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The number of irreducibles in the plethysm $s_\lambda[s_m]$

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Abstract. We give a formula for the number of irreducibles (with multiplicity) in the decomposition of the plethysm $s_\lambda[s_m]$ of Schur functions in terms of the number of lattice points in certain rational polytopes. In the case where $\lambda = n$ consists of a single part, we will give a combinatorial interpretation of this number as the cardinality of a set of matrices modulo permutation equivalence. This is also the setting of Foulkes' conjecture, and our results allow us to state a weaker version that only involves comparing the cardinalities of such sets, rather than the multiplicities of irreducible representations.

Keywords: plethysm, symmetric functions, lattice point counting, representation theory of symmetric groups, wreath product.

1 Introduction

Let Λ denote the ring of symmetric functions over $\mathbb{Q}$. *Plethysm* is a binary operation $(f, g) \mapsto f[g]$ on Λ, introduced by Littlewood [15] in 1936. In modern language, it is most easily expressed in terms of the power sum symmetric functions p_m as the unique operation [16] satisfying

1. for $n, m \geq 1$, $p_n[p_m] = p_{nm}$;
2. for $m \geq 1$, $g \mapsto p_m[g]$ is a $\mathbb{Q}$-algebra homomorphism $\Lambda \to \Lambda$;
3. for $g \in \Lambda$, $f \mapsto f[g]$ is a $\mathbb{Q}$-algebra homomorphism $\Lambda \to \Lambda$.

The decomposition of the plethysm of Schur functions

$$s_\lambda[s_\mu] = \sum_{\nu \vdash nm} a^{\nu}_{\lambda,\mu} s_\nu \tag{1.1}$$

for partitions $\lambda \vdash n, \mu \vdash m$ is of particular importance. Schur functions correspond to irreducible representations of symmetric groups, and from this point of view it can be shown that the *plethysm coefficients* $a^{\nu}_{\lambda,\mu}$ are non-negative integers [11, Section 5.4]. Various formulas and algorithms have been developed to compute these coefficients [5, 13, 25], though from a computational complexity perspective this is known to be a hard

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problem in general [8]. Recent works study plethysm via the representation theory of partition algebras [3], party algebras [19], and geometric complexity theory [7, 8].

We focus on the case $\mu = m$ and study instead the sum

$$\sum_{\nu \vdash nm} a^{\nu}_{\lambda,m}, \tag{1.2}$$

which is the number of irreducibles in the decomposition (1.1), counted with multiplicity.

1.1 Notation

Before stating our results, we recall some notions from the representation theory of finite groups and symmetric groups to fix notation.

Unless otherwise stated, n and m shall denote positive integers. We use $\#S$ and $|S|$ to denote the cardinality of a finite set S. For a finite group G, we let

$$\langle \chi, \phi \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\phi(g)}$$

denote the inner product of class functions χ and ϕ of G. We omit the subscript G if the group is clear from context. For a subgroup $H \leq G$, we write $\operatorname{Ind}_H^G \chi$ for the induction of a character χ from H to G and $\operatorname{Res}_H^G \chi$ for the restriction of a character χ of G to H. For a quotient $G \twoheadrightarrow K$, denote by $\operatorname{Inf}_K^G \chi$ the inflation of a character χ from K to G.

Let $G \wr \mathfrak{S}_n$ denote the *wreath product* of G with the symmetric group $\mathfrak{S}_n$. By definition, it is the semidirect product $G^n \rtimes \mathfrak{S}_n$, with $\mathfrak{S}_n$ acting on G^n by permuting the coordinates. Following [11, Section 4.1], we write an element of $G \wr \mathfrak{S}_n$ in the form $(f;\sigma)$, where $f \in G^n$ and $\sigma \in \mathfrak{S}_n$. For $\sigma \in \mathfrak{S}_n$, we shall sometimes write σ for $(1;\sigma)$, where $1 \in G^n$ is the identity element. We now specialize to $G = \mathfrak{S}_m$. For $1 \leq i \leq n$, define

$$\mathcal{P}_i = \{(i-1)m+1, \ldots, im\}. \tag{1.3}$$

We have inclusions $\mathfrak{S}_m^n \leq \mathfrak{S}_m \wr \mathfrak{S}_n \leq \mathfrak{S}_{nm}$, where $\mathfrak{S}_m^n$ and $\mathfrak{S}_m \wr \mathfrak{S}_n$ are identified with the stabilizers of $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ and $\{\mathcal{P}_1, \ldots, \mathcal{P}_n\}$ respectively under the natural $\mathfrak{S}_{nm}$-actions.

Given a partition λ, we let $\ell(\lambda)$ denote its length. We write m^n for the partition of nm consisting of n occurrences of m. We generally use $\lambda \vdash n$ to index the irreducible representations of $\mathfrak{S}_n$ and $\rho \vdash n$ for cycle types of elements of $\mathfrak{S}_n$. Thus we write $\chi^{\lambda}(\rho)$ for the value of the irreducible character χ^{λ} of $\mathfrak{S}_n$ at an element of cycle type ρ, see [11, Section 2.3]. In particular, recall that $\chi^n = 1$ and $\chi^{1^n} = \operatorname{sgn}$ are the trivial and sign characters of $\mathfrak{S}_n$ respectively. The plethysm coefficients in our case are then given [11, Section 5.4] by

$$a^{\nu}_{\lambda,m} = \langle \chi^{\nu}, \operatorname{Ind}_{\mathfrak{S}_m \wr \mathfrak{S}_n}^{\mathfrak{S}_{nm}} \operatorname{Inf}_{\mathfrak{S}_n}^{\mathfrak{S}_m \wr \mathfrak{S}_n} \chi^{\lambda} \rangle.$$

1.2 Statement of results

We now state our main results. Let $M(n,m)$ denote the set of $n \times n$ matrices with non-negative integer entries whose row and column sums are all equal to m. We shall identify $\mathfrak{S}_n$ with the group of $n \times n$ permutation matrices.

Definition 1.1. Define the function $N^m : \mathfrak{S}_n \to \mathbb{Z}$ by

$$N^m(\sigma) = \#\{A \in M(n,m) \mid \sigma A^\mathsf{T} = A\}$$

for $\sigma \in \mathfrak{S}_n$, where A^T denotes the transpose of A.

It is easy to see that N^m is a class function of $\mathfrak{S}_n$. Without the transpose, it would be the permutation character of $M(n,m)$. The main result is as follows:

Theorem 1.2. *For integers $n, m \geq 1$, N^m is a character of $\mathfrak{S}_n$. Moreover for $\lambda \vdash n$,*

$$\langle \chi^\lambda, N^m \rangle = \sum_{\nu \vdash nm} a^\nu_{\lambda,m}.$$

We shall prove this result in the next section. As we shall explain further in Section 4, for fixed $\sigma \in \mathfrak{S}_n$, $N^m(\sigma)$ is an Ehrhart quasipolynomial in m. Thus for fixed $\lambda \vdash n$, the sum (1.2) is quasipolynomial in m. Related to this, according to [14], the function $s \mapsto a^{s\nu}_{n,sm}$ is a quasipolynomial by a deep result, but is not an Ehrhart quasipolynomial. In the case $n = 3$, these quasipolynomials were computed in [1] using the decomposition of $s_\lambda[s_m]$ for $\lambda \vdash 3$. With the aid of a computer, Theorem 1.2 enables us to compute the quasipolynomials (1.2) for all $\lambda \vdash n$ with $n \leq 6$.

Example 1.3. For $n = 6$ and $\lambda = 6$, we have computed using SageMath [24] that

$$\begin{aligned}\sum_{\nu \vdash 6m} a^\nu_{6,m} = \frac{243653}{1434705592320000}m^{15} + \frac{243653}{31882346496000}m^{14} + \frac{91173671}{573882236928000}m^{13} \\ + \frac{5954623}{2942985830400}m^{12} + \frac{3895930519}{220723937280000}m^{11} + \frac{149644967}{1337720832000}m^{10} \\ + \frac{1072677673}{2006581248000}m^{9} + \frac{14723521}{7431782400}m^{8} + \frac{350041981}{59719680000}m^{7} + O(m^6),\end{aligned}$$

where we have omitted the trailing terms whose coefficients have period greater than 1.

Asymptotics of (1.2) as $m \to \infty$ were studied in [10]. We recover and slightly extend one of their results in Theorem 4.4 by studying the dimensions of the polytopes involved.

We call two matrices $A, B \in M(n,m)$ *permutation equivalent* and write $A \sim B$ if A can be transformed into B by row and column permutations. Let $T(n,m) = \{A \in M(n,m) \mid A \sim A^\mathsf{T}\}$ denote the subset of matrices that are permutation equivalent to their transpose. A major open problem in algebraic combinatorics is to find a combinatorial interpretation of plethysm coefficients $a^\nu_{\lambda,\mu}$ [20, Problem 9]. We have the following combinatorial interpretation of the sum (1.2) in the case $\lambda = n$:

Theorem 1.4. *When $\lambda = n$, the sum (1.2) is equal to the cardinality of $T(n,m)/\sim$:*

$$\sum_{\nu \vdash nm} a^{\nu}_{n,m} = \langle 1, N^m \rangle_{\mathfrak{S}_n} = \# T(n,m)/\sim.$$

Example 1.5. We compute that $s_3[s_3] = s_9 + s_{72} + s_{63} + s_{522} + s_{441}$. Correspondingly, there are five elements of $T(3,3)/\sim$, represented by

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

In the above setting, Foulkes [9] conjectured in 1950 that if $n \leq m$, then $a^{\nu}_{n,m} \leq a^{\nu}_{m,n}$. Brion [4] proved this for $n \ll m$ in 1993. Among recent works on Foulkes' conjecture, [7] verifies it for an infinite family, and a stable version was proven in [3] via partition algebras. Together with the previous theorem, Foulkes' conjecture implies

Conjecture 1.6. *If $n \leq m$, then $\# T(n,m)/\sim \, \leq \# T(m,n)/\sim$.*

2 Proof of Theorem 1.2

In this section we prove Theorem 1.2. Define the function $\theta : \mathfrak{S}_{nm} \to \mathbb{C}$ by

$$\theta(\sigma) = \#\{\tau \in \mathfrak{S}_{nm} \mid \tau^2 = \sigma\}.$$

It is well-known that the irreducible representations of $\mathfrak{S}_{nm}$ can be realized over $\mathbb{R}$, see e.g. [11, Theorem 2.1.12]. Hence [12, Corollary 23.17] implies

$$\theta = \sum_{\nu \vdash nm} \chi^{\nu},$$

and is in particular a character. We compute by Frobenius reciprocity

$$\begin{aligned} \sum_{\nu \vdash nm} a^{\nu}_{\lambda,m} &= \langle \theta, \operatorname{Ind}^{\mathfrak{S}_{nm}}_{\mathfrak{S}_m \wr \mathfrak{S}_n} \operatorname{Inf}^{\mathfrak{S}_m \wr \mathfrak{S}_n}_{\mathfrak{S}_n} \chi^{\lambda} \rangle = \langle \operatorname{Res}^{\mathfrak{S}_{nm}}_{\mathfrak{S}_m \wr \mathfrak{S}_n} \theta, \operatorname{Inf}^{\mathfrak{S}_m \wr \mathfrak{S}_n}_{\mathfrak{S}_n} \chi^{\lambda} \rangle \\ &= \frac{1}{m!^n n!} \sum_{(f;\sigma) \in \mathfrak{S}_m \wr \mathfrak{S}_n} \theta(f;\sigma) \chi^{\lambda}(\sigma) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \left(\frac{1}{m!^n} \sum_{f \in \mathfrak{S}_m^n} \theta(f;\sigma) \right) \chi^{\lambda}(\sigma). \end{aligned}$$

The expression in parentheses simplifies as

$$\frac{1}{m!^n} \sum_{f \in \mathfrak{S}_m^n} \theta(f;\sigma) = \frac{1}{m!^n} \sum_{f \in \mathfrak{S}_m^n} \#\{\tau \in \mathfrak{S}_{nm} \mid \tau^2 = (f;\sigma)\} = \frac{1}{m!^n} \#\{\tau \in \mathfrak{S}_{nm} \mid \tau^2 \sigma^{-1} \in \mathfrak{S}_m^n\}.$$

Thus it suffices to show that for $\sigma \in \mathfrak{S}_n$,

$$\frac{1}{m!^n}\#\{\tau \in \mathfrak{S}_{nm} \mid \tau^2\sigma^{-1} \in \mathfrak{S}_m^n\} = N^m(\sigma).$$

Now fix $\sigma \in \mathfrak{S}_n$ and recall the definition of $\mathcal{P}_i$ in (1.3). The proof is completed by

Lemma 2.1. *The map $F : \{\tau \in \mathfrak{S}_{nm} \mid \tau^2\sigma^{-1} \in \mathfrak{S}_m^n\} \to \{A \in M(n,m) \mid \sigma A^\mathsf{T} = A\}$ given by*

$$F(\tau)_{ij} = \#(\mathcal{P}_i \cap \tau\mathcal{P}_j)$$

is $m!^n$-to-1 and surjective.

Proof. We first show that F has the stated codomain. Let $\tau \in \mathfrak{S}_{nm}$ be such that $\tau^2\sigma^{-1} \in \mathfrak{S}_m^n$, or equivalently $\tau^2\mathcal{P}_i = \mathcal{P}_{\sigma(i)}$ for each $1 \le i \le n$. Let $A = F(\tau)$ and for $1 \le i, j \le n$, set $\mathcal{Q}_{ij} = \mathcal{P}_i \cap \tau\mathcal{P}_j$. Then

(i) for each $1 \le i \le n$, $\{\mathcal{Q}_{ij} \mid 1 \le j \le n\}$ is a partition of $\mathcal{P}_i$ with $\#\mathcal{Q}_{ij} = A_{ij}$; and

(ii) for each $1 \le i, j \le n$, τ restricts to a bijection $\mathcal{Q}_{ij} \to \mathcal{Q}_{\sigma(j)i}$.

Indeed (i) is clear, and (ii) follows from $\tau\mathcal{Q}_{ij} = \tau\mathcal{P}_i \cap \tau^2\mathcal{P}_j = \tau\mathcal{P}_i \cap \mathcal{P}_{\sigma(j)} = \mathcal{Q}_{\sigma(j)i}$. It follows from (i) that A has row sums equal to m. The condition $\sigma A^\mathsf{T} = A$ is equivalent to $A_{ij} = A_{\sigma(j)i}$ for $1 \le i, j \le n$, which holds by (ii).

Conversely let $A \in M(n,m)$ be such that $\sigma A^\mathsf{T} = A$. We want to count $\tau \in \mathfrak{S}_{nm}$ such that $\tau^2\sigma^{-1} \in \mathfrak{S}_m^n$ and $F(\tau) = A$. Reversing the above, giving such τ is equivalent to giving $\mathcal{Q}_{ij}$ for $1 \le i, j \le n$ satisfying (i), and bijections $\mathcal{Q}_{ij} \to \mathcal{Q}_{\sigma(j)i}$ as in (ii). There are

$$\binom{m}{A_{i1}, \ldots, A_{in}}$$

partitions of $\mathcal{P}_i$ satisfying (i), and $A_{ij}!$ bijections in (ii) since $A_{ij} = A_{\sigma(j)i}$. Thus there are

$$\prod_{i=1}^n \binom{m}{A_{i1}, \ldots, A_{in}} \prod_{1 \le i,j \le n} A_{ij}! = m!^n$$

such τ as required. $\square$

3 $(G \wr C_2)$-sets and representations

Let G be a finite group and let C_2 denote the cyclic group of order 2, whose generator we shall suggestively denote by T. Elements of $G \wr C_2$ have the form $(g, h; 1)$ or $(g, h; \mathsf{T})$ for $g, h \in G$. In this section we collect some results on $(G \wr C_2)$-actions and representations. We will obtain Theorem 1.4 as a consequence of more general results.

3.1 Irreducible characters of $G \wr C_2$

We describe the irreducible characters of $G \wr C_2$ in terms of the irreducible characters of G. For characters χ_1 and χ_2 of G, let $\chi_1 \boxtimes \chi_2$ denote their external tensor product. It is a character of $G \times G$ with

$$(\chi_1 \boxtimes \chi_2)(g, h) = \chi_1(g)\chi_2(h).$$

Suppose V is a representation of G. Then $V \otimes V$ is a representation of $G \wr C_2$ with action

$$(g, h; 1)(v \otimes w) = gv \otimes hw; \qquad (1, 1; \mathsf{T})(v \otimes w) = w \otimes v.$$

If χ is the character of V, then the character $\widetilde{\chi}$ of $V \otimes V$ is given by

$$\widetilde{\chi}(g, h; 1) = \chi(g)\chi(h); \qquad \widetilde{\chi}(g, h; \mathsf{T}) = \chi(gh).$$

Thus given a character χ of G, we can define the character $\widetilde{\chi}$ of $G \wr C_2$ by the above. By [11, Theorem 4.4.3], there are three types of irreducible characters of $G \wr C_2$, these are:

1. $\widetilde{\chi}$, where χ is an irreducible character of G;
2. $\widetilde{\chi} \cdot \mathrm{Inf}_{C_2}^{G \wr C_2} \mathrm{sgn}$, where χ is an irreducible character of G;
3. $\mathrm{Ind}_{G \times G}^{G \wr C_2}(\chi_1 \boxtimes \chi_2)$, where χ_1, χ_2 are distinct irreducible characters of G.

By a case analysis, we obtain

Proposition 3.1. *Let ψ be a character of $G \wr C_2$. Then $\psi^{\mathsf{T}}(g) = \psi(g, 1; \mathsf{T})$ is a virtual character of G. Moreover if χ is an irreducible character of G, then*

$$\langle \chi, \psi^{\mathsf{T}} \rangle = \langle \widetilde{\chi}, \psi \rangle - \langle \widetilde{\chi} \cdot \mathrm{Inf}_{C_2}^{G \wr C_2} \mathrm{sgn}, \psi \rangle.$$

3.2 Orbit-counting with a twist

Let X be a finite $(G \wr C_2)$-set. For $x \in X$ and $g, h \in G$, we write $gx = (g, 1; 1)x$, $xh = (1, h^{-1}; 1)x$, and $x^{\mathsf{T}} = (1, 1; \mathsf{T})x$. In this way, X is equipped with left and right G-actions and an involution compatible with each other, i.e.

1. $g(xh) = (gx)h$, which we simply denote by gxh; and
2. $(gxh)^{\mathsf{T}} = h^{-1}x^{\mathsf{T}}g^{-1}$.

Conversely given X with left and right G-actions and an involution $x \mapsto x^{\mathsf{T}}$ satisfying the above, we have a $(G \wr C_2)$-action on X.

Let $(G \times G)\backslash X$ denote the set of $(G \times G)$-orbits of X. For $x, y \in X$, write $x \sim y$ if they lie in the same $(G \times G)$-orbit, i.e. if $(G \times G) \cdot x = (G \times G) \cdot y$. The set $(G \times G)\backslash X$ inherits a C_2-action, given by

$$((G \times G) \cdot x)^{\mathsf{T}} = (G \times G) \cdot x^{\mathsf{T}}.$$

We let $((G \times G)\backslash X)^{C_2}$ denote the subset of fixed points under the C_2-action. This is a set we are interested in counting. Our main example is the following:

Example 3.2. Let $G = \mathfrak{S}_n$ and $X = M(n,m)$. We have a natural $(\mathfrak{S}_n \wr C_2)$-action on X, where $\mathfrak{S}_n$ acts by permuting rows on the left and permuting columns on the right, while T acts by taking transposes. Two matrices $A, B \in M(n,m)$ are permutation equivalent precisely when they lie in the same $(\mathfrak{S}_n \times \mathfrak{S}_n)$-orbit. We have $((\mathfrak{S}_n \times \mathfrak{S}_n)\backslash X)^{C_2} = T(n,m)/\sim$.

For an arbitrary finite $(G \wr C_2)$-set X as before, define

$$N(g) = \#\{x \in X \mid gx^{\mathsf{T}} = x\}.$$

For $s \in X$, let $\mathrm{Stab}_{G\times G}(s)$ denote the stabilizer of s under the $(G \times G)$-action and define

$$N^s(g) = \#\{x \in (G \times G) \cdot s \mid gx^{\mathsf{T}} = x\}.$$

It is easy to see that N and N^s are class functions of G, and N^s only depends on the $(G \times G)$-orbit of s. We can say more:

Proposition 3.3. *For $s \in X$, N^s is a virtual character of G. Moreover if χ is an irreducible character of G, then*

$$\langle \chi, N^s \rangle = \frac{1}{|\mathrm{Stab}_{G\times G}(s)|} \sum_{\substack{g,h \in G \\ gs^{\mathsf{T}}h^{-1} = s}} \chi(gh).$$

Applying this to the trivial character of G, we obtain

Corollary 3.4. *For $s \in X$, we have*

$$\langle 1, N^s \rangle = \begin{cases} 1 & \text{if } s \sim s^{\mathsf{T}} \\ 0 & \text{otherwise.} \end{cases}$$

This gives the following generalization of the Cauchy–Frobenius lemma:

Theorem 3.5. *N is a virtual character of G. Furthermore, we have*

$$\#((G \times G)\backslash X)^{C_2} = \langle 1, N \rangle = \frac{1}{|G|} \sum_{g \in G} N(g).$$

Proof of Theorem 1.4. The first equality of Theorem 1.4 is a special case of Theorem 1.2 with $\lambda = n$. The second equality follows immediately from Theorem 3.5 with G and X as in Example 3.2. □

Remark 3.6. The equality $\#T(n,m)/\sim = \sum_{\nu \vdash nm} a^{\nu}_{n,m}$ was the original motivation for this paper. OEIS sequence A333737 gives the number of non-negative integer symmetric matrices with equal row sums, up to permutation equivalence. This is related to, but not exactly the sequence we have, as $n \times n$ matrices that are permutation equivalent to their transpose need not be permutation equivalent to a symmetric matrix when $n \geq 6$. See [6, Example 1] for a counterexample.

Remark 3.7. A related quantity to (1.2) is the number of irreducibles in the decomposition of $(s_m)^n$. This is equal to $N^m(1_{\mathfrak{S}_n})$, i.e. the number of $n \times n$ non-negative integer symmetric matrices with row sums equal to m. This follows from Theorem 1.2 and [22, Corollary 7.12.5]. Alternatively, a bijective proof can be given by using Young's rule (take $\mu = m^n$ in [22, Corollary 7.12.4]) and the RSK algorithm [22, Theorems 7.11.5, 7.13.1].

4 Properties of N^m

Now we shall describe some properties of the characters N^m. For $\sigma \in \mathfrak{S}_n$, we write $N_\sigma(m)$ for $N^m(\sigma)$ when we want to emphasize that it is a function of m. If σ is of cycle type $\rho \vdash n$, we also set $N^m(\rho) = N_\rho(m) = N^m(\sigma)$. Thus, Theorem 1.2 asserts that

$$\sum_{\nu \vdash nm} a^{\nu}_{\lambda,m} = \sum_{\rho \vdash n} z_\rho^{-1} \chi^\lambda(\rho) N^m(\rho),$$

where $z_\rho = \prod_{i \geq 1} i^{m_i} m_i!$, with m_i being the number of parts of ρ equal to i.

4.1 Lattice points in polytopes

We study the N_σ as defined above from the point of view of Ehrhart theory, see [2] or [21, Chapter 4] for an introduction to this subject.

Recall that a *rational convex polytope* $\mathcal{P}$ is the convex hull of finitely many points with rational coordinates in some Euclidean space $\mathbb{R}^n$. Given such a polytope $\mathcal{P}$, let $L_{\mathcal{P}}(t) = \#(t\mathcal{P} \cap \mathbb{Z}^n)$ and let $\mathrm{Vol}(\mathcal{P})$ denote its relative volume. A *quasipolynomial of degree n* is a function $f : \mathbb{Z} \to \mathbb{C}$ of the form $f(t) = c_n(t)t^n + \ldots + c_1(t)t + c_0(t)$ where $c_i : \mathbb{Z} \to \mathbb{C}$ are periodic functions with $c_n \neq 0$. By Ehrhart's theorem [2, Theorem 3.23], $L_{\mathcal{P}}$ is a quasipolynomial, known as the *Ehrhart quasipolynomial* of $\mathcal{P}$, of degree equal to the dimension of $\mathcal{P}$.

We now apply this theory in our context. For a set S, let $M_n(S)$ denote the set of $n \times n$ matrices with entries in S. For $\sigma \in \mathfrak{S}_n$, define the rational convex polytope

$$\mathcal{P}(\sigma) = \{A \in M_n(\mathbb{R}_{\geq 0}) \mid A \text{ has row sums equal to 1 and } \sigma A^{\mathsf{T}} = A\} \subseteq M_n(\mathbb{R}).$$

Then for $\sigma \in \mathfrak{S}_n$, $N_\sigma(m) = L_{\mathcal{P}(\sigma)}(m)$, so N_σ is a quasipolynomial of degree equal to the dimension of $\mathcal{P}(\sigma)$. We have

Proposition 4.1. *For $\rho \vdash n$, the degree of N_ρ is*

$$\deg N_\rho = \sum_{1 \le i < j \le \ell(\rho)} \gcd(\rho_i, \rho_j) + \sum_{1 \le i \le \ell(\rho)} \left\lfloor \frac{\rho_i - 1}{2} \right\rfloor.$$

Corollary 4.2. *For $\rho \vdash n$, we have $(-1)^{\deg N_\rho} = (-1)^{\frac{n(n-1)}{2}} \varepsilon_\rho^{n-1}$, where $\varepsilon_\rho = (-1)^{n-\ell(\rho)}$.*

Corollary 4.3. *For $\rho \vdash n$, we have $\deg N_\rho \le \frac{n(n-1)}{2}$ with equality if and only if $\rho = 1^n$.*

We deduce from this and a similar analysis of the second-highest degree term the following asymptotics for (1.2), c.f. [10, Theorem 3.7 (i)]:

Theorem 4.4. *For $\lambda \vdash n$, as $m \to \infty$, we have*

(i) $\displaystyle\sum_{\nu \vdash nm} a^\nu_{\lambda,m} \sim \frac{\chi^\lambda(1)}{n!} \mathrm{Vol}(\mathcal{P}(1_{\mathfrak{S}_n})) m^{\frac{n(n-1)}{2}}$*; and*

(ii) $\displaystyle\sum_{\nu \vdash nm} a^\nu_{\lambda,m} = \chi^\lambda(1) \sum_{\nu \vdash nm} a^\nu_{n,m} + O\left(m^{\frac{(n-1)(n-2)}{2}}\right).$

The fact that N_σ is a quasipolynomial allows us to extend its definition to all integers, and hence define a class function N^m for all $m \in \mathbb{Z}$.

Proposition 4.5. *For $n \ge 1$ and $m \in \mathbb{Z}$, we have*

$$N^m = (-1)^{\frac{n(n-1)}{2}} \operatorname{sgn}^{n-1} \cdot N^{-m-n}$$

as virtual characters of $\mathfrak{S}_n$. Furthermore, $N^m = 0$ for $-n+1 \le m \le -1$.

Proof Sketch. This follows by Ehrhart–Macdonald reciprocity [2, Theorem 4.1] and Corollary 4.2. □

4.2 A decomposition of N^m

Let $n, m \ge 1$ be integers. As in Section 3, there is a natural way to decompose the character N^m of $\mathfrak{S}_n$ as a sum

$$N^m = \sum_{\mathcal{C} \in T(n,m)/\sim} N^{\mathcal{C}},$$

where for $\mathcal{C} \in T(n,m)/\sim$, we set $N^{\mathcal{C}}(\sigma) = \#\{A \in \mathcal{C} \mid \sigma A^\mathsf{T} = A\}$. If $S \in \mathcal{C}$, then in the notation of Section 3, $N^{\mathcal{C}} = N^S$ and it is a virtual character of $\mathfrak{S}_n$ by virtue of Proposition 3.1. Theorem 1.2 implies that for $\lambda \vdash n$,

$$\sum_{\nu \vdash nm} a^\nu_{\lambda,m} = \sum_{\mathcal{C} \in T(n,m)/\sim} \langle \chi^\lambda, N^{\mathcal{C}} \rangle.$$

This allows us to measure the contribution of each $\mathcal{C} \in T(n,m)/\sim$ to the sum (1.2).

Proposition 4.6. *For $\mathcal{C} \in T(n,m)/\sim$, we have $\langle 1, N^{\mathcal{C}} \rangle = 1$.*

Remark 4.7. If $\mathcal{C}_1 \in T(n,m_1)/\sim$ and $\mathcal{C}_2 \in T(n,m_2)/\sim$ are such that there are $A_1 \in \mathcal{C}_1, A_2 \in \mathcal{C}_2$, and a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $f(A_1) = A_2$, then $N^{\mathcal{C}_1} = N^{\mathcal{C}_2}$.

Remark 4.8. Unfortunately $N^{\mathcal{C}}$ is not a character in general, for example for the matrix in [6, Example 1]. By Proposition 3.3, we see that $-\chi^\lambda(1) \le \langle \chi^\lambda, N^{\mathcal{C}} \rangle \le \chi^\lambda(1)$. By Theorem 1.4 and Theorem 4.4,

$$\lim_{m\to\infty} \frac{1}{|T(n,m)/\sim|} \sum_{\mathcal{C}\in T(n,m)/\sim} \langle \chi^\lambda, N^{\mathcal{C}} \rangle = \chi^\lambda(1).$$

Thus when m is large, for most $\mathcal{C} \in T(n,m)/\sim$, $N^{\mathcal{C}}$ will be the character of the regular representation of $\mathfrak{S}_n$.

5 The case $m = 2$

We conclude by using our results to study the case $m = 2$.

Let $n_1, n_2 \ge 1$. For $\mathcal{C}_1 = [A_1] \in T(n_1, 2)/\sim$ and $\mathcal{C}_2 = [A_2] \in T(n_2, 2)/\sim$, we define their sum $\mathcal{C}_1 + \mathcal{C}_2 \in T(n_1 + n_2, 2)/\sim$ to be the equivalence class of the block diagonal matrix $A_1 \oplus A_2$. We call $\mathcal{C} \in T(n,2)/\sim$ *irreducible* if it cannot be written as a nontrivial sum $\mathcal{C}_1 + \mathcal{C}_2$. They are represented by matrices of the form

$$\begin{pmatrix} 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \\ 1 & & 1 \\ & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & & & \\ 1 & & 1 & & \\ & 1 & & 1 & \\ & & 1 & & 1 \\ & & & 1 & 1 \end{pmatrix}, \dots$$

Furthermore, each $\mathcal{C} \in T(n,2)/\sim$ can be expressed as a sum of irreducibles, unique up to ordering. We thus have a bijection between $T(n,2)/\sim$ and the set of partitions of n, sending an equivalence class $\mathcal{C}$ to the partition $\lambda_{\mathcal{C}} \vdash n$ recording the sizes of the irreducible summands.

Example 5.1. If $\mathcal{C} \in T(5,2)/\sim$ is the equivalence class of

$$\begin{pmatrix} 1 & 1 & & & \\ 1 & & 1 & & \\ & 1 & 1 & & \\ & & & 2 & \\ & & & & 2 \end{pmatrix},$$

then $\lambda_{\mathcal{C}} = (3,1,1)$.

Proposition 5.2. *For $\mathcal{C} \in T(n,2)/\sim$, we have*

$$\langle \mathrm{sgn}, N^{\mathcal{C}} \rangle_{\mathfrak{S}_n} = \begin{cases} 1 & \text{if all parts of } \lambda_{\mathcal{C}} \text{ are odd} \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 5.3. *We have*

(i) $\langle 1, N^2 \rangle_{\mathfrak{S}_n} = \sum_{\nu \vdash 2n} a^{\nu}_{n,2} = \#\{\text{partitions of } n\}$; *and*

(ii) $\langle \mathrm{sgn}, N^2 \rangle_{\mathfrak{S}_n} = \sum_{\nu \vdash 2n} a^{\nu}_{1^n,2} = \#\{\text{partitions of } n \text{ into odd parts}\}$.

While the decompositions of $s_n[s_2]$ and $s_{1^n}[s_2]$ are well-known [17, page 138], the above corollary was derived independently of these. As in Remark 4.8, for general m, it is possible but rare that $\langle \mathrm{sgn}, N^{\mathcal{C}} \rangle = -1$. We hope that a better understanding of this will lead to a combinatorial interpretation of (1.2) for $\lambda = 1^n$.

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