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Volume and Lattice Points of Generalized Pitman–Stanley Flow Polytopes

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Abstract. The Pitman–Stanley polytope is well-studied and related to plane partitions of skew shape with entries 0 and 1. We study a generalization of this polytope related to plane partitions with entries $0, 1, \ldots, m$. We study formulas for the volume and lattice points of this polytope. These formulas have rich combinatorics including connections to standard Young tableaux of rectangular shape. We also give a combinatorial proof for a special case, partially answering a question of Pitman and Stanley.

Résumé. Nous étudions des formules pour le nombre de points à coordonnées entières et le volume d'une généralisation du polytope de Pitman–Stanley reliée aux partitions de plan avec entrées $\{0, 1, \ldots, m\}$. Ces formules établissent des liens avec les tableaux standards de Young de forme rectangulaire. Nous présentons une preuve combinatoire pour le cas $n = 2$, répondant ainsi partiellement à une question de Pitman et Stanley.

Keywords: Pitman–Stanley polytope, plane partitions, tableaux, Fuss–Catalan

1 Introduction

The eponymous Pitman–Stanley polytope introduced in [13] is a well-studied polytope in geometric, algebraic, and enumerative combinatorics. This polytope is defined as follows. For a positive integer n and vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{N}^n$, let

$$\mathrm{PS}_n(\mathbf{a}, \mathbf{b}) := \{\mathbf{x} \in \mathbb{R}^n_{\geq 0} \quad | \quad b_1 + \cdots + b_i \leq x_1 + \cdots + x_i \leq a_1 + \cdots + a_i \ \text{ for } i = 1, \ldots, n\},$$

i.e., $\mathrm{PS}_n(\mathbf{a}, \mathbf{b})$ consists of the nonnegative vectors $\mathbf{x}$ in $\mathbb{R}^n$ that are between vectors $\mathbf{a}$ and $\mathbf{b}$ in *dominance order*. Recall that, for $\mathbf{v}, \mathbf{w} \in \mathbb{N}^n$, we say that $\mathbf{v}$ *dominates* $\mathbf{w}$ if $\sum_{j=1}^{i} v_j \geq \sum_{j=1}^{i} w_j$ for every $i = 1, \ldots, n$. The polytope $\mathrm{PS}_n(\mathbf{a}) := \mathrm{PS}_n(\mathbf{a}, \mathbf{0})$ is an example

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of a generalized permutahedron [10], and a flow polytope [1]. The skew case $\mathrm{PS}_n(\mathbf{a},\mathbf{b})$ is related to lattice path matroids [2, Section 4.5].

Pitman and Stanley showed the lattice points of $\mathrm{PS}_n(\mathbf{a},\mathbf{b})$ correspond to *plane partitions* of a skew shape $\theta(\mathbf{a},\mathbf{b}) = (a_1+\cdots+a_n,\ldots,a_1+a_2,a_1)/(b_1+\ldots+b_n,\ldots,b_1+b_2,b_1)$ with entries $0,1$. In their paper, they mentioned in passing a generalization of their polytope whose lattice entries correspond to plane partitions of shape $\theta(\mathbf{a},\mathbf{b})$ with entries $0,1,\ldots,m$. We call this the *generalized Pitman–Stanley polytope* defined as follows:

$$\mathrm{PS}_n^m(\mathbf{a},\mathbf{b}) := \{(x_{ij})_{1\le i\le n, 1\le j\le m} \in \mathbb{R}^{nm}_{\ge 0} \;\mid\; b_1+\cdots+b_i \le x_{1m}+x_{2m}+\ldots+x_{im} \le \cdots \\ \cdots \le x_{11}+x_{21}+\ldots+x_{i1} \le a_1+\cdots+a_i \text{ for } i=1,\ldots,n\}.$$

In [3], we started an in-depth study of this polytope. We showed that $\mathrm{PS}_n^m(\mathbf{a},\mathbf{b})$ is integrally equivalent to a flow polytope of a grid graph $G(n,m)$ (the flow on the horizontal edges correspond to the coordinates of the polytope, see Figure 1, left) and studied its vertices and faces. In this extended abstract, we study formulas for its volume and lattice points.

Pitman and Stanley [13] found the following nonnegative formulas for the volume and lattice points of $\mathrm{PS}_n(\mathbf{a})$.

$$\operatorname{vol}\mathrm{PS}_n(\mathbf{a}) = \sum_{\mathbf{j}} \binom{n}{j_1,\ldots,j_n} a_1^{j_1}\cdots a_n^{j_n},\; \#(\mathrm{PS}_n(\mathbf{a})\cap\mathbb{Z}^n) = \sum_{\mathbf{j}} \left(\!\!\binom{a_i+1}{j_1}\!\!\right)\left(\!\!\binom{a_2}{j_2}\!\!\right)\cdots\left(\!\!\binom{a_n}{j_n}\!\!\right),$$

where $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$ and the sum is over compositions $\mathbf{j}=(j_1,\ldots,j_n)$ of n that dominate $\mathbf{1}$. The number of such compositions is $C_n := \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number which counts Dyck paths of size n, Standard Young tableaux (SYT) of shape (n^2) (or shape (2^n), among others. Their formula for lattice points implies that the Ehrhart polynomial $\#(\mathrm{PS}_n(t\mathbf{a})\cap\mathbb{Z}^n)$ of this polytope has nonnegative coefficients in t.

Straight shape case $\mathrm{PS}_n^m(\mathbf{a},\mathbf{0})$: Using the connection between the generalized Pitman–Stanley polytope and flow polytopes, we use the *Lidskii volume formula* of the latter [1, 9] to give a nonnegative formula for the volume of $\mathrm{PS}_n^m(\mathbf{a}) := \mathrm{PS}_n^m(\mathbf{a},\mathbf{0})$ (see Theorem 2.2). The number of terms of the formula is the *Fuss–Catalan number* $FC_{n,m} := \frac{1}{nm+1}\binom{(m+1)n}{n}$ which count m-Dyck paths of size n [14]. Another important ingredient of the formula are certain multiplicities $K_{G(n,m)}(\mathbf{j}-\mathbf{1})$ that are *mixed volumes* and count several interesting objects including: certain integer flows on the graphs $G(n,m)$, certain tuples of generalized Dyck paths, and the number of SYT of shape n^{m+1} with prescribed last (or first) row. In Section 2, we give bijections among these objects (Propositions 2.6 and 2.7), other interpretations, and enumerative properties. The latter include a determinantal formula for the generating polynomial of these multiplicities (Corollary 2.17).

The case of $\mathbf{a}=\mathbf{1}$ is particularly interesting as shown algebraically in [13, Theorem 14].

$$\operatorname{vol} \mathrm{PS}_n^m(\mathbf{1}) = \mathrm{SYT}(n^m) \cdot \prod_{i=1}^{m} i!(n+i)^{n-m-1+i}, \tag{1.1}$$

where $\mathrm{SYT}(n^m)$ is the number of SYT of shape $m \times n$. In particular $\operatorname{vol} \mathrm{PS}_n(\mathbf{1}) = (n+1)^{n-1}$, the number of parking functions of size n. Pitman and Stanley gave in [13] a combinatorial proof for the case $m = 1$ and asked for a combinatorial proof for general m and n. In Section 2.3, we use our volume formula to give a combinatorial proof for the case $n = 2$ (which has $\binom{m+2}{3}$ vertices), where the volume is $C_m \prod_{i=1}^{m} i!(2+i)^{1-m+i}$.

In Theorem 3.1, we also give a nonnegative formula for the number of lattice points of $\mathrm{PS}_n^m(\mathbf{a})$, that reduces to Pitman–Stanley's formula for $m = 1$ and also implies our volume formula (2.1). The summation formula is in terms of SYT of shape n^{m+1} and their *natural descents*. Calculations suggest that $\mathrm{PS}_n^m(\mathbf{a})$ is Ehrhart positive but this is not evident from our lattice point formula.

Skew shape case $\mathrm{PS}_n^m(\mathbf{a}, \mathbf{b})$: Lattice points of $\mathrm{PS}_n^m(\mathbf{a}, \mathbf{b})$ correspond to plane partitions with bounded entries, their number is given by a determinant formula of Kreweras [6]:

$$\#(\mathrm{PS}_n^m(\mathbf{a}, \mathbf{b}) \cap \mathbb{Z}^{mn}) \;=\; \det \left[\binom{a_1 + \cdots + a_i - b_1 - \cdots - b_j + m}{i - j + m} \right]_{i,j=1}^{n}. \tag{1.2}$$

Note that from the subtractions in the binomials above, these formulas for generic $\mathbf{a}$ and $\mathbf{b}$ are piecewise polynomial. In Theorem 3.5 we give nonnegative formulas for $\#(\mathrm{PS}_n^m(\mathbf{a}, \mathbf{b}) \cap \mathbb{Z}^{mn})$. Two salient features of these formulas are that (i) they are in terms of SYT of shape n^{m+2} and their descents, and (ii) for fixed n, there are C_n possible formulas depending on the relative order of the partial sums of the a_is and b_is. From the lattice point formulas one can also extract a formula for the volume of $\mathrm{PS}_n^m(\mathbf{a}, \mathbf{b})$ (Corollary 3.6). These lattice point formulas for $\mathrm{PS}_n^m(\mathbf{a})$ and $\mathrm{PS}_n^m(\mathbf{a}, \mathbf{b})$ are obtained using Stanley's theory of *marked order polytopes* [13, 11] and not from the Lidskii formula for lattice points of flow polytopes.

2 The Lidskii volume formula for $PS_n^m(\mathbf{a},\mathbf{b})$

2.1 Lidskii volume formulas for $PS_n^m(\mathbf{a})$

The Lidskii formula can be used to calculate the volume of a flow polytope [1, 9]. Considering we have shown that $\mathrm{PS}_n^m(\mathbf{a})$ is a flow polytope [3, Theorem 3.4], we can use the Lidskii volume formula of the latter [1, 9] to calculate the volume of $PS_n^m(\mathbf{a})$.

Definition 2.1. *For the grid graph $G(n,m)$ and a composition $\mathbf{j} = (j_1, \ldots, j_n)$ of mn, let $K_{G(n,m)}(\mathbf{j} - \mathbf{1}) := K_{G(n,m)}(\mathbf{c})$, where $\mathbf{c}$ is the concatenation of $\mathbf{j} - \mathbf{1}$ of length n, of $-\mathbf{1}$ of length $n(m-1)$ and $\mathbf{0}$ of length $n+1$, be the number of integral flows on $G(n,m)$ where the net flow* *of the leftmost vertices of $G(n,m)$ is $\mathbf{j}-\mathbf{1}$, of the rightmost vertices is $\mathbf{0}$ (including the sink), and all internal vertices of the graph have net flow -1. Note that we do not draw the sink and the edges adjacent to the sink in future figures as they carry no flow.*

Theorem 2.2.

$$\operatorname{vol} \mathrm{PS}_n^m(a_1,\ldots,a_n) = \sum_{\mathbf{j}} \binom{mn}{j_1,\ldots,j_n} a_1^{j_1}\cdots a_n^{j_n} K_{G(n,m)}(\mathbf{j}-\mathbf{1}), \tag{2.1}$$

where the sum is over compositions $\mathbf{j}=(j_1,\ldots,j_n)$ of mn that dominate $m\mathbf{1}$.

Proof Sketch. $\mathrm{PS}_n^m(\mathbf{a})$ is integrally equivalent to the flow polytope $\mathcal{F}_{G(n,m)}(\mathbf{c})$ where $\mathbf{c} = (\mathbf{a},\mathbf{0},-\sum_i a_i)$ indicates netflow a_i for vertices on the left boundary of the graph, zero netflow for every vertex in $G(n,m)$ that is not on the left boundary or is a sink of the graph. We can thus apply the Lidskii volume formula [1, Theorem 38] to $\mathrm{PS}_n^m(\mathbf{a})$. □

Example 2.3. *For $n=m=2$, we have from Figure 2 that*

$$\operatorname{vol} \mathrm{PS}_2^2(a_1,a_2) = \binom{4}{4,0}\cdot a_1^4\cdot 2 + \binom{4}{3,1}\cdot a_1^3a_2\cdot 2 + \binom{4}{2,2}\cdot a_1^2a_2^2\cdot 1 = 2a_1^4+8a_1^3a_2+6a_1^2a_2^2.$$

2.2 Interpretations of $K_{G(n,m)}(\mathbf{j}-\mathbf{1})$

We now give different combinatorial interpretations for $K_{G(n,m)}(\mathbf{j}-\mathbf{1})$, but first, we show a recursion.

Lemma 2.4. *We have that $K_{G(n,m)}(\mathbf{j}-\mathbf{1}) = \sum_{\mathbf{k}} K_{G(n,m-1)}(\mathbf{k}-\mathbf{1})$ where the sum is over compositions $\mathbf{k}$ of $(m-1)n$ such that $\mathbf{k}$ dominates $(m-1)\mathbf{1}$ and is dominated by $\mathbf{j}-\mathbf{1}$.*

Proof sketch. The sum of flows in the first column of $G(n,m)$ is $n(m-1)$, and the vector of the flows in the first column must be dominated by $\mathbf{j}-\mathbf{1}$. □

Definition 2.5. *We call a Dyck path that stays above (but may touch) the line $y=\frac{1}{m}x$ an m-*Dyck *path. We say an m-Dyck path is i-*stepping *if between every two North steps, there are at least i East steps. We call a set of $m+1$ non-crossing m-Dyck paths from $(0,0)$ to (mn,n) $\{P_0,P_1,\ldots,P_m\}$ a* lengthening m-Dyck set *for n if P_i is above or equal to P_{i+1} everywhere, and if P_i is i-stepping for every i. We let $\mathcal{D}_{m,n}$ be the collection of lengthening m-Dyck sets for n.*

Proposition 2.6. *$K_{G(n,m)}(\mathbf{j}-\mathbf{1})$ counts sets of Dyck paths $\{P_0,P_1,\ldots,P_m\}$ in $\mathcal{D}_{m,n}$ where the lengths of horizontal steps of P_0 are given by $\mathbf{j}$ (starting from the top right corner).*

Proof sketch. The flows counted by $K_{G(n,m)}(\mathbf{j}-\mathbf{1})$ (see Definition 2.1) are uniquely characterized by $\mathbf{j}$ and the flows on the horizontal edges of $G(n,m)$. Moreover, $x_{1k}+x_{2k}+$

$\ldots + x_{nk}$, the sum of the flows on the horizontal edges in the kth column of the graph is $n(m-k)$ for each $1 \le k \le m$. Let P_k be the Dyck path where the lengths of horizontal steps (starting from the top right corner of the grid) are the flows of the horizontal edges in the kth column of the graph plus k, and let $P_0 = \mathbf{j}$. Then $\{P_0, P_1, \ldots, P_m\} \in \mathcal{D}_{m,n}$. □

Proposition 2.7. *We have that $K_{G(n,m)}(\mathbf{j}-\mathbf{1})$ counts the number of SYT of shape n^{m+1} where the $(m+1)$th row has values $(j_1+1, j_1+j_2+2, \ldots, j_1+\cdots+j_{n-1}+n-1, n(m+1))$. This implies that $\#\mathcal{D}_{m,n} = \mathrm{SYT}(n^{m+1})$.*

Proof sketch. We use Proposition 2.6 to give a bijection with such SYTs. Consider a standard Young tableau Y of shape n^{m+1} with entries y_{ij} for $1 \le i \le m+1$ and $1 \le j \le n$. Now build an auxiliary $(m+1) \times n$ array X where $x_{i,1} = y_{i,1}$ for all $1 \le i \le m+1$ and $x_{i,j} = y_{i,j} - y_{i,j-1} - s_{i,j}$ for all $2 \le j \le n$ and $1 \le i \le m+1$ where

$$s_{i,j} := \#\{y_{i',j'} \mid y_{i,j-1} < y_{i',j'} < y_{i,j} \text{ such that } j' < j \text{ and } i' > i\}.$$

Now consider an $(m+1) \times n$ array Z where $z_{i,j} = x_{i,j} + m - i$. Each row gives the horizontal steps of a Dyck path (starting from the top right corner). One can show that this set of Dyck paths is in $\mathcal{D}_{m,n}$, and that the top Dyck path (given by the last row of Z) has horizontal steps of length $(y_{m+1,1}-1, y_{m+1,2}-y_{m+1,1}-1, y_{m+1,3}-y_{m+1,2}-1, \ldots, y_{m+1,n}-y_{m+1,n-1}-1)$. One can then show that this gives $\#\mathcal{D}_{m,n} = \mathrm{SYT}(n^{m+1})$. □

Example 2.8. *Consider the following standard Young tableau where $m = 3$ and $n = 6$.*

$$Y = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 4 & 5 & 11 & 14 & 16 \\ \hline 2 & 6 & 9 & 15 & 18 & 19 \\ \hline 3 & 8 & 12 & 17 & 21 & 22 \\ \hline 7 & 10 & 13 & 20 & 23 & 24 \\ \hline \end{array}, \qquad X = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 3 & 2 & 1 \\ 3 & 4 & 3 & 4 & 3 & 1 \\ 7 & 3 & 3 & 7 & 3 & 1 \end{pmatrix}, \qquad Z = \begin{pmatrix} 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 4 & 2 & 4 & 3 & 2 \\ 3 & 4 & 3 & 4 & 3 & 1 \\ 6 & 2 & 2 & 6 & 2 & 0 \end{pmatrix}$$

(Ignore for now the highlighted cells.) Proposition 2.7 yields arrays X and Z, the latter which corresponds to the set of $m+1$ lengthening m-Dyck paths in Figure 1, center. Subtracting 1 from every entry in X yields the flows on horizontal edges in $G(6,3)$ in Figure 1, right.

Corollary 2.9. *We have that $\mathrm{SYT}(n^{m+1}) = \sum_{\mathbf{j}} K_{G(n,m)}(\mathbf{j}-\mathbf{1})$.*

Corollary 2.9 could be recovered as a particular case by combining results [9, Theorem 6.6] and [8, Theorem 3.11]; however, the bijective proof of the statement is new.

Remark 2.10. *The numbers $K_{G(n,m)}(\mathbf{j}-\mathbf{1})$ are also* mixed-volumes *and therefore are log-concave in* root directions *by the* Alexandrov–Fenchel inequalities *e.g. [5, Proposition 11].*

Remark 2.11. *The role that rectangular SYT play in this volume formula and the lattice point formulas below is analogous to the role that shifted SYT of staircase shape play in volume and lattice point formulas for* Gelfand–Tsetlin polytopes *[10, 7].*

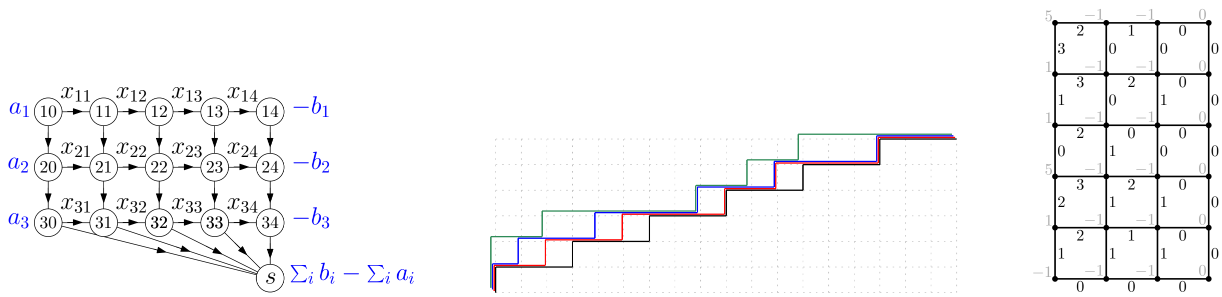

Figure 1: The grid graph $G(3,4)$ that realizes $\mathrm{PS}_3^4(\mathbf{a},\mathbf{b})$ as the flow polytope $\mathcal{F}_{G(3,4)}(\mathbf{a},\mathbf{b})$. Set of generalized Dyck paths from Example 2.8 and the corresponding flow on $G(6,3)$.

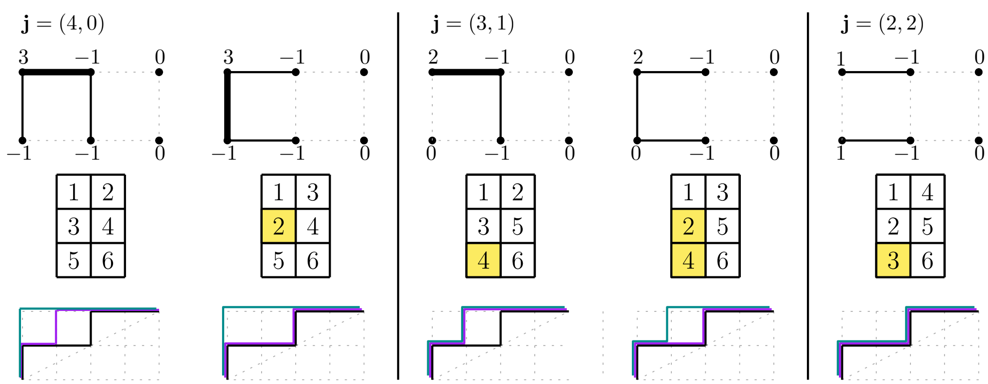

Figure 2: Objects counted by $K_{G(2,2)}(\mathbf{j}-\mathbf{1})$ for different $\mathbf{j}$: certain integer flows on $G(2,2)$ (thick line segments carry two units of flow and with the sink and the edges adjacent to it omitted as they carry no flow), the SYT of shape $(2,2,2)$ with descents highlighted in yellow, and tuples of generalized Dyck paths.

Example 2.12. *For $n = m = 2$, we have the five SYT in Figure 2 and so for* $\mathbf{j}$=$(4,0)$,$(3,1)$,$(2,2)$, $K_{G(2,2)}(\mathbf{j}-\mathbf{1}) = 2,2,1$, *respectively, which match the coefficients in* (2.3).

From the rectangular tableaux interpretation above and a result in the next section (Lemma 2.19) we get the following result.

Corollary 2.13. *The number of different (first) last rows of SYT of shape n^{m+1} is equal to the Fuss–Catalan number $F_{n,m} = \frac{1}{nm+1}\binom{(m+1)n}{n}$.*

We give another interpretation of $K_{G(n,m)}(\mathbf{j}-\mathbf{1})$ in terms of tableaux which is an analogue of [10, Corollary 15.4] for shifted SYT of staircase shape and the associahedron.

Definition 2.14. *Given a composition $\mathbf{j} = (j_1,\ldots,j_n)$ of mn that dominates $m\mathbf{1}$ and an index $1 \leq i \leq n$, let $\Lambda^i_{\mathbf{j}}$ be the set of partitions $\lambda^i = (\lambda^i_1, \lambda^i_2, \ldots)$ of $\sum_{k=1}^i j_i$ such that $\lambda^i_k = m$ for all $k \leq i$, and $\lambda^i_k \leq m$ for all other parts.*

Lemma 2.15. *We have that* $K_{G(n,m)}(\mathbf{j}-\mathbf{1}) = \sum_{(\lambda^1,\ldots,\lambda^n)\in\Lambda^1\times\cdots\times\Lambda^n} \prod_{k=1}^n \mathrm{SYT}(\lambda^k/\lambda^{k-1})$ *where* λ^0 *is empty.*

Proof sketch. By Proposition 2.7 and taking the transpose, $K_{G(n,m)}(\mathbf{j}-\mathbf{1})$ counts SYT of shape $(m+1)^n$ where the $(m+1)th$ column has values $(j_1+1, j_1+j_2+2, \ldots, j_1+\cdots+j_{n-1}+n-1, n(m+1))$. By removing this fixed column and reindexing the resulting tableaux to have entries in $[nm]$, this is also the number of SYT of shape $n\times m$ where the entries in the first k rows are in $\{1,\ldots,\sum_{i=1}^k j_i\}$. Each such tableau can be decomposed into skew tableaux having entries in $[\sum_{k=1}^i j_i]\setminus[\sum_{k=1}^{i-1} j_i]$ giving the desired formula. $\square$

Example 2.16. *Let* $n=m=3$, *and* $\mathbf{j}=(5,3,1)$. *Then* $\Lambda^1_{\mathbf{j}} = \{(3,2),(3,1,1)\}$, $\Lambda^2_{\mathbf{j}} = \{(3,3,2)\}$ *and* $\Lambda^3_{\mathbf{j}} = \{(3,3,3)\}$. *Therefore, we have that* $K_{G(n,m)}(\mathbf{j}-\mathbf{1})$ *is*

$$\mathrm{SYT}\left(\ydiagram{3,2}\right)\cdot\mathrm{SYT}\left(\ydiagram{2+1,2}\right)\cdot\mathrm{SYT}(\square)+\mathrm{SYT}\left(\ydiagram{3,1,1}\right)\cdot\mathrm{SYT}\left(\ydiagram{1+2,1+1}\right)\cdot\mathrm{SYT}\left(\square\right) = 5\cdot3\cdot1+6\cdot2\cdot1 = 27.$$

We finish the section with a determinant formula for the generating polynomial of the numbers $K_{G(n,m)}(\mathbf{j}-\mathbf{1})$. This is an analogue of Postnikov's result for the number of shifted staircase SYT with prescribed diagonal [10, Theorem 15.1].

Corollary 2.17. *For positive integers n and m and nonnegative integers* $a_i \geq 0$ *we have that*

$$\sum_{\mathbf{j}} K_{G(n,m)}(\mathbf{j}-\mathbf{1})\frac{a_1^{j_1}}{j_1!}\cdots\frac{a_n^{j_n}}{j_n!} = \det\left[\frac{(a_1+\cdots+a_i)^{j-i+m}}{(j-i+m)!}\right]_{i,j=1}^n.$$

Proof. This follows from Theorem 2.2 and the determinant formula for the volume of $\mathrm{PS}_n^m(\mathbf{a})$ derived from (1.2). $\square$

Example 2.18. *Continuing Example 2.3:* $2\frac{a_1^4}{4!}+2\frac{a_1^3a_2}{3!1!}+1\frac{a_1^2a_2^2}{2!2!} = \det\begin{bmatrix}\frac{1}{2}a_1^2 & \frac{1}{3!}a_1^3\\ a_1+a_2 & \frac{1}{2}(a_1+a_2)^2\end{bmatrix}.$

2.3 About combinatorial proofs of the volume formula

We now use the different interpretations of $K_{G(n,m)}(\mathbf{j}-\mathbf{1})$ to give a combinatorial proof of equations (1.1) and (2.1) for $\mathbf{a}=\mathbf{1}$ and $n=2$, i.e., we will show that

$$\mathrm{SYT}(n^m)\cdot\prod_{i=1}^m i!(n+i)^{n-m-1+i} = \sum_{\mathbf{j}}\binom{mn}{j_1,\ldots,j_n}K_{G(n,m)}(\mathbf{j}-\mathbf{1}), \tag{2.2}$$

where the sum is over compositions $\mathbf{j}$ of mn that dominate $m\mathbf{1}$.

Lemma 2.19. *For* $\mathbf{a}=(1,\ldots,1)$ *the number of terms of the Lidskii formula for* $\mathcal{F}_{G(n,m)}$ *equals the Fuss–Catalan number* $F_{n,m} = \frac{1}{nm+1}\binom{(m+1)n}{n}$.

Proof. Such $\mathbf{j}$'s can be thought of as the lengths of the horizontal segments in m-Dyck paths (starting from the top right corner). It is known that such m-Dyck paths are counted by Fuss–Catalan numbers [14]. $\square$

Lemma 2.20. *We have that* $\prod_{k=1}^{m} k!(2+k)^{1-m+k} = \sum_{i=0}^{m}(i+1)\binom{m}{i}$.

Proof sketch. One can show that $\prod_{k=1}^{m} k!(2+k)^{1-m+k} = (m+2)2^{m-1}$ by induction on m. Note that $(m+2)2^{m-1}$ is equal to the number of parts among all compositions of $m+1$ (see, e.g., A001792 in the OEIS). We can partition compositions by their number of parts, say i. The number of parts in all compositions with i parts is $i\binom{m}{i-1}$. So $\prod_{k=1}^{m} k!(2+k)^{1-m+k} = \sum_{i=1}^{m+1} i\binom{m}{i-1}$, and thus the result holds. $\square$

Lemma 2.21. *We have that* $K_{G(2,m)}(2m-i-1,i-1) = \frac{i+1}{m+1}\binom{2m-i}{m}$ *for* $0 \le i \le m$.

Proof sketch. The (multi-parameter) Fuss–Catalan numbers $A_{m-i}(2,i+1) := \frac{i+1}{m+1}\binom{2m-i}{m}$ are known to satisfy the recurrence $A_{m-i}(2,i+1) = A_{m-i}(2,i) + A_{m-i-1}(2,i+1) + \ldots + A_0(2,m)$, i.e., the recurrence given in Lemma 2.4. $\square$

With Lemma 2.21, it is easy to directly check that equation (2.2) holds when $n = 2$. We now give a combinatorial interpretation for it.

Theorem 2.22. *The volume of* $PS_2^m(1,1)$ *is* $\mathrm{SYT}(2^m)\prod_{k=1}^{m} k!(2+k)^{1-m+k}$ *where* $\mathrm{SYT}(2^m) = C_m$*, the mth Catalan number.*

Proof. Let us understand the left-hand side of equation (2.2) first. By Proposition 2.7, we know that $\mathrm{SYT}(2^m) = \#\mathcal{D}_{m-1,2}$. For each path in each set of paths, elongate each horizontal step by one (including horizontal steps of length 0). Let $\{P'_0, P'_1, \ldots, P'_{m-1}\}$ be one of the resulting sets of paths. Then we have that P'_i is an m-Dyck path that is $(i+1)$-stepping. Note that adding an m-Dyck path P^* to this set such that P^* is above or equal to P'_0 everywhere would yield a set of paths that is in $\mathcal{D}_{m,2}$. From Lemma 2.20, we know that $\prod_{k=1}^{m} k!(2+k)^{1-m+k} = \sum_{i=0}^{m}(i+1)\binom{m}{i}$. We can think of the right-hand side as counting the number of elements in compositions of $m+1$, and splitting this according to the number of parts. Now associate the number of parts to the path P'' with horizontal steps $2m-i$ and i (from the top right corner), and add $(i+1)\binom{m}{i}$ copies of P'' to every set $\{P'_0, P'_1, \ldots, P'_{m-1}\}$ above such that this new path is above or equal to P'_0. For the sets where it is not possible to add P'', distribute those copies of P'' evenly among the $K_{G(2,m)}(2m-i-1,i-1)$ sets where it is possible to add it. (Note this is by Proposition 2.6.) We now count how many copies of each set of paths in $\mathcal{D}_{m,2}$ we have obtained by doing this process. For a set of paths $\{Q_0, Q_1, \ldots, Q_m\} \in \mathcal{D}_{m,2}$ where Q_0 has horizontal steps of length $2m-i$ and i for some $0 \le i \le m$, this set will be counted $(i+1)\cdot\binom{m}{i}\cdot C_m/K_{G(2,m)}(2m-i-1,i-1)$. By Lemma 2.21, this yields $\binom{2m-i}{i}$ just as in the right-hand side. $\square$

Example 2.23. *Let $n = 2$, $m = 3$. The left-hand side of equation (2.2) is $C_3 \prod_{k=1}^{3} k!(2+k)^{1-3+k}$ where C_3 corresponds to the sets of paths in $\mathcal{D}_{2,2}$ shown in Figure 2. We now elongate each horizontal step by one to obtain the following sets of paths in the top of Figure 3.*

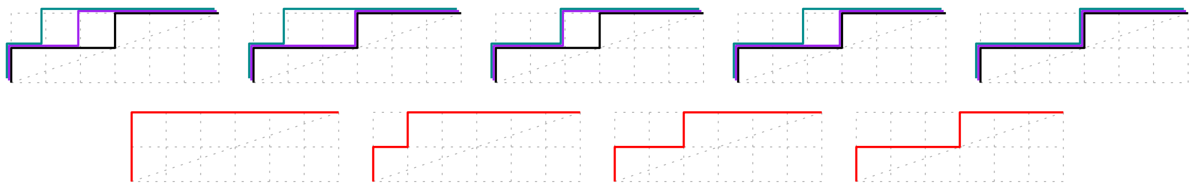

Figure 3: Top: Sets in $\mathcal{D}_{2,2}$ where each horizontal step is elongated by 1. Bottom: Dyck paths with horizontal steps $7-i$ and $i-1$ for $1 \le i \le 4$ associated to compositions.

Moreover, $\prod_{k=1}^{3} k!(2+k)^{1-3+k} = 20$ counts the number of elements in the compositions of 4, namely one composition with one element (4), three compositions with two elements $(3,1)$, $(1,3)$, $(2,2)$, three with three elements $(2,1,1), (1,2,1), (1,1,2)$, and one with four $(1,1,1,1)$. To each of these composition, we associate the Dyck path with horizontal steps of lengths $7-i$ and $i-1$ where i is the number of elements in that composition as in the bottom of Figure 3.

We add those paths (with multipliers) to the sets of paths at the top of Figure 3 if doing so yields a set of paths in $\mathcal{D}_{3,2}$. For example, the first path in the bottom of Figure 3 can be added to each set in the top, and since there is one composition with one element, we count the resulting sets of paths one times each. The third path in the bottom can only be added to the third, fourth and fifth sets in the top. Since there are three compositions with three elements, we want the total contribution to be $5 \cdot 3 \cdot 3$, but as we only add this path to three sets, we count the resulting sets 15 times each.

On the right-hand side, possible $\mathbf{j}$'s need to be such that $j_1 \ge 3$ and $j_1 + j_2 = 6$, namely $(3,3)$, $(4,2)$, $(5,1)$ and $(6,0)$. Recall that $K_G(\mathbf{j}-\mathbf{1})$ counts the number of sets in $\{P_0, P_1, P_2, P_3\} \in \mathcal{D}_{3,2}$ where P_0 has horizontal steps of lengths j_1 and j_2. Such sets are counted with multiplicity $\binom{6}{j_1, j_2}$, namely 20, 15, 6 and 1 for our different $\mathbf{j}$'s.

With the second interpretation of $K_{G(n,m)}(\mathbf{j}-\mathbf{1})$ in Lemma 2.15, it is easy to directly prove the Lidskii formula when $n = 2$.

Theorem 2.24.
$$C_m \prod_{k=1}^{m} k!(2+k)^{1-m+k} = \sum_{\mathbf{j}} \binom{2m}{j_1, j_2} K_G(\mathbf{j}-\mathbf{1}).$$

Proof. Consider $\mathbf{j} = (2m - j_2, j_2)$ for some $0 \le j_2 \le m$. Then $\Lambda^1_{\mathbf{j}} = \{(m, m-j_2)\}$ and $\lambda^2_{\mathbf{j}} = \{(m,m)\}$. Thus

$$K_{G(n,m)}(\mathbf{j}-\mathbf{1}) = \mathrm{SYT}(m, m-j_2) \times \mathrm{SYT}((m,m)/(m,m-j_2)) = \frac{j_2+1}{m+1}\binom{2m-j_2}{m}$$

by using the *hook length formula* [12, Corollary 7.21.6]. Therefore, we have that

$$\frac{1}{C_m}\sum_{j_2=0}^{m}\binom{2m}{j_1,j_2}K_G(2m-j_2-1,j_2-1)=\frac{1}{C_m}\sum_{\mathbf{j}}\binom{2m}{j_1,j_2}\frac{j_2+1}{m+1}\binom{2m-j_2}{m}$$
$$=\sum_{j_2=0}^{m}(j_2+1)\binom{m}{j_2}=\prod_{k=1}^{m}k!(2+k)^{1-m+k}$$

where the last line follows by Lemma 2.20. □

3 Lattice point formulas in terms of tableaux

We give lattice point formulas for $\mathrm{PS}_n^m(\mathbf{a})$ and $\mathrm{PS}_n^m(\mathbf{a},\mathbf{b})$. Although there are Lidskii formulas for lattice points of flow polytopes, the internal vertices with zero flow of $G(n,m)$, can make these formulas be alternating and also they do not cover net flows like $(\mathbf{a},\mathbf{0},-\mathbf{b},\ldots)$. Instead, we use Stanley's theory of *marked order polytopes* (see [13, Section 3],[7]), since $\mathrm{PS}_n^m(\mathbf{a})$ and $\mathrm{PS}_n^m(\mathbf{a},\mathbf{b})$ are marked order polytopes of the poset $\mathbf{n}\times(\mathbf{m}+\mathbf{1})$ with last n-chain marked, and $\mathbf{n}\times(\mathbf{m}+\mathbf{2})$ with first and last n-chains marked, respectively.

We state the formulas in terms of rectangular SYT. Given a SYT T of shape λ, a *(natural) descent* of T is a value i of T such that $i+1$ occurs strictly above i in T [4]. Note that there is a more common notion of descents of a tableau [12, Section 7.19].

3.1 Straight shape case

Given a standard Young tableau T of shape n^{m+1} with last row $c_1<c_2<\cdots<c_n=(m+1)n$. For $i=1,\ldots,n$, let $d_i(T)$ be the number of descents of T between and including c_{i-1} up to c_i (where $c_0=0$). The following result is an application of a general result of marked order polytopes [13, Theorem 5], and it implies the volume formula (2.1).

Theorem 3.1.

$$\#(PS_n^m(\mathbf{a})\cap\mathbb{Z}^{nm})=\sum_{T\in SYT(n^{m+1})}\prod_{i=1}^{n}\left(\!\!\binom{a_i-d_i(T)+1}{c_i-c_{i-1}-1}\!\!\right), \tag{3.1}$$

where $c_1<c_2<\cdots<c_n=(m+1)n$ is the last row of T and $d_i(T)$ are as defined above.

Example 3.2. *For $n=2$ and $m=2$, consider the tableaux of shape $(2,2,2)$ in Figure 2 with descents sets $\varnothing,\{2\},\{4\},\{2,4\},\{3\}$, respectively. From the last row of each tableau, the associated compositions are $(4,0),(4,0),(3,1),(3,1),(2,2)$, and descent statistics (d_1,d_2) are $(0,0),(1,0),(0,1),(1,1),(0,1)$. This gives rise to the formula for $\#(PS_2^2(a_1,a_2)\cap\mathbb{Z}^4)$ being*

$$\left(\!\!\binom{a_1+1}{4}\!\!\right)+\left(\!\!\binom{a_1}{4}\!\!\right)+\left(\!\!\binom{a_1+1}{3}\!\!\right)\left(\!\!\binom{a_2}{1}\!\!\right)+\left(\!\!\binom{a_1}{3}\!\!\right)\left(\!\!\binom{a_2}{1}\!\!\right)+\left(\!\!\binom{a_1+1}{2}\!\!\right)\left(\!\!\binom{a_2}{2}\!\!\right).$$

Example 3.3. *Consider the SYT T in (2.8) where $m = 3$ and $n = 6$ and whose descents are highlighted in yellow. From the last row of T, the associated composition is $\mathbf{j} = (6,2,2,6,2,0)$ and T has descents $\{3, \mathbf{7}, 8, \mathbf{10}, \mathbf{13}, 15, 17, \mathbf{20}\}$ with $(d_1, d_2, d_3, d_4, d_5, d_6) = (1,2,1,3,1,0)$. The contribution of this tableau in (3.1) is $\left(\!\binom{a_1-1+1}{6}\!\right)\left(\!\binom{a_2-2+1}{2}\!\right)\left(\!\binom{a_3-1+1}{2}\!\right)\left(\!\binom{a_4-3+1}{6}\!\right)\left(\!\binom{a_5-1+1}{2}\!\right)$.*

The Ehrhart polynomial $\mathrm{PS}_n^1(t\mathbf{a})$ is known to be positive in t [13, Corollary 10] which can be seen from the case $m = 1$ of (3.1) since $d_i \le 1$. For general m, we don't have this restriction so our formula does not imply positivity for $\mathrm{PS}_n^m(t\mathbf{a})$. However, calculations up to $m, n \le 8$ suggest positivity for the latter.

Conjecture 3.4. *As a polynomial in a_i, we have that $\#(\mathrm{PS}_n^m(\mathbf{a}) \cap Z^{mn})$ is in $\mathbb{N}[\mathbf{a}]$. In particular, $\mathrm{PS}_n^m(\mathbf{a})$ is Ehrhart positive.*

3.2 Skew shape case

Given $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{Z}^n_{\ge 0}$, let $\alpha_i = \sum_{j=1}^{i} a_j$ and $\beta_i = \sum_{j=1}^{i} b_j$. Let $\mathrm{SYT}_{\mathbf{a},\mathbf{b}}(n^{m+2})$ be the set of SYT of shape n^{m+2} such that the first and last row have the same relative order as the numbers $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$. We denote this relative order by $\pi(T) = (\pi_1, \dots, \pi_{2n})$. Note that in the generic case where $a_i > 0$ and $b_i > 0$, then there are Catalan ($\mathrm{SYT}(n,n) = C_n$) many different sets $\mathrm{SYT}_{\mathbf{a},\mathbf{b}}(n^{m+2})$.

Consider a SYT T of shape n^{m+2} with entries $1 = c_1 < c_2 < \cdots < c_{2n} = n(m+2)$ in the first and last row of T. For $i = 1, \dots, 2n$, let $d_i(T)$ be the number of descents of T between and including c_{i-1} up to c_i (where $c_0 = 0$). The following result was sketched for $m = 1$ in the arxiv version of [13].

Theorem 3.5. *$\#(PS_n^m(\mathbf{a},\mathbf{b}) \cap \mathbb{Z}^{nm}) = \sum_{T \in \mathrm{SYT}_{\mathbf{a},\mathbf{b}}(n^{m+2})} \prod_{i=1}^{2n} \left(\!\binom{\pi_i - \pi_{i-1} - d_i(T)+1}{c_i - c_{i-1} - 1}\!\right)$, where $1 = c_1 < c_2 < \cdots < c_{2n} = n(m+2)$ are the entries in the first and last row of T, $d_i(T)$ are as defined above, (π_i) is the relative order of α_is and β_is from T where $c_0 = \pi_0 = 0$.*

From the lattice point formula above, we can extract a volume formula for $\mathrm{PS}_n^m(\mathbf{a},\mathbf{b})$.

Corollary 3.6. $\operatorname{vol} PS_n^m(\mathbf{a},\mathbf{b}) = \sum_T \binom{mn}{c_1 - c_0 - 1, \dots, c_{2n} - c_{2n-1} - 1} \prod_{i=1}^{2n} (\pi_i - \pi_{i-1})^{c_i - c_{i-1} - 1}$, *where the sum is over T in $\mathrm{SYT}_{\mathbf{a},\mathbf{b}}(n^{m+2})$; c_i, $d_i(T)$, and (π_i) are the same as in Theorem 3.5.*

Example 3.7 ([13, pp. 25-26])**.** *For $n = 2$ and $m = 1$, we have five SYT of shape 2^3 in Figure 2. From the order of the entries on the first and last row, the first four tableaux correspond to the ordering $\beta_1 < \beta_2 < \alpha_1 < \alpha_2$ and the last one to the ordering $\beta_1 < \alpha_1 < \beta_2 < \alpha_2$. This gives*

$$\#(PS_2^1(\mathbf{a},\mathbf{b}) \cap \mathbb{Z}^2) = \begin{cases} A & \text{if } \alpha_1 > \beta_2 \\ B & \text{if } \alpha_1 \le \beta_2 \end{cases}, \quad \operatorname{vol} PS_2^1(\mathbf{a},\mathbf{b}) = \begin{cases} A' & \text{if } \alpha_1 > \beta_2 \\ B' & \text{if } \alpha_1 \le \beta_2 \end{cases},$$

where $A = \left(\!\binom{\alpha_1-\beta_2+1}{2}\!\right) + \left(\!\binom{\alpha_1-\beta_2+1}{1}\!\right)\left(\!\binom{\alpha_2-\alpha_1}{1}\!\right) + \left(\!\binom{\beta_2-\beta_1}{1}\!\right)\left(\!\binom{\alpha_1-\beta_2+1}{1}\!\right) + \left(\!\binom{\beta_2-\beta_1}{1}\!\right)\left(\!\binom{\alpha_2-\alpha_1}{1}\!\right)$, $B = \left(\!\binom{\alpha_1-\beta_1+1}{1}\!\right)\left(\!\binom{\alpha_2-\beta_2+1}{1}\!\right)$, $A' = \binom{2}{2}(\alpha_1-\beta_2)^2 + \binom{2}{1,1}(\alpha_1-\beta_2)(\alpha_2-\alpha_1) + \binom{2}{1,1}(\beta_2-\beta_1)(\alpha_1-\beta_2) + \binom{2}{1,1}(\beta_2-\beta_1)(\alpha_2-\alpha_1)$, and $B' = \binom{2}{1,1}(\alpha_1-\beta_1)(\alpha_2-\beta_2)$.

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