2,2),(2,5),(1,1),(1,3),(1,5)).$$

Then the recurrence definition of $\Phi(\cdot)$, we have

$$\Phi(D_1) = I_4(\Phi(D_2)) = ((4,4),(3,5),(2,2),(2,5),(1,1),(1,3),(1,5)),$$

$$\Phi(D) = \downarrow (\Phi(D_1)) = ((3,4),(3,5),(2,2),(2,5),(1,1),(1,3),(1,5)).$$

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