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Simple modules for affine type A KLR algebras via skew Specht modules

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Abstract. Cuspidal systems parameterise KLR algebra representations via root partitions π, where simple modules $L(\pi)$ arise as heads of proper standard modules. We define a combinatorial dilation map and construct skew Young diagrams $\zeta(\pi)$ associated to each root partition π in an arbitrary convex order in affine type A. These skew diagrams have the property that the corresponding skew Specht module $S^{\zeta(\pi)}$ has simple head $L(\pi)$ and a filtration by proper standard modules. Our results represent a far more explicit and combinatorial construction of these modules than was previously known.

Keywords: KLR algebras, simple modules, skew Specht modules, root partitions

1 Introduction

Fix $e \in \mathbb{Z}_{>1}$, and let $\Phi_+ = \Phi_+^{\mathrm{re}} \sqcup \Phi_+^{\mathrm{im}}$ be the positive root system of type $\mathtt{A}_{e-1}^{(1)}$, where $I = \{\alpha_0, \ldots, \alpha_{e-1}\}$ is the set of simple roots, Φ_+^{re} is the set of real roots, and $\Phi_+^{\mathrm{im}} = \{d\delta \mid d \in \mathbb{Z}_{>0}\}$ is the set of imaginary roots, with $\delta = \alpha_0 + \cdots + \alpha_{e-1}$ being the null root. We additionally write $\Psi := \Phi_+^{\mathrm{re}} \sqcup \{\delta\}$ for the set of indivisible roots. This root system data is fundamental in the representation theory of the Kac–Moody Lie algebra $\widehat{\mathfrak{sl}}_e(\mathbb{C})$ [4].

For any field $\mathbb{F}$ and $\omega \in \mathbb{Z}_{\geqslant 0}I$, there is an associated KLR algebra R_ω over $\mathbb{F}$. This family of algebras categorifies the positive part of the quantum group $U_q(\widehat{\mathfrak{sl}}_e(\mathbb{C}))$, see [5, 12]. The representation theory of KLR algebras is studied via *cuspidal systems*, as in [8, 10, 7, 13], which are associated with PBW bases for the quantum group [3]. We will now briefly explain the setup.

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Fix a convex preorder $\succcurlyeq$ on Φ_+. For $\omega \in \mathbb{Z}_{\geqslant 0}I$, a *Kostant partition* of ω is a tuple of non-negative integers $\boldsymbol{K} = (K_\beta)_{\beta\in\Psi}$ such that $\sum_{\beta\in\Psi} K_\beta \beta = \omega$. If $\beta_1 \succ \cdots \succ \beta_t$ are the members of Ψ such that $K_{\beta_1} \neq 0$, then we write $\boldsymbol{K}$ in the form $\boldsymbol{K} = (\beta_1^{K_{\beta_1}} \mid \cdots \mid \beta_t^{K_{\beta_t}})$. The convex preorder $\succcurlyeq$ induces a bilexicographic partial order $\geqslant_{\mathrm{b}}$ on $\Xi(\omega)$, the set of all Kostant partitions of ω. We write $\Pi(\omega)$ for the set of all *root partitions* of ω; these are pairs $\pi = (\boldsymbol{K}, \boldsymbol{\nu})$, where

$$\boldsymbol{K} = (\beta_1^{K_{\beta_1}} \mid \cdots \mid \beta_u^{K_{\beta_u}} \mid \delta^{K_\delta} \mid \beta_{u+1}^{K_{\beta_{u+1}}} \mid \cdots \mid \beta_t^{K_{\beta_t}}) \in \Xi(\omega), \text{and}$$
$$\boldsymbol{\nu} = (\nu^{(1)} \mid \cdots \mid \nu^{(e-1)}) \text{ is an } (e-1)\text{-multipartition of } K_\delta.$$

To each $\beta \in \Phi_+^{\mathrm{re}}$, we associate a simple *cuspidal* R_β-module $L(\beta)$, and to each $(e-1)$-multipartition $\boldsymbol{\nu}$ of $d \in \mathbb{Z}_{>0}$, we associate a simple *semicuspidal* $R_{d\delta}$-module $L(\boldsymbol{\nu})$. Then, to each $\pi \in \Pi(\omega)$, we associate a proper standard module $\bar{\Delta}(\pi)$ which is an ordered induction product of these simple semicuspidal modules. The module $\bar{\Delta}(\pi)$ has a self-dual simple head $L(\pi)$, and $\{L(\pi) \mid \pi \in \Pi(\omega)\}$ is a complete and irredundant set of simple R_ω-modules up to isomorphism and grading shift. To be precise, if π is as above, then $\bar{\Delta}(\pi)$ is (up to grading shift) the induction product of the following semicuspidal simple modules:

$$\bar{\Delta}(\pi) = L(\beta_1)^{\circ K_{\beta_1}} \circ \cdots \circ L(\beta_u)^{\circ K_{\beta_u}} \circ L(\boldsymbol{\nu}) \circ L(\beta_{u+1})^{\circ K_{\beta_{u+1}}} \circ \cdots \circ L(\beta_t)^{\circ K_{\beta_t}}.$$

In the literature, the semicuspidal modules $L(\boldsymbol{\nu})$ are not presented directly – rather, their existence is established via categorification, or they are constructed through Morita equivalences with symmetric groups and Schur algebras. In this extended abstract our primary goal is to use *skew Specht modules* to render a more direct and combinatorial-flavoured description of semicuspidal and simple R_ω-modules.

We study combinatorial and algebraic interactions between cuspidal systems for Khovanov–Lauda–Rouquier (KLR) algebras and skew Specht modules, and apply these connections to construct simple, semicuspidal, and proper standard KLR modules as (quotients of) explicit skew Specht modules.

Finally, we note that skew Specht modules were already used in [1], where the simple modules indexed by *real roots* were shown to be isomorphic to skew Specht modules. Our results here may thus be seen as an extension of this framework to the simple semicuspidal modules $L(\boldsymbol{\nu})$, and subsequently to the simple modules $L(\pi)$ indexed by arbitrary root partitions.

2 Multipartition combinatorics

Let $\mathcal{N} = \mathbb{Z} \times \mathbb{Z}$. We refer to elements of $\mathcal{N}$ as *nodes*, and by convention, we visually represent nodes as boxes in a $\mathbb{Z} \times \mathbb{Z}$ array, so that the node (x, y) is a box in the xth row and yth column of the array. In this orientation, a positive increase in the x component corresponds to a southward move, and a positive increase in the y component corresponds to an eastward move. We write

$$(x,y) \searrow (x',y') \text{ provided } x' \geqslant x \text{ and } y' \geqslant y.$$

We define the single-unit north, east, south and west translations of nodes, respectively, by setting

$$\mathtt{N}(x,y) := (x-1,y), \quad \mathtt{E}(x,y) := (x,y+1), \quad \mathtt{S}(x,y) := (x+1,y), \quad \mathtt{W}(x,y) := (x,y-1).$$

We define a *residue* function $\mathrm{res} : \mathcal{N} \to \mathbb{Z}/e\mathbb{Z}$ on nodes by setting:

$$\mathrm{res}((x,y)) := \overline{y-x}.$$

A *skew diagram* is a finite subset $\tau \subseteq \mathcal{N}$ such that $(x,y) \searrow (x',y') \searrow (x'',y'')$ and $(x,y),(x'',y'') \in \tau$ implies $(x',y') \in \tau$.

A *partition* λ *of* charge $\kappa \in \mathbb{Z}$ is a skew diagram which is either empty, or contains the *corner* node $(1,\kappa+1)$, such that $(1,\kappa+1) \searrow (x,y)$ for all $(x,y) \in \lambda$. Pictorially, λ is a northwest-aligned array of boxes (often called a *Young diagram*) with the node $(1,\kappa+1)$ at the northwest corner. Partitions of a given charge are in bijection with integer partitions; writing

$$\lambda_a := |\{(a,x) \mid x \in \mathbb{Z}_{\geqslant 0}\} \cap \lambda|$$

for all $a \in \mathbb{Z}_{>0}$, we have that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$, and $\sum \lambda_a = |\lambda|$. Thus we will often abuse notation and write $\lambda = (\lambda_1, \lambda_2, \ldots)$. We collect repeated parts in partitions, writing $(4, 2^3, 1^2)$ as shorthand for the partition $(4,2,2,2,1,1)$, for example. We remark that the empty set $\varnothing$ is a partition of charge κ for any $\kappa \in \mathbb{Z}$. If we refer to a partition without mentioning a specific charge, we will take the charge to be 0 by default.

For partitions $\mu \subseteq \lambda$ of charge κ, we set $\lambda/\mu \subseteq \mathcal{N}$ to be the set difference $\lambda \backslash \mu$, and call this a *skew partition of charge* κ. We remark that λ/μ is a skew diagram, and consider the choice of λ and μ to be part of the data of λ/μ. Every skew diagram can be realised as (some translation of) a skew partition.

More generally, for a *level* $\ell \in \mathbb{Z}_{>0}$, we write

$$\mathcal{N}_\ell := \bigsqcup_{t \in [1,\ell]} \mathcal{N} = \mathcal{N}^{(1)} \sqcup \cdots \sqcup \mathcal{N}^{(\ell)},$$

labelling subsets and nodes in the constituent copies of $\mathcal{N}$ in $\mathcal{N}_\ell$ with parenthesised superscripts, so that $(x,y)^{(r)} \in \mathcal{N}^{(r)}$.

We extend the definitions above to higher levels as follows. We will say that $\boldsymbol{\tau} = (\tau^{(1)} \mid \cdots \mid \tau^{(\ell)})$ is a *skew ℓ-diagram* provided that each component $\tau^{(i)}$ is a skew diagram. For $\omega \in \mathbb{Z}_{\geqslant 0} I$ we write

$$\Lambda^\ell(\omega) := \{\text{skew diagram } \boldsymbol{\tau} \subseteq \mathcal{N}_\ell \mid \mathrm{cont}(\boldsymbol{\tau}) = \omega\}.$$

For a *multicharge* $\boldsymbol{\kappa} = (\kappa_1 \mid \cdots \mid \kappa_\ell) \in \mathbb{Z}^\ell$, we say that $\boldsymbol{\lambda} = (\lambda^{(1)} \mid \cdots \mid \lambda^{(\ell)})$ is an *ℓ-multipartition of multicharge* $\boldsymbol{\kappa}$ provided that each component $\lambda^{(i)}$ is a partition of charge κ_i. We extend this to define *skew ℓ-multipartitions of multicharge* $\boldsymbol{\kappa}$ in the obvious way. We will visually depict skew diagrams, multipartitions, and skew multipartitions in rows, with component labels increasing from left to right.

For a finite set $S \subseteq \mathcal{N}_\ell$, the *content* of S is defined by:

$$\mathrm{cont}(S) := \sum_{u \in S} \alpha_{\mathrm{res}(u)} \in \mathbb{Z}_{\geqslant 0} I.$$

Let $\boldsymbol{\kappa} = (\kappa_1 \mid \cdots \mid \kappa_\ell) \in \mathbb{Z}^\ell$ be a multicharge of level ℓ. For a fixed multipartition $\boldsymbol{\rho} \in \mathcal{N}_\ell$, we write

$$\Lambda^{\boldsymbol{\kappa}}_{+/+}(\omega) := \{\text{skew multipartition } \boldsymbol{\tau} \subset \mathcal{N}_\ell \text{ of multicharge } \boldsymbol{\kappa} \mid \mathrm{cont}(\boldsymbol{\tau}) = \omega\} \supseteq \Lambda^{\boldsymbol{\kappa}}_{+}(\omega).$$

We may define tableaux, and standard tableaux, of shape a given skew diagram $\boldsymbol{\tau}$ or skew multipartition $\boldsymbol{\lambda}/\boldsymbol{\mu}$, in the usual way, filling the corresponding Young diagram with the numbers $1, \ldots, n$. We often think of a tableau as a bijection $\mathtt{t} : \{1, \ldots, n\} \to |\boldsymbol{\tau}|$, where the Young diagram $|\boldsymbol{\tau}|$ is a set of nodes in $\mathcal{N}_\ell$. We let $\mathrm{Std}(\boldsymbol{\tau})$ denote the set of standard tableaux of shape $\boldsymbol{\tau}$. The *leading $\boldsymbol{\tau}$-tableau* $\mathtt{t}^{\boldsymbol{\tau}}$ is the tableau filled with $\{1, \ldots, n\}$ in order from left-to-right along the first row, and then the second row, and so on, of the first component, and then moving in like fashion through the second component, and so on.

For $\boldsymbol{\tau} \in \Lambda^\ell(\omega)$, we define the following *residue sequence*, (or *content sequence*) corresponding to the leading $\boldsymbol{\tau}$-tableau $\mathtt{t}^{\boldsymbol{\tau}}$:

$$\boldsymbol{i}^{\boldsymbol{\tau}} := (\mathrm{cont}(\mathtt{t}^{\boldsymbol{\tau}}(1)), \ldots, \mathrm{cont}(\mathtt{t}^{\boldsymbol{\tau}}(|\boldsymbol{\tau}|))) \in I^\omega.$$

3 KLR algebras and their skew Specht modules

3.1 KLR algebras

Fix a field $\mathbb{F}$ of characteristic $p \geqslant 0$. Let $\omega \in \mathbb{Z}_{\geqslant 0} I$, and set $m = \mathrm{ht}(\omega)$. We set

$$I^\omega := \{\boldsymbol{i} \in I^{\mathrm{ht}(\omega)} \mid i_1 + \cdots + i_m = \omega\}.$$

When convenient, we abuse notation and identify I with $\mathbb{Z}/e\mathbb{Z}$.

As in [5, 12], the KLR (or *quiver Hecke*) algebra (of type $\mathtt{A}^{(1)}_{e-1}$) is the unital $\mathbb{Z}$-graded $\mathbb{F}$-algebra R_ω generated by

$$\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^\omega\} \cup \{y_1, \ldots, y_m\} \cup \{\psi_1, \ldots, \psi_{m-1}\},$$

subject to the following relations.

$$1_{\boldsymbol{i}}1_{\boldsymbol{j}} = \delta_{\boldsymbol{i},\boldsymbol{j}}1_{\boldsymbol{i}}; \qquad \sum_{\boldsymbol{i}\in I^\omega} 1_{\boldsymbol{i}} = 1; \qquad y_r 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}} y_r; \qquad y_r y_s = y_s y_r; \qquad \psi_r 1_{\boldsymbol{i}} = 1_{s_r\boldsymbol{i}}\psi_r;$$

$$\psi_r y_s = y_s \psi_r \ (\text{if } s \neq r, r+1); \qquad \psi_r\psi_s = \psi_s\psi_r \ (\text{if } |r-s| > 1);$$

$$\psi_r y_{r+1} 1_{\boldsymbol{i}} = (y_r\psi_r + \delta_{i_r,i_{r+1}})1_{\boldsymbol{i}}; \qquad y_{r+1}\psi_r 1_{\boldsymbol{i}} = (\psi_r y_r + \delta_{i_r,i_{r+1}})1_{\boldsymbol{i}};$$

$$\psi_r^2 1_{\boldsymbol{i}} = \begin{cases} 0 & \text{if } i_r = i_{r+1};\\ (y_r - y_{r+1})1_{\boldsymbol{i}} & \text{if } i_{r+1} = i_r + 1;\\ (y_{r+1} - y_r)1_{\boldsymbol{i}} & \text{if } i_{r+1} = i_r - 1;\\ -(y_{r+1} - y_r)^2 1_{\boldsymbol{i}} & \text{if } i_{r+1} \neq i_r \text{ and } e = 2;\\ 1_{\boldsymbol{i}} & \text{otherwise;} \end{cases}$$

$$(\psi_{r+1}\psi_r\psi_{r+1} - \psi_r\psi_{r+1}\psi_r)1_{\boldsymbol{i}} = \begin{cases} 1_{\boldsymbol{i}} & \text{if } i_{r+2} = i_r = i_{r+1} - 1;\\ -1_{\boldsymbol{i}} & \text{if } i_{r+2} = i_r = i_{r+1} + 1;\\ (-y_r + 2y_{r+1} - y_{r+2})1_{\boldsymbol{i}} & \text{if } i_{r+2} = i_r \neq i_{r+1};\\ 0 & \text{otherwise.} \end{cases}$$

The $\mathbb{Z}$-grading is given by:

$$\deg_{\mathbb{Z}}(1_{\boldsymbol{i}}) = 0; \qquad \deg_{\mathbb{Z}}(y_r) = 2; \qquad \deg_{\mathbb{Z}}(\psi_r 1_{\boldsymbol{i}}) = \begin{cases} -2 & \text{if } i_r = i_{r+1};\\ 1 & \text{if } i_r = i_{r+1} \pm 1;\\ 2 & \text{if } i_r \neq i_{r+1} \text{ and } e = 2;\\ 0 & \text{otherwise.} \end{cases}$$

For $(L_0, L_1, \dots, L_{e-1}) \in \mathbb{Z}^e_{\geqslant 0}$, we let $\Lambda = L_0\Lambda_0 + L_1\Lambda_1 + \cdots + L_{e-1}\Lambda_{e-1}$, and define the *cyclotomic KLR algebra* R^Λ_ω to be the quotient of R_ω by the two-sided ideal generated by the elements

$$\{y_1^{L_{i_1}} 1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^\omega\}.$$

3.2 Skew Specht modules

All R_ω-modules that we work with are graded, and all module homomorphisms will be (not necessarily degree 0) graded maps of graded modules. As we are not primarily concerned with grading shifts or graded decomposition numbers here, we use the notation $[M : L] \in \mathbb{Z}_{\geqslant 0}$ to indicate the ungraded multiplicity of a simple module L as a composition factor of a module M. We write $M \cong N$ to indicate that two R_ω-modules are isomorphic, and $M \approx N$ to indicate that they are isomorphic up to some grading shift.

Let $\omega \in \mathbb{Z}_{\geqslant 0}I$, and let $\boldsymbol{\tau} \in \Lambda^\ell(\omega)$ be a skew diagram of content ω. We define the *(row) skew Specht module* $\mathbf{S}^{\boldsymbol{\tau}}$ to be the graded R_ω-module generated by the vector $v^{\boldsymbol{\tau}}$ in degree zero, and subject to the following relations.

(i) $1_{\boldsymbol{i}} v^{\boldsymbol{\tau}} = \delta_{\boldsymbol{i},\boldsymbol{i}^{\boldsymbol{\tau}}} v^{\boldsymbol{\tau}}$ for all $\boldsymbol{i} \in I^\omega$;

(ii) $y_r v^{\boldsymbol{\tau}} = 0$ for all $r \in [1, \mathrm{ht}(\omega)]$;

(iii) $\psi_r v^{\boldsymbol{\tau}} = 0$ for all $r \in [1, \mathrm{ht}(\omega) - 1]$ such that $\mathtt{Et}^{\boldsymbol{\tau}}(r) = \mathtt{t}^{\boldsymbol{\tau}}(r+1)$;

(iv) $g^u v^{\boldsymbol{\tau}} = 0$ for all $u \in \boldsymbol{\tau}$ such that $\mathtt{S}u \in \boldsymbol{\tau}$.

The element $g^u \in R_\omega$ above is the *Garnir element*, see [9, Section 5], [11, Section 4]. The description of this element is rather technical, and not needed for our purposes, so we refer the reader to the above papers for the definition.

We note that these skew Specht modules are natural generalisations of Specht modules $S^{\boldsymbol{\lambda}}$ and $S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ indexed by multipartitions [9] and skew multipartitions [11], respectively. In fact, up to grading shift, our skew Specht module $\mathbf{S}^{\boldsymbol{\tau}}$ is isomorphic to the skew Specht module $S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ of [11], for any $\boldsymbol{\lambda}, \boldsymbol{\mu}$ yielding a skew multipartition 'equivalent' to the skew diagram $\boldsymbol{\tau}$. Thus, up to grading shift, we may freely work with either $\mathbf{S}^{\boldsymbol{\tau}}$ – where we do not have multipartitions $\boldsymbol{\lambda}, \boldsymbol{\mu}$ in mind – or the more concrete $S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$.

Proposition 3.1 ([9, Corollary 6.24], [11, Theorem 5.1])**.** *Let $\omega \in \mathbb{Z}_{\geqslant 0}I$, $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \Lambda^{\kappa}_{+/+}(\omega)$. Then $S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ has a homogeneous $\mathbb{F}$-basis*

$$\{v^{\mathtt{t}} := \psi^{\mathtt{t}} v^{\boldsymbol{\lambda}/\boldsymbol{\mu}} = 1_{\boldsymbol{i}^{\mathtt{t}}} \psi^{\mathtt{t}} v^{\boldsymbol{\lambda}/\boldsymbol{\mu}} \mid \mathtt{t} \in \mathrm{Std}(\boldsymbol{\lambda}/\boldsymbol{\mu})\}, \quad \textit{where } \deg_{\mathbb{Z}}(v^{\mathtt{t}}) = \deg_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathtt{t}).$$

Here, $\deg_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ is combinatorial degree function on tableaux, defined in [11].

3.3 Cuspidal systems

A *convex preorder* on Φ_+ is a binary relation $\succcurlyeq$ on Φ_+ which, for all $\beta, \gamma, \nu \in \Phi_+$ satisfies the following:

(i) $\beta \succcurlyeq \beta$ (reflexivity);

(ii) $\beta \succcurlyeq \gamma$ and $\gamma \succcurlyeq \nu$ imply $\beta \succcurlyeq \nu$ (transitivity);

(iii) $\beta \succcurlyeq \gamma$ or $\gamma \succcurlyeq \beta$ (totality);

(iv) $\beta \succcurlyeq \gamma$ and $\beta + \gamma \in \Phi_+$ imply $\beta \succcurlyeq \beta + \gamma \succcurlyeq \gamma$ (convexity);

(v) $\beta \succcurlyeq \gamma$ and $\gamma \succcurlyeq \beta$ if and only if $\beta = \gamma$ or $\beta, \gamma \in \Phi_+^{\mathrm{im}}$ (imaginary equivalence).

We write $\beta \succ \gamma$ if $\beta \succcurlyeq \gamma$ and $\gamma \not\succcurlyeq \beta$. Then (iii) and (v) together imply that $\succ$ restricts to a total order on $\Psi = \Phi_+^{\mathrm{re}} \sqcup \{\delta\}$. We also write $\beta \approx \gamma$ if $\beta \succcurlyeq \gamma$ and $\gamma \succcurlyeq \beta$, that is if $\beta \approx \gamma$, $\beta \neq \gamma$ if and only if $\beta = m\delta$, $\gamma = m'\delta$, for some $m \neq m' \in \mathbb{Z}_{>0}$.

The proposition below, paraphrased from [2, Example 2.14(ii)] and [10, Example 3.5], provides a way to generate convex preorders on Φ_+.

Proposition 3.2. *Let $(V, \geqslant)$ be a totally ordered $\mathbb{Q}$-vector space. Let $h : \mathbb{Z}I \to V$ be a $\mathbb{Z}$-linear map such that $\beta \mapsto \frac{h(\beta)}{\mathrm{ht}(\beta)}$ is injective on $\Psi \subseteq \mathbb{Z}I$. Then the relation*

$$\beta \succcurlyeq \gamma \iff \frac{h(\beta)}{\mathrm{ht}(\beta)} \geqslant \frac{h(\gamma)}{\mathrm{ht}(\gamma)}$$

defines a convex preorder on Φ_+.

Following [8, 10, 7, 13], for $m \in \mathbb{Z}_{>0}$, $\beta \in \Psi$, we say an $R_{m\beta}$-module M is *semicuspidal* provided that for all $0 \neq \theta_1, \theta_2 \in \mathbb{Z}_{\geqslant 0}I$ with $\theta_1 + \theta_2 = m\beta$, we have $\mathrm{Res}^{m\beta}_{\theta_1,\theta_2} M \neq 0$ only if θ_1 is a sum of positive roots $\preccurlyeq \beta$ and θ_2 is a sum of positive roots $\succcurlyeq \beta$. Here $\mathrm{Res}^{m\beta}_{\theta_1,\theta_2}$ is the restriction functor from $R_{m\beta}$-mod to $R_{\theta_1} \otimes R_{\theta_2}$-mod. We say moreover that M is *cuspidal* if $m = 1$ and the comparisons above are strict. Cuspidal and semicuspidal modules are key building blocks in the representation theory of R_ω.

As explained in [8], for $\beta \in \Phi_+^{\mathrm{re}}$, $m \in \mathbb{Z}_{>0}$, there is a unique 'real' simple semicuspidal $R_{m\beta}$-module denoted $L(\beta^m)$. On the other hand, the 'imaginary' simple semicuspidal $R_{m\delta}$-modules are more plentiful; they may be indexed by $(e-1)$-multipartitions of m:

$$\{L(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} = (\lambda^{(1)} \mid \cdots \mid \lambda^{(e-1)}) \text{ a multipartition of } m\}. \tag{3.1}$$

There is some amount of freedom in this choice of labelling, see for instance [10, Section 21]. Our choices are made for compatibility with *row* Specht modules and e-restricted partition labels, and differ for instance with the labels chosen in [6] by reversing the order of components in $\boldsymbol{\lambda}$.

Let $\omega \in \mathbb{Z}_{\geqslant 0}I$. Recall from the introduction that to each root partition $\pi \in \Pi(\omega)$, we may associate a proper standard module $\bar{\Delta}(\pi)$ as an ordered induction product of simple semicuspidal modules. More precisely, if

$$\pi = ((\beta_1^{K_{\beta_1}} \mid \cdots \mid \beta_u^{K_{\beta_u}} \mid \delta^{K_\delta} \mid \beta_{u+1}^{K_{\beta_{u+1}}} \mid \cdots \mid \beta_t^{K_{\beta_t}}), \boldsymbol{\lambda}),$$

then $\bar{\Delta}(\pi)$ is (up to grading shift) the induction product of semicuspidal simple modules:

$$\bar{\Delta}(\pi) := L(\beta_1^{K_{\beta_1}}) \circ \cdots \circ L(\beta_u^{K_{\beta_u}}) \circ L(\boldsymbol{\lambda}) \circ L(\beta_{u+1}^{K_{\beta_{u+1}}}) \circ \cdots \circ L(\beta_t^{K_{\beta_t}}).$$

The module $\bar{\Delta}(\pi)$ has a self-dual simple head $L(\pi)$, and $\{L(\pi) \mid \pi \in \Pi(\omega)\}$ is a complete and irredundant set of simple R_ω-modules up to isomorphism and grading shift, as explained in [8, 10, 7, 13]. So to construct the proper standard modules, it suffices to construct the simple semicuspidal modules associated to multiples of positive roots.

4 Constructing the simple modules

We say that a nonempty skew diagram $\xi \subseteq \mathcal{N}_\ell$ is a *ribbon* provided that ξ is a nonempty connected skew diagram, contained within one component of $\mathcal{N}_\ell$, containing no 2×2 squares. In [1] it was shown that, for all $\beta \in \Phi_+^{\mathrm{re}}$, there exists an explicit ribbon $\zeta(\beta)$ of content β such that $\mathbf{S}^{\zeta(\beta)} \approx L(\beta)$ (recall that we use $\approx$ to signify isomorphism up to grading shift). For any $a \in \mathbb{Z}/e\mathbb{Z}$, we define the height L positive root $\alpha(a, L) := \alpha_{\bar{a}} + \alpha_{\overline{a+1}} + \cdots + \alpha_{\overline{a+L-1}}$.

As above, associated to each real root β, there is a simple cuspidal R_β-module $L(\beta)$.

Definition 4.1. *Let $\kappa \in \mathbb{Z}$, $\beta = \alpha(a, L) \in \Psi$, where $a \in \mathbb{Z}/e\mathbb{Z}$, $L \in \mathbb{Z}_{>0}$. Let $u \in \mathcal{N}$ be a node with $\mathrm{res}(u) = a$. We define $\zeta(\beta, u)$ to be the ribbon constructed by beginning with the node u, then iteratively adding nodes of residue $\overline{a+1}, \ldots, \overline{a+L-1}$ step-by-step, either to the east or to the north of the previous node, as follows. After i steps of this process, we have constructed a ribbon of content $\alpha(a, i)$. Then in step $i+1$ we either:*

- *add the node of residue $\overline{a+i}$ to the north of the $\overline{a+i-1}$-node if $\alpha(a, i) \succ \beta$, or*
- *add the node of residue $\overline{a+i}$ to the east of the $\overline{a+i-1}$-node if $\alpha(a, i) \prec \beta$.*

After the Lth step, we have constructed the ribbon $\zeta(\beta, u)$ of content β, with southwesternmost node u.

Lemma 4.2. *Let u_i denote a node of residue $i \in \mathbb{Z}/e\mathbb{Z}$. Then the set $\{\zeta(\delta, u_i) \mid i \in \mathbb{Z}/e\mathbb{Z}\}$ consists of e different ribbons of content δ. The ribbons may be distinguished by the number of rows it contains, ranging from 1 to e. Any ribbon $\zeta(\delta, u)$ is equivalent to one of these e ribbons.*

We now choose a distinguished set of *cuspidal* ribbons.

Definition 4.3.

(i) For $\beta = \alpha(a, L) \in \Phi_+^{\mathrm{re}}$, we fix some $u \in \mathcal{N}$ with $\mathrm{res}(u) = a$, and set $\zeta(\beta) := \zeta(\beta, u)$.

(ii) For $i \in [0, e-1]$, we fix some $u \in \mathcal{N}$ such that $\zeta(\delta, u)$ consists of $i+1$ rows, and set $\zeta_i := \zeta(\delta, u)$.

Example 4.4. *In order to help visualise our constructions, we will make use of the following ongoing example. Let $e = 4$, and define a convex preorder $\succcurlyeq$ on Φ_+ following Proposition 3.2, letting $h : \mathbb{Z}I \to \mathbb{Q}^2$ be the $\mathbb{Z}$-linear map given by setting*

$$h(\alpha_0) = (1, 0), \qquad h(\alpha_1) = (-1, -1), \qquad h(\alpha_2) = (2, 1), \qquad h(\alpha_3) = (-2, 0),$$

and taking the usual total lexicographic order on $\mathbb{Q}^2$:

$$(x, y) \geqslant (x', y') \qquad \Longleftrightarrow \qquad x > x' \text{ or } x = x' \text{ and } y \geqslant y',$$

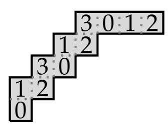
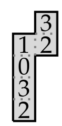
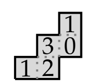
for all $(x, y), (x', y') \in \mathbb{Q}^2$. For $\beta, \gamma \in \Phi_+$, we then set

$$\beta \succcurlyeq \gamma \qquad \Longleftrightarrow \qquad \frac{h(\beta)}{\mathrm{ht}(\beta)} \geqslant \frac{h(\gamma)}{\mathrm{ht}(\gamma)}.$$

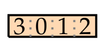
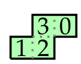
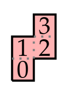
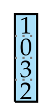
To each $\beta \in \Phi_+^{\mathrm{re}}$, we associate the ribbon $\zeta(\beta)$ of content β via the algorithm described in Definition 4.1 such that $\mathbf{S}^{\zeta(\beta)} \approx L(\beta)$. The corresponding ribbons $\zeta(\beta_i)$ for the indivisible real positive roots

$$\beta_1 = \alpha_2 + \alpha_3 + \alpha_0, \quad \beta_2 = 2\delta + \alpha_0 + \alpha_1 + \alpha_2, \quad \beta_3 = \delta + \alpha_2 + \alpha_3, \quad \beta_4 = \delta + \alpha_1$$

in Φ_+, where $\beta_1 \succ \beta_2 \succ \beta_3 \succ \beta_4$, are

$\zeta(\beta_1) =$  $\zeta(\beta_2) =$  $\zeta(\beta_3) =$  $\zeta(\beta_4) =$ ,

and the e distinct δ-ribbons corresponding to the null root δ are

$\zeta_0 =$  $\zeta_1 =$  $\zeta_2 =$  $\zeta_3 =$ .

Proposition 4.5 ([1, Proposition 8.4])**.** *Let $\beta \in \Phi_+^{\mathrm{re}}$ and $K \in \mathbb{Z}_{>0}$. Up to grading shift, the real semicuspidal self-dual simple module $L(\beta^K)$ is isomorphic to the skew Specht module $\mathbf{S}^{\zeta(\beta)^K}$.*

Our first main result establishes an analogous result for the thornier *imaginary* simple semicuspidal modules.

Definition 4.6. *Let ν be a partition of d, and let $i \in [0, e-1]$. We define the skew diagram $\mathrm{dil}_i(\nu) \in \Lambda(d\delta)$ by setting*

$$\mathrm{dil}_i(\nu) := \bigsqcup_{(x,y)\in\nu} \mathtt{W}^{(x-1)(e-i-1)}\mathtt{S}^{(x-1)(i+1)}\mathtt{E}^{(y-1)(e-i)}\mathtt{N}^{(y-1)i}\zeta_i.$$

*We refer to $\mathrm{dil}_i(\nu)$ as the i-*dilation *of ν. Note that $\mathrm{dil}_i((1)) = \zeta_i$.*

Given an $(e-1)$-multipartition $\boldsymbol{\nu} = (\nu^{(1)} \mid \cdots \mid \nu^{(e-1)})$ of some $K \in \mathbb{Z}_{>0}$, we then set

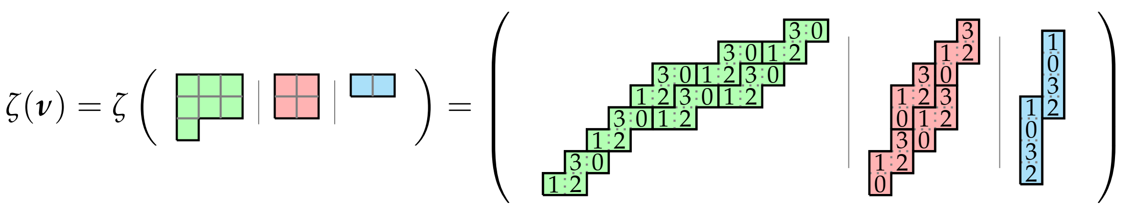
$$\zeta(\boldsymbol{\nu}) := (\mathrm{dil}_1(\nu^{(1)}) \mid \cdots \mid \mathrm{dil}_{e-1}(\nu^{(e-1)})) \in \Lambda^{(e-1)}(K\delta).$$

The definition of i-dilation above is slightly difficult to parse at a glance, but is easily described via the following heuristic. One associates to every node $u \in \nu$ a shifted copy $\xi(u)$ of ζ_i, which we think of as an i-dilation of the node u. The ribbons $\xi(u)$ should be chosen such that

- $\xi((1,1)) = \zeta_i$;
- if $u, \mathtt{E}u \in \nu$, then the northeasternmost node in $\xi(u)$ is one step to the west of the southwesternmost node in $\xi(\mathtt{E}u)$; and
- if $u, \mathtt{S}u \in \nu$, then the southwesternmost node in $\xi(u)$ is one step to the north of the northeasternmost node in $\xi(\mathtt{S}u)$.

Theorem 4.7. *Let $\boldsymbol{\nu}, \boldsymbol{\mu}$ be $(e-1)$-multipartitions of d. Then $\mathbf{S}^{\zeta(\boldsymbol{\nu})}$ is an indecomposable semicuspidal $R_{d\delta}$-module, with simple semicuspidal head $\mathrm{hd}(\mathbf{S}^{\zeta(\boldsymbol{\nu})}) \approx L(\boldsymbol{\nu})$, and $[\mathbf{S}^{\zeta(\boldsymbol{\nu})} : L(\boldsymbol{\mu})] = d^{\mathrm{RoCK}}_{\boldsymbol{\nu},\boldsymbol{\mu}}$, well-studied decomposition numbers of Specht modules in (level one) RoCK blocks of defect d.*

Example 4.8. *Continuing Example 4.4, let $\boldsymbol{\nu} = ((3^2,1) \mid (2^2) \mid (2))$ be a 3-multipartition of 13. Then we have:*



Then $\mathbf{S}^{\zeta(\boldsymbol{\nu})}$ is an indecomposable semicuspidal $R_{13\delta}$-module with $\mathrm{hd}(\mathbf{S}^{\zeta(\boldsymbol{\nu})}) \approx L(\boldsymbol{\nu})$.

Definition 4.9. *Let $\pi = (\boldsymbol{K}, \boldsymbol{\nu}) = ((\beta_1^{K_{\beta_1}} \mid \cdots \mid \beta_u^{K_{\beta_u}} \mid \delta^{K_\delta} \mid \beta_{u+1}^{K_{\beta_{u+1}}} \mid \cdots \mid \beta_t^{K_{\beta_t}}), \boldsymbol{\nu}) \in \Pi(\omega)$ be a root partition of $\omega \in \mathbb{Z}_{\geqslant 0} I$. Writing $k = e - 1 + \sum_{\beta \in \Phi_+^{\mathrm{re}}} K_\beta$, we define the skew diagram $\zeta(\pi)$ by setting*

$$\zeta(\pi) := \left(\zeta(\beta_1)^{K_{\beta_1}} \mid \cdots \mid \zeta(\beta_u)^{K_{\beta_u}} \mid \zeta(\boldsymbol{\nu}) \mid \zeta(\beta_{u+1})^{K_{\beta_{u+1}}} \mid \cdots \mid \zeta(\beta_t)^{K_{\beta_t}}\right) \in \Lambda^k(\omega).$$

Recall that we have a bilexicographic partial order $\geqslant_{\mathrm{b}}$ on $\Xi(\omega)$, the set of all Kostant partitions of ω. We extend this to a *bilexicographic dominance* partial order $\geqslant_{\mathrm{bd}}$ on $\Pi(\omega)$, by incorporating the dominance order on multipartitions. That is,

$$\pi \geqslant_{\mathrm{bd}} \pi' \quad \Longleftrightarrow \quad \boldsymbol{K} \geqslant_{\mathrm{b}} \boldsymbol{K}' \quad \text{or} \quad \boldsymbol{K} = \boldsymbol{K}' \text{ and } \boldsymbol{\nu} \trianglerighteq^D \boldsymbol{\nu}',$$

where $\pi = (\boldsymbol{K}, \boldsymbol{\nu})$ and $\pi' = (\boldsymbol{K}', \boldsymbol{\nu}')$.

Finally, we may state our main theorem, which shows that there exists a cuspidal-theoretic approach to KLR representation theory in which skew Specht modules play a role analogous to proper standard modules. We recall once more that we use $\approx$ to signify isomorphism up to grading shift.

Theorem 4.10. *Let $\pi = (\boldsymbol{K}, \boldsymbol{\nu}) \in \Pi(\omega)$. Then the following statements hold.*

(i) The skew Specht module $\mathbf{S}^{\zeta(\pi)}$ is indecomposable with simple head

$$\mathrm{hd}(\mathbf{S}^{\zeta(\pi)}) \approx L(\pi),$$

so that $\{\mathrm{hd}(\mathbf{S}^{\zeta(\pi)}) \mid \pi \in \Pi(\omega)\}$ gives a complete and irredundant set of simple R_ω-modules up to grading shift.

(ii) For $\sigma \in \Pi(\omega)$, we have $[\mathbf{S}^{\zeta(\pi)} : L(\pi)] = 1$, and $[\mathbf{S}^{\zeta(\pi)} : L(\sigma)] > 0$ only if $\sigma \leqslant_{\mathrm{bd}} \pi$. Moreover, for all root partitions of the form $(\boldsymbol{K}, \boldsymbol{\mu}) \in \Pi(\omega)$, we have $[\mathbf{S}^{\zeta(\pi)} : L(\boldsymbol{K}, \boldsymbol{\mu})] = d^{\mathrm{RoCK}}_{\boldsymbol{\nu},\boldsymbol{\mu}}$.

(iii) There exists a surjection $\mathbf{S}^{\zeta(\pi)} \twoheadrightarrow \bar{\Delta}(\pi)$, and $\mathbf{S}^{\zeta(\pi)}$ has a filtration by proper standard modules of the form $\bar{\Delta}(\boldsymbol{K}, \boldsymbol{\mu})$, where $(\mathbf{S}^{\zeta(\pi)} : \bar{\Delta}(\boldsymbol{K}, \boldsymbol{\mu})) = d^{\mathrm{RoCK}}_{\boldsymbol{\nu},\boldsymbol{\mu}}$.

We note that the isomorphism in part (i) of the theorem is an isomorphism of graded modules up to some shift which can be computed; however, that shift is dependent on the choice of convex preorder, so that there is not a nice formula for this shift in general.

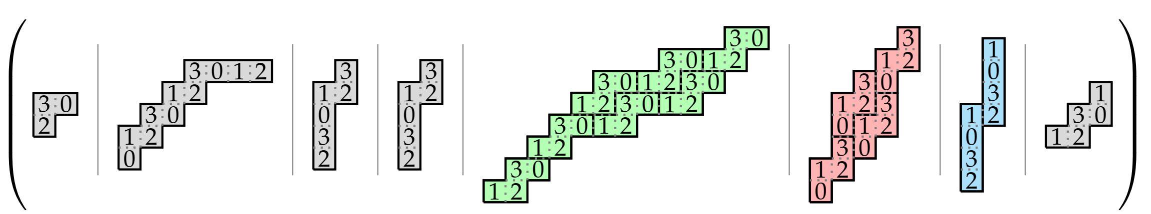
By comparison with the proper standard module $\bar{\Delta}(\pi)$, the Specht module $\mathbf{S}^{\zeta(\pi)}$ has a direct presentation via generators and relations and an explicit combinatorial basis. One may view the above parameterisation of simple KLR modules via skew diagrams as an affine type generalisation of *multisegment* parameterisation of Hecke algebra representations [14].

Example 4.11. *Continuing with Examples 4.4 and 4.8, take for instance the root partition $\pi \in \Pi(20\alpha_0 + 20\alpha_1 + 22\alpha_2 + 21\alpha_3)$ defined as*

$$\pi = \Big(\Big(\alpha_2 + \alpha_3 + \alpha_0 \mid 2\delta + \alpha_0 + \alpha_1 + \alpha_2 \mid (\delta + \alpha_2 + \alpha_3)^2 \mid \delta^{13} \mid \delta + \alpha_1\Big), \Big((3^2, 1) \mid (2^2) \mid (2)\Big)\Big).$$

Then, in consideration of the ribbons in Example 4.4, we have that

$\zeta(\pi) =$



*Then $\mathbf{S}^{\zeta(\pi)}$ is an indecomposable $R_{20\alpha_0+20\alpha_1+22\alpha_2+21\alpha_3}$-*module *with simple head* $\mathrm{hd}(\mathbf{S}^{\zeta(\pi)}) \approx L(\pi)$.

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